Slanted matrices, Sampling, and Banach frames

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Trends in Harmonic Analysis, Strobl, Austria
Notation and Motivation

- Sampling in shift invariant spaces
- Sampling operator (matrix)

Main result

Conclusions

- Bonus Results

References
Sampling in shift invariant spaces

- \( \Phi = \{ \varphi_1, \ldots, \varphi_n \} \) - generator, nice function(s);
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- $(\Phi, X, M)$ - sampling model.
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**Definition**

A sampling model is **stable** if

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\|(f \ast M)(X)\|_p \sim \|f\|_p \quad \text{for all } f \in \mathcal{V}^p(\Phi). \tag{1.1}
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A sampling model is **stable** if

$$\|(f * M)(X)\|_p \sim \|f\|_p \quad \text{for all } f \in V^p(\Phi). \quad (1.1)$$

Stability is preserved by all reasonable perturbations for a **fixed** $p$, [AK, AAK].

What if we change $p$?
The sampling operator (matrix) $A$ is given by $(\Phi_k \ast M)(X)$; in the simplest case, $a_{jk} = \varphi(x_j - k)$.
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It is known that (1.1) is equivalent to

$$\|\mathbf{A}c\|_p \sim \|c\|_p \quad \text{for all } c \in \ell^p.$$  (1.2)
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It is known that (1.1) is equivalent to

$$\|Ac\|_p \sim \|c\|_p \quad \text{for all } c \in \ell^p.$$  

Does (1.2) remain valid for all $p$?
Main Result

Theorem (ABK)

Let $A$ be a matrix with sufficient off-slant decay and satisfying (1.2) for some $p \in [1, \infty]$. Then $A$ satisfies (1.2) for all $p \in [1, \infty]$. Moreover, a universal lower bound exists and can be estimated.
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Proof.

$2 \rightarrow p$. Very easy in a Hilbert space:

$$\|c\|^2 \sim \langle \mathbb{A}c, \mathbb{A}c \rangle = \langle \mathbb{A}^*\mathbb{A}c, c \rangle$$

implies invertibility of $\mathbb{A}^*\mathbb{A}$ in $\ell^2$, invertibility in $\ell^p$ follows from Wiener’s Lemma, and, hence, $\mathbb{A}$ is left invertible in all $\ell^p$. 

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General case. Very hard: over 5 pages of proof. Involves $p \rightarrow \infty$, $\infty \rightarrow p$, and Cesaro means.
Conclusions

- If \((\Phi, X, M)\) is a nicely localized sampling model which is stable for some \(p\), then it is stable for all \(p\).
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- If $\mathcal{G}$ is a nicely localized $p$-frame for some $p$, then it is a Banach frame for all $p$. 
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- If \(\mathcal{G}\) is a nicely localized \(p\)-frame for some \(p\), then it is a Banach frame for all \(p\).
- Other applications: differential and difference equations, filter banks, etc.
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For first order: $1 - \gamma(X)$ in $\ell^\infty$.

For second order: $\frac{1}{2} (1 - \gamma^2(X))$ in $\ell^\infty$. 
References


The papers are available via [http://www.math.niu.edu/~krishtal/](http://www.math.niu.edu/~krishtal/) or from ArXiV.