On slanted matrices, frames, and sampling

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August 27, 2007

SPIE Optics + Photonics 2007: Wavelets XII
Outline

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   - Sampling in shift invariant spaces
   - Sampling operator (matrix)

2. Main result

3. Conclusions
   - Bonus Results

4. References
Sampling in shift invariant spaces

- \( \Phi = \{\varphi_1, \ldots, \varphi_n\} \) - generator, nice function(s);
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- $(\Phi, X, M)$ - sampling model.
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**Definition**

A sampling model is **stable** if

$$\| (f \ast M)(X) \|_p \sim \| f \|_p \quad \text{for all } f \in V^p(\Phi). \quad (1.1)$$
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A sampling model is **stable** if

$$\| (f * M)(X) \|_p \sim \| f \|_p \quad \text{for all} \ f \in V^p(\Phi). \ (1.1)$$

Stability is preserved by all reasonable perturbations for a fixed $p$, [AK, AAK].
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What if we change $p$?
Sampling operator (matrix)

The sampling operator (matrix) $A$ is given by $(\Phi_k \ast M)(X)$; in the simplest case, $a_{jk} = \varphi(x_j - k)$. 

It is known that (1.1) is equivalent to $\|A c\|_p \sim \|c\|_p$ for all $c \in \ell_p$. (1.2)

Does (1.2) remain valid for all $p$?

Is $A$ automatically left invertible, i.e., can the dual frame method be used for reconstruction?
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The sampling operator (matrix) $\mathcal{A}$ is given by $(\Phi_k * M)(X)$; in the simplest case, $a_{jk} = \varphi(x_j - k)$.

\[
\begin{pmatrix}
  & & \\
  & 0 & \\
 0 & & 0
\end{pmatrix}
\]

It is known that (1.1) is equivalent to

\[\|\mathcal{A}c\|_p \sim \|c\|_p \quad \text{for all } c \in \ell^p. \quad (1.2)\]
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Does (1.2) remain valid for all $p$?

Is $\mathbf{A}$ automatically left invertible, i.e., can the dual frame method be used for reconstruction?
**Theorem (ABK)**

Let $A$ be a matrix with sufficient off-slant decay and satisfying

(1.2) *for some* $p \in [1, \infty]$. *Then* $A$ *satisfies* (1.2) *for all* $p \in [1, \infty]$. *Moreover, a universal* lower bound exists and can be estimated.
Main Result

Theorem (ABK)

Let $A$ be a matrix with sufficient off-slant decay and satisfying (1.2) for some $p \in [1, \infty]$. Then $A$ satisfies (1.2) for all $p \in [1, \infty]$. Moreover, a universal lower bound exists and can be estimated.

Proof.
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Proof.

$2 \to p$. Very easy in a Hilbert space:

$$\|c\|^2 \sim \langle Ac, Ac \rangle = \langle A^*Ac, c \rangle$$

implies invertibility of $A^*A$ in $\ell^2$, invertibility in $\ell^p$ follows from Wiener’s Lemma, and, hence, $A$ is left invertible in all $\ell^p$. 
Theorem (ABK)

Let $\mathbb{A}$ be a matrix with sufficient off-slant decay and satisfying (1.2) for some $p \in [1, \infty]$. Then $\mathbb{A}$ satisfies (1.2) for all $p \in [1, \infty]$. Moreover, a universal lower bound exists and can be estimated.

Proof.

$2 \to p$. Very easy in a Hilbert space:

$$\|c\|^2 \sim \langle \mathbb{A}c, \mathbb{A}c \rangle = \langle \mathbb{A}^*\mathbb{A}c, c \rangle$$

implies invertibility of $\mathbb{A}^*\mathbb{A}$ in $\ell^2$, invertibility in $\ell^p$ follows from Wiener’s Lemma, and, hence, $\mathbb{A}$ is left invertible in all $\ell^p$.

General case. Very hard: over 5 pages of proof. Involves $p \to \infty$, $\infty \to p$, and Cesaro means. $\mathbb{A}^*$ is NOT used.
Conclusions

- If $(\Phi, X, M)$ is a nicely localized sampling model which is stable for some $p$, then it is stable for all $p$. 
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- If $G$ is a nicely localized $p$-frame for some $p$, then it is a Banach frame for all $p$. 

Other applications: differential and difference equations, filter banks, etc.
Conclusions

- If \((Φ, X, M)\) is a nicely localized sampling model which is stable for some \(p\), then it is stable for all \(p\).
- If \(G\) is a nicely localized \(p\)-frame for some \(p\), then it is a Banach frame for all \(p\).
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For first order: \( 1 - \gamma(X) \) in \( \ell^\infty \).
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For first order: \(1 - \gamma(X)\) in \(\ell^\infty\).

For second order: \(\frac{1}{2}(1 - \gamma^2(X))\) in \(\ell^\infty\).
References


The papers are available via http://www.math.niu.edu/~krishtal/ or from ArXiV.