THE SPECTRUM OF INFINITE DIMENSIONAL HAMILTONIAN OPERATORS[*]

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Abstract This is a survey of the authors' recent work on the spectral theory of infinite dimensional Hamiltonian operators. Three kinds of spectra, i.e., the point, residual and continuous spectra, are studied. Characterizations of these spectra are obtained for some classes of infinite dimensional Hamiltonian operators. Conditions for the spectra to be empty are also given, and some conditions are necessary and sufficient ones. Moreover, several examples are presented to illustrate these results.

Key words non-self-adjoint operators; infinite dimensional Hamiltonian operators; point spectrum; residual spectrum; continuous spectrum

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Introduction

Partial differential equations (PDE's) have wide applications in science, engineering and technology. Many PDE's, especially many of those from mechanics, can be written as infinite dimensional Hamiltonian systems $\dot{u} = Hu$, where u is a function of a single variable t and taking values in $X \times X$ with X being a Hilbert space, and H is an infinite dimensional Hamiltonian operator. See, for example, [1], [2], [19] and [20]. The now commonly accepted concept of infinite dimensional Hamiltonian system was introduced in 1978 or so, and many famous scholars such as Arnold [4], [5], Gel'fand [7], Lax [12], Magri [13] and Olver [15] have made important contributions to the development and final formulation of this concept. Infinite dimensional Hamiltonian systems are usually continuum mechanics including fluid, plasma and elastic media, and have origins in stability theory, motive power theory, elasticity theory, compound material mechanics, crushing mechanics, etc.

Definition Suppose that X is a Hilbert space and $H : \mathcal{D}(H) \subseteq X \times X \to X \times X$ is a closed densely defined (linear) operator. If $(JH)^* = JH$, then H is called an **infinite dimensional Hamiltonian operator**, where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

with I being the identity operator on X and 0 the zero operator on X, and $(JH)^*$ is the adjoint of JH.

Usually, solution structures of infinite dimensional Hamiltonian systems are determined by the spectral properties of the corresponding infinite dimensional Hamiltonian operators.

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Infinite dimensional Hamiltonian operators are non-self-adjoint operators, and there is no complete theory on their spectrum. Actually, the spectrum of such an operator may be the entire complex plane, or contain a non-empty residual part, which causes many difficulties in the investigation of the spectrum and in solving the corresponding problems in applications.

However, infinite dimensional Hamiltonian operators have a certain type of self-adjointness. We use the J-self-adjoint operator theory of Glazman [8] and the subdivision of the spectrum of Stone [16] to study the spectrum of infinite dimensional Hamiltonian operators.

Even though infinite dimensional Hamiltonian operators are only a special class of non-self-adjoint operators, we believe that a good understanding of their spectrum should provide valuable insights into the spectral theory of general non-self-adjoint operators.

For an operator A, its domain, range, null space, spectrum, point spectrum, residual spectrum, continuous spectrum and resolvent set are denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\sigma(A)$, $P_\sigma(A)$, $R_\sigma(A)$, $C_\sigma(A)$ and $\rho(A)$, respectively. Moreover, the set of complex numbers, the empty set and the set of all bounded operators on a Hilbert space X are written as $\mathbb{C}$, $\emptyset$ and $\mathcal{L}(X)$, respectively.

Throughout this paper, X is always an infinite dimensional Hilbert space.

This paper is organized as follows: Section 1 deals with the point spectrum of infinite dimensional Hamiltonian operators, Section 2 handles the residual spectrum, and Section 3 treats the continuous spectrum.

1 The point spectrum

It is well known that the point spectrum of an operator corresponds to the set of its eigenvalues. Thus, the research of the point spectrum is an important part of Hilbert-Schmidt's Expansion Theorem. Generally speaking, the study of the point spectrum is relatively simple, and to some extent is the foundation of that of other spectra.

Now let us give an equivalent definition of infinite dimensional Hamiltonian operators.

Lemma 1.1 A densely defined operator $H : \mathcal{D}(H) \subseteq X \times X \to X \times X$, $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, is an infinite dimensional Hamiltonian operator if and only if A, B, C and D are all closed densely defined operators, $D = -A^*$, and B and C are both self-adjoint operators.

The following theorem is on the point spectrum of diagonal infinite dimensional Hamiltonian operators. See Lemma 1 in [11].

Theorem 1.2 If $H = \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator, then $P_\sigma(H) = P_\sigma(A) \cup P_\sigma(-A^*)$.

Further, results on the point spectrum of upper triangular infinite dimensional Hamiltonian operators follow. See Theorem 3 and Corollary 4 in [3], respectively.

Theorem 1.3 If $H = \begin{bmatrix} A & C \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator, then

$$P_\sigma(H) = \{\lambda \in \mathbb{C} : \lambda \in P_\sigma(A)\} \cup \{\lambda \in \mathbb{C} : \lambda \in P_\sigma(-A^*), R(C_1) \cap R(\lambda I - A) \neq \emptyset\},$$

where $C_1 = C|_{(\mathcal{N}(\lambda I + A^*) \cap \mathcal{D}(C)) \setminus \{0\}}$ is the restriction of C to $(\mathcal{N}(\lambda I + A^*) \cap \mathcal{D}(C)) \setminus \{0\}$.

Corollary 1.4 If $H = \begin{bmatrix} A & C \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator with $\mathcal{D}(A^*) \subseteq \mathcal{D}(C)$, then

$$P_\sigma(H) = \{\lambda \in \mathbb{C} \colon \lambda \in P_\sigma(A)\} \cup \{\lambda \in \mathbb{C} \colon \lambda \in P_\sigma(-A^*), R(C_2) \cap R(\lambda I - A) \neq \emptyset\},$$

where $C_2 = C|_{\mathcal{N}(\lambda I + A^*) \setminus \{0\}}$.

2 The residual spectrum

Hilbert-Schmidt's Expansion Theorem is the mathematical foundation of the separation of the variables method. It is well known that the Expansion Theorem has two conditions, which are the compactness and the self-adjointness of the operator. These two conditions can guarantee the calculations of coefficients of series expansion and completeness of the system of eigenfunctions.

However, infinite dimensional Hamiltonian operators are non-self-adjoint, which does not satisfy the conditions of Hilbert-Schmidt's Expansion Theorem, and hence we can't directly use Hilbert-Schmidt's Expansion Theorem and should expand the range of its application.

The system of eigenfunctions of self-adjoint operators is orthogonal, but that of infinite dimensional Hamiltonian operators is symplectic orthogonal, so the calculations of coefficients of series expansion has no problem.

Provided that we want to consider the completeness of the system of eigenfunctions, we should first investigate whether the residual spectrum of the operator studied is empty. It is well known that the residual spectrum of some non-self-adjoint operators with certain special properties is empty, such as J-self-adjoint [8], transport [18] and u-scalar operators [17]. In fact, the properties of infinite dimensional Hamiltonian operators are very similar to them, and the residual spectrum of many of them is empty too, which gives us an illusion that the residual spectrum of all infinite dimensional Hamiltonian operators is empty. Frankly speaking, the problem confuses us for a long time that the residual spectrum of which kind of infinite dimensional Hamiltonian operators is non-empty. Because of the importance of this problem, we have studied it for several years and found that there exist infinite dimensional Hamiltonian operators with non-empty residual spectrum indeed, and a necessary and sufficient condition for their residual spectrum to be empty is obtained.

Moreover, some infinite dimensional Hamiltonian operators with non-empty residual spectrum are first constructed, which plays an important role in further study of infinite dimensional Hamiltonian systems and clarifies some ideas.

Theorem 2.1 If $H : \mathcal{D}(H) \subseteq X \times X \to X \times X$ is an infinite dimensional Hamiltonian operator, then $R_\sigma(H) = \emptyset$ if and only if $P_\sigma(H)$ is symmetric about the imaginary axis.

This theorem is just Theorem 2.2.4 in [6], and in order to prove it, we need the following lemmas, which are Lemmas 2.2.2 and 2.2.3 in [6].

Lemma 2.2 If A is a closed densely defined operator in X, then the following results hold:
(1) if $\lambda \in R_\sigma(A)$, then $\overline{\lambda} \in P_\sigma(A^*)$;
(2) if $\lambda \in P_\sigma(A)$, then $\overline{\lambda} \in P_\sigma(A^*) \cup R_\sigma(A^*)$.

Lemma 2.3 If $H : \mathcal{D}(H) \subseteq X \times X \to X \times X$ is an infinite dimensional Hamiltonian operator, then we have:

(1) $\lambda \in R_\sigma(H)$ if and only if $-\lambda \in P_\sigma(H^*)$;

(2) $f \in \mathcal{N}(\lambda I - H)$ if and only if $Jf \in \mathcal{N}(-\lambda I - H^*)$.

As a equivalent proposition of Theorem 2.1, we have:

Corollary 2.4 If $H : \mathcal{D}(H) \subseteq X \times X \to X \times X$ is an infinite dimensional Hamiltonian operator, then $R_\sigma(H) \neq \emptyset$ if and only if $P_\sigma(H)$ is not symmetric about the imaginary axis.

Remark For an operator which is not an infinite dimensional Hamiltonian operator, Theorem 2.1 is not necessary valid. Let's see an example: Let $B : l^2 \to l^2$ be an operator with $Bx = (4x_2, x_1, x_2, \cdots)$ for all $x = (x_1, x_2, x_3 \cdots) \in l^2$. It is easily verified that B is not an infinite dimensional Hamiltonian operator. However, B is not only an one-to-one mapping but also an operator with $\mathcal{R}(B) \subseteq l^2$ and $\overline{\mathcal{R}(B)} \subset l^2$. Therefore $0 \in R_\sigma(B)$. By classical method, we obtain $P_\sigma(B) = \{2, -2\}$. From the discussion above, we know that $R_\sigma(B) \neq \emptyset$ even though $P_\sigma(B)$ is symmetric about the imaginary axis.

Noted that the space $L^2 \times L^2$, which is relative to energy functions, is often used in practice, for instance, the domain of infinite dimensional Hamiltonian operators arisen in mechanics is contained in a functional space. So it is very important to study the residual spectrum of infinite dimensional Hamiltonian operators in a functional space, especially whether there exist such examples that the residual spectrum of them is non-empty in $L^2 \times L^2$ and how to judge whether it is empty before solving the mechanical problems. To solve these problems, we have studied the residual spectrum of triangular infinite dimensional Hamiltonian operators from an operator matrices point of view, and obtained a characterization of it. Then, using this characterization, several infinite dimensional Hamiltonian operators with non-empty residual spectrum are constructed in $L^2 \times L^2$. Also, it is easy to see that the conclusion of Theorem 2.1 is valid for these operators.

Corresponding to the result on the point spectrum of diagonal infinite dimensional Hamiltonian operators in section 2, we have the following result on the residual spectrum. See Theorem 1 in [11].

Theorem 2.5 If $H = \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator, then

$$R_\sigma(H) = \{\lambda \in \mathbb{C} : \lambda \in R_\sigma(A), \lambda \notin P_\sigma(-A^*)\} \cup \{\lambda \in \mathbb{C} : \lambda \in R_\sigma(-A^*), \lambda \notin P_\sigma(A)\}.$$

The results on the residual spectrum of upper triangular infinite dimensional Hamiltonian operators are as follows, and see Theorem 5 and Proposition 6 in [3].

Theorem 2.6 If $H = \begin{bmatrix} A & C \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator, then

$$\begin{aligned} R_\sigma(H) = & \left\{\lambda \in \mathbb{C} : \lambda \notin P_\sigma(A), \lambda \notin P_\sigma(-A^*), \overline{\mathcal{R}(\lambda I - A) + \mathcal{R}(C_3)} \neq X\right\} \\ & \cup \left\{\lambda \in \mathbb{C} : \lambda \notin P_\sigma(A), \mathcal{R}(C_1) \cap \mathcal{R}(\lambda I - A) = \emptyset, \overline{\mathcal{R}(\lambda I - A) + \mathcal{R}(C_3)} \neq X\right\} \\ & \cup \left\{\lambda \in \mathbb{C} : \lambda \notin P_\sigma(A), \lambda \notin P_\sigma(-A^*), \overline{\mathcal{R}(\lambda I + A_1)} \neq X\right\} \\ & \cup \left\{\lambda \in \mathbb{C} : \lambda \notin P_\sigma(A), \mathcal{R}(C_1) \cap \mathcal{R}(\lambda I - A) = \emptyset, \overline{\mathcal{R}(\lambda I + A_1)} \neq X\right\}, \end{aligned}$$

where $A_1 = A^*|_{\mathcal{D}(A^*)\cap\mathcal{D}(C)}$, $C_1 = C|_{(\mathcal{N}(\lambda I + A^*)\cap\mathcal{D}(C))\setminus\{0\}}$ and $C_3 = C|_{\mathcal{D}(A^*)\cap\mathcal{D}(C)}$.

Corollary 2.7 If $H = \begin{bmatrix} A & C \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator with $\mathcal{D}(A^*) \subseteq \mathcal{D}(C)$, then

$$\begin{aligned} R_\sigma(H) \ = \ & \left\{\lambda \in \mathbb{C}\colon \lambda \notin P_\sigma(A), \lambda \notin P_\sigma(-A^*), \overline{\mathcal{R}(\lambda I - A) + \mathcal{R}(C_4)} \neq X\right\} \\ & \cup \left\{\lambda \in \mathbb{C}\colon \lambda \notin P_\sigma(A), \mathcal{R}(C_2) \cap \mathcal{R}(\lambda I - A) = \emptyset, \overline{\mathcal{R}(\lambda I - A) + \mathcal{R}(C_4)} \neq X\right\} \\ & \cup \{\lambda \in \mathbb{C}\colon \lambda \notin P_\sigma(A), \lambda \in R_\sigma(-A^*)\} \\ & \cup \left\{\lambda \in \mathbb{C}\colon \lambda \notin P_\sigma(A), \mathcal{R}(C_2) \cap \mathcal{R}(\lambda I - A) = \emptyset, \overline{\mathcal{R}(\lambda I + A^*)} \neq X\right\}, \end{aligned}$$

where $C_2 = C|_{\mathcal{N}(\lambda I + A^*)\setminus\{0\}}$ and $C_4 = C|_{\mathcal{D}(A^*)}$.

In the proceeding results, we have gotten some characterizations for the residual spectrum of infinite dimensional Hamiltonian operators, thus we have seen that the residual spectrum of them may be not empty so far. An example in the Hilbert space $l^2 \times l^2$ is as follows.

Example 2.8 Suppose that $X = l^2[1, +\infty)$ and the operator $A : X \to X$ is defined by $Ax = (x_1 + 2x_2, x_1, x_2, \cdots)$ for all $x = (x_1, x_2, x_3, \cdots) \in X$. then $H = \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator on $X \times X$ and $R_\sigma(H) \neq \emptyset$.

This example is just Example 2.1.4 in [6]. The following are examples in the functional space $L^2 \times L^2$. For details, see Example 1 in [11] and Examples 2 and 3 in [3].

Example 2.9 Suppose that $X = L^2[0, +\infty)$ and $A : \mathcal{D}(A) \subseteq X \to X$ is a closed densely defined operator and is defined by $Ax = x'$ *for all* $x \in \mathcal{D}(A)$, where

$$\mathcal{D}(A) = \{x \in X : x \text{ is absolutely continuous}, x(0) = 0, x' \in X\},$$

then $H = \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator in $X \times X$ and $R_\sigma(H) \neq \emptyset$.

Example 2.10 Suppose that $X = L^2[0, +\infty)$ and $A : \mathcal{D}(A) \subseteq X \to X$ is the same as in Example 2.9. then $H = \begin{bmatrix} A & I \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator in $X \times X$ and $R_\sigma(H) \neq \emptyset$.

Example 2.11 Suppose that $X = L^2[0, +\infty)$, $A : \mathcal{D}(A) \subseteq X \to X$ is the same as in Example 2.9 and $C : \mathcal{D}(C) \subseteq X \to X$ is a closed densely defined operator and is defined by $Cy = y''$ for all $y \in \mathcal{D}(C)$, where

$$\mathcal{D}(C) = \{y \in X : y' \text{ is absolutely continuous}, y(0) = 0, y', y'' \in X\},$$

then $H = \begin{bmatrix} A & C \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator in $X \times X$ and $R_\sigma(H) \neq \emptyset$.

3 The continuous spectrum

We have obtained some results on the point and residual spectrum of infinite dimensional Hamiltonian operators in two sections above. Separate the variables for an infinite dimensional Hamiltonian system, we can correspondingly get an infinite dimensional Hamiltonian operator. And then we can get the formal solution of the system, but does the formal solution converge to the solution of the system? So we should investigate the continuous spectrum of the operator before studying the convergence of the formal solution, i.e., the completeness of the system of eigenfunctions just as that for self-adjoint operators.

For infinite dimensional Hamiltonian operators, we have gotten a necessary and sufficient condition of $R_\sigma(H) = \emptyset$, and many infinite dimensional Hamiltonian operators have the property of $R_\sigma(H) = \emptyset$, therefore we directly assume $R_\sigma(H) = \emptyset$ in the following discussion sometimes.

First of all, we will discuss the continuous spectrum of upper triangular infinite dimensional Hamiltonian operators.

Theorem 3.1 If $H = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator with $P_\sigma(H)^c \subseteq \rho(A) \cap \rho(A_1)$, then $C_\sigma(H) = \emptyset$, where $P_\sigma(H)^c = \mathbb{C} \backslash P_\sigma(H)$, $A_1 = -A^*|_{\mathcal{D}(B)\cap\mathcal{D}(A^*)}$.

Theorem 3.2 Suppose that $R_\sigma(H) = \emptyset$ and $A = A_1$, if $H = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator with $C_\sigma(H) = \emptyset$, then $P_\sigma(H)^c \subseteq \rho(A)$, where $A_1 = -A^*|_{\mathcal{D}(B)\cap\mathcal{D}(A^*)}$.

These two theorems are Theorems 1 and 2 in [9], respectively. The following corollary can be easily obtained from Theorem 3.2.

Corollary 3.3 Suppose that $R_\sigma(H) = \emptyset$, $A^* = -A$ and $\mathcal{D}(A) \subseteq \mathcal{D}(B)$, if $H = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator with $C_\sigma(H) = \emptyset$, then $P_\sigma(H)^c \subseteq \rho(A)$.

From Theorem 3.1 and Theorem 3.2, we immediately obtain the following theorem, which is a necessary and sufficient condition of $C_\sigma(H) = \emptyset$ under the assumptions of Theorem 3.2. See Theorem 3 in [9].

Theorem 3.4 Suppose that $R_\sigma(H) = \emptyset$, $A = A_1$ and $H = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator, then $C_\sigma(H) = \emptyset$ if and only if $P_\sigma(H)^c \subseteq \rho(A)$, where $A_1 = -A^*|_{\mathcal{D}(B)\cap\mathcal{D}(A^*)}$.

The following lemmas are necessary to prove the theorems above, which are just Lemmas 1 and 2 in [9].

Lemma 3.5 For the infinite dimensional Hamiltonian operator $H = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix}$, if $P_\sigma(H)^c \subseteq \rho(A) \cap \rho(A_1)$, then $1 \in \rho(K_\lambda)$ for any a $\lambda \in P_\sigma(H)^c$, where $K_\lambda = \hat{B}A_\lambda^{-1}$, $A_\lambda = \begin{bmatrix} \lambda I - A & 0 \\ 0 & \lambda I - A_1 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$ and $A_1 = -A^*|_{\mathcal{D}(B)\cap\mathcal{D}(A^*)}$.

Lemma 3.6 Suppose that $H = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator, if $P_\sigma(H)^c \subseteq \rho(A) \cap \rho(A_1)$, then $P_\sigma(H)^c \subseteq \rho(H)$.

Finally, we will generalize the results above to the case of general infinite dimensional Hamiltonian operators with the help of the following lemmas.

Lemma 3.7 Suppose that $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(X)$, $C \in \mathcal{L}(X)$ and $D \in \mathcal{L}(X)$. Consider $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}(X \times X)$, if A and D are invertible then the following assertions are equivalent:

(1) M is invertible in $\mathcal{L}(X \times X)$;
(2) $A - BD^{-1}C$ is invertible in $\mathcal{L}(X)$;
(3) $D - CA^{-1}B$ is invertible in $\mathcal{L}(X)$.

In addition, under the condition (2), we have

$$M^{-1} = \begin{bmatrix} \mathrm{E}^{-1} & -\mathrm{E}^{-1}BD^{-1} \\ -D^{-1}C\mathrm{E}^{-1} & D^{-1}(I + C\mathrm{E}^{-1}BD^{-1}) \end{bmatrix}, \textit{where } \mathrm{E} = A - BD^{-1}C.$$

Remark Lemma 3.7 is a special case of Lemma 2.1 in [14].

Lemma 3.8 Suppose that $H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator with $P_\sigma(H)^c \subseteq \rho(A_1) \cap \rho(A_2)$. For any a $\lambda \in P_\sigma(H)^c$, if $1 \in \rho(B(\lambda I - A_2)^{-1}C(\lambda I - A_1)^{-1})$ then $1 \in \rho(\tilde{K}_\lambda)$, where $\tilde{K}_\lambda = \tilde{B}\tilde{A}_\lambda^{-1}$, $\tilde{A}_\lambda = \begin{bmatrix} \lambda I - A_1 & 0 \\ 0 & \lambda I - A_2 \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, $A_1 = A|_{\mathcal{D}(A) \cap \mathcal{D}(C)}$ and $A_2 = -A^*|_{\mathcal{D}(B) \cap \mathcal{D}(A^*)}$.

This lemma is Lemma 2 in [10], and the following results are Theorems 1, 2 and 3 in [10], respectively.

Theorem 3.9 Suppose that $H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator with $P_\sigma(H)^c \subseteq \rho(A_1) \cap \rho(A_2)$. If $1 \in \rho(B(\lambda I - A_2)^{-1}C(\lambda I - A_1)^{-1})$ for any a $\lambda \in P_\sigma(H)^c$, then $C_\sigma(H) = \emptyset$, where the operators A_1 and A_2 are the same as in Lemma 3.8.

Theorem 3.10 Suppose that $H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator with $P_\sigma(H)^c \subseteq \rho(A_1) \cap \rho(A_2)$ and $R_\sigma(H) = \emptyset$, and both B and C are invertible operators. If $C_\sigma(H) = \emptyset$, then $1 \in \rho(B(\lambda I - A_2)^{-1}C(\lambda I - A_1)^{-1})$ for any a $\lambda \in P_\sigma(H)^c$, where the operators A_1 and A_2 are the same as in Lemma 3.8.

Analogously to the upper triangular case, we can obtain the following theorem under the assumptions of Theorem 3.10.

Theorem 3.11 If the hypotheses of Theorem 3.10 is satisfied, then $C_\sigma(H) = \emptyset$ if and only if $1 \in \rho(B(\lambda I - A_2)^{-1}C(\lambda I - A_1)^{-1})$ for any a $\lambda \in P_\sigma(H)^c$.

Finally, we give two more examples.

Example 3.12 Suppose that $X = L^2[0,1]$, the operator $A = B$ and $A : \mathcal{D}(A) \subseteq X \to X$ is defined by $Ax = x''$, $x \in \mathcal{D}(A)$, where

$$\mathcal{D}(A) = \{x \in X : x' \, is\, absolutely\, continuious\,, x(0) = x(1) = 0, x', x'' \in X\}$$

then $H = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator in $X \times X$ and $C_\sigma(H) = \emptyset$.

This Example is Example in [9], and the following can be found in [10].

Example 3.13 Suppose that $X = L^2[0,1]$ and the operator $A : \mathcal{D}(A) \subseteq X \to X$ is defined the same as in Example 3.12, then $H = \begin{bmatrix} A & 0 \\ I & -A^* \end{bmatrix}$ is an infinite dimensional Hamiltonian operator in $X \times X$ and $C_\sigma(H) = \emptyset$.

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