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# TITCHMARSH-WEYL COEFFICIENTS FOR ODD-ORDER LINEAR HAMILTONIAN SYSTEMS

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ABSTRACT. We define the Titchmarsh-Weyl coefficient for an odd-order linear Hamiltonian system $Jy' - B(x)y = \lambda A(x)y$ in an intrinsic manner and without taking a limit of regular problems. This follows the method of our earlier work for the even order case. We consider here the case of one regular endpoint and one singular endpoint which is in the limit point case. We associate with the system and boundary conditions at the regular point a Hilbert space and a symmetric operator **B**. A difficulty is caused by the possible existence of solutions to $Jy' - B(x)y = A(x)f$ with $||y|| = 0$ and $||f|| \neq 0$. It is shown how the space of such $y$ affects both the definition of the Hilbert space and operator. The odd-order case causes special difficulties since there are two associated operators. The regular even order case is illustrated to show the dependence of both the Hilbert space and associated self-adjoint operator on the boundary conditions.

## 1. INTRODUCTION

In [11] we defined the Titchmarsh-Weyl matrix $M(\lambda)$ for generalized formally selfadjoint systems in terms of separated selfadjoint boundary conditions where an arbitrary deficiency index was allowed. A special case of these systems are Hamiltonian systems of the form

$$Jy' - B(x)y = \lambda A(x)y \tag{1.1}$$

with a skew hermitian invertible matrix $J$ and symmetric matrices $B(x)$ and $A(x)$. For the case of a half open interval $I = [a, b)$ we made the essential assumption that the number of positive and negative eigenvalues of the hermitian matrix $-iJ$ were equal, and hence the investigations were restricted to even order systems.

In this paper we show that the methods used in [11] can be extended to Hamiltonian systems of arbitrary order. By an affine transformation we can transform the matrix $J$ to be of the form

$$J = \begin{pmatrix} 0 & 0 & -E_t \\ 0 & iE_k & 0 \\ E_t & 0 & 0 \end{pmatrix} \tag{1.2}$$

where $E_l$ is the $l \times l$ unit matrix, and $t \geqslant 1$. For $k = 0$ we have the even order case treated in [11], and all results of [11] turn out to be special cases of the theorems we

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prove in this paper in the limit point case. The general case can be proved along similar lines.

A general scalar symmetric differential equation $L[y] = \lambda wy$ can be put in the form (1.1) as shown by Walker [25]. Here $L$ is of the form

$$(1.3)\quad L[y] = \sum_{k=0}^{n} (-1)^k (p_{n-k} y^{(k)})^{(k)} + i \sum_{s=0}^{m} (-1)^s [(q_{m-s} y^{(s)})^{(s-1)} + (q_{m-s} y^{(s-1)})^{(s)}]$$

For $n \geq m$ the equation is of even order $2n$ and for $n < m$ the equation is of odd order $2m-1$. Actually the form (1.3) requires smoothness of the coefficients. With minimal conditions on the coefficients of local integrability as in Section 2, the terms of (1.3) must be grouped to form quasi-derivatives. This is done in Walker [25]. More general symmetric differential expressions than (1.3) have been considered by Everitt and Zettl [9] where the point is discussed of grouping of derivatives to form quasi-derivatives which allow minimal conditions on the coefficients.

Also, if $N$ is a differential expression of the same form as $L$, but with order less that that of $L$, then it was shown by Schneider [24] that the equation $L[y] = N[y]$ can be put in the form (1.1). This was a special case of the theory of S-Hermitian boundary-eigenvalue problems.

While we are mainly concerned here with singular odd order Hamiltonian systems, and the existence of the Titchmarsh-Weyl coefficient which plays a fundamental role in the study of the spectrum of singular systems, the results of Section 3 are applicable to regular even order systems which have widespread applications.

In order to compare the results of this paper with those in [11] our paper is organized in the following way. In section 2 we make the basic assumptions and introduce the notation we use. In Section 3 we digress from the main theme to consider the regular case. Here we show how to obtain a Hilbert space formulation of regular problems. For simplicity we consider only the even order case with separated boundary conditions. It will be seen that the Hilbert space formulation depends not only on (1.1), but also possibly on the boundary conditions. Since these problems have a discrete spectrum, the spectral resolution of the associated self adjoint operator **A** yields an eigenfunction expansion for all elements of the Hilbert space. In a sense, which we now explain, this section may be regarded as a completion of the regular theory presented in Chapter 9 of [1] and Chapter 7, Section 11 of [18] - see in particular Theorem 9.7.4 of [1] and Theorem 11.5 of [18]. In the first place, (1.1) is considered in [18] only for $A(x)$ of the form

$$A(x) = \begin{pmatrix} K(x) & 0 \\ 0 & 0 \end{pmatrix}, \quad K(x) > 0, a.e.,$$

and in both [1] and [18], eigenfunction expansions are obtained only for absolutely continuous functions $y(x)$ which satisfy the associated boundary conditions and $Jy' - B(x)y = A(x)f$ for some $f$ satisfying $\int A|f|^2\, dx < \infty$. The convergence here is uniform and the series for $y'(x)$ may be differentiated termwise with convergence in a square integrable sense. These conditions require the function $y$ be in the domain of the associated operator discussed in Section 3. There is no development of an eigenfunction expansion of an arbitrary element of an associated Hilbert space. While there is an associated Hilbert space in [1] ( defined as $\mathbb{E}(\Delta)$ below), it is pointed out in Section 3, and also by Atkinson, if the space $S(\Delta)$ defined below is $\neq \{0\}$, then one cannot have an eigenfunction expansion for all elements of $\mathbb{E}(\Delta)$. The appropriate reduced Hilbert space is determined in Section 3.

Section 4 explains the definition of the limit point concept for our equation and summarizes some equivalent properties. The main result in section 5 is Theorem 5.3 where we prove the unique existence of the nonsquare (in the case $k \neq 0$) Titchmarsh-Weyl matrix $M(\lambda)$ for given boundary conditions at the regular endpoint $a$. This construction is in terms of the square integrable solution of (1.1) and not as a limiting procedure as has sometimes been done in the even order case, e.g., as in Krall [14] or[15]. $M(\lambda)$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}$ and the number of rows and columns interchange when $\lambda$ changes from $\operatorname{Im}\lambda > 0$ to $\operatorname{Im}\lambda < 0$. The further results we prove in this section reduce to the corresponding results of [11] for the case $k = 0$. In section 6 we construct a resolvent by means of the Titchmarsh-Weyl matrix $M(\lambda)$ in Theorem 6.3 and Theorem 6.22. This resolvent again coincides with the resolvent constructed in the case $k = 0$ in section 5 of [11] and in section 7 we define a symmetric operator $\mathbf{B}$ and its adjoint operator $\mathbf{B}^*$ in a suitable Hilbert space $\mathbb{H}$ by means of the resolvent defined in section 6. $\mathbf{B}$ turns out to be equivalent to a singular boundary eigenvalue problem for the Hamiltonian system (1.1). The upper halfplane $\mathbb{C}^+$ of $\mathbb{C}$ belongs to the resolvent set of $\mathbf{B}$ whereas the lower halfplane $\mathbb{C}^-$ belongs to the residual spectrum of $\mathbf{B}$. For the adjoint operator $\mathbf{B}^*$ the upper halfplane is the resolvent set and the lower halfplane is contained in the point spectrum. In the last section 8 we consider a Hamiltonian system of even order $2(2t + k)$ following an idea of H. D. Niessen in [17] where for the matrix $-i\hat{J}$ the number of positive and negative eigenvalues coincide. Hence in this case the corresponding $\hat{M}(\lambda)$ matrix is a Nevanlinna matrix and we show that this matrix is completely determined by our matrix $M(\lambda)$ of section 5. Although $M(\lambda)$ is nonsquare for $k \geq 1$, its relation to $\hat{M}(\lambda)$ reveals a square Nevanlinna matrix which is a submatrix of $M(\lambda)$ ( the matrix $m_2(\lambda)$ in Section 8). These nonsquare matrices will play an important role in the study of odd order systems with two singular endpoints and the corresponding construction of the resolvent.

The general construction and properties of the Titchmarsh-Weyl matrix $M(\lambda)$ for an odd order symmetric linear Hamiltonian system and associated operators does not seem to have been given in the literature although some partial results have been given. The odd order scalar equation has been thoroughly investigated by Everitt [6] and Everitt and Kumar [7, 8]. For the construction of $M(\lambda)$ for odd order systems as a sequential limit, results are given in Hinton and Shaw [10] for the limit point case, but there is no development of the associated Hilbert space theory. When the system is not in either the maximal or minimal deficiency case, these $M(\lambda)$ matrices obtained by sequential limits may not correspond to self adjoint problems. This was pointed out in [12] and [19].

There is a general theory for the construction of selfadjoint extensions of a symmetric linear relation, e.g., see Bennewitz [2], Coddington and Dijksma [4] or Dijksma, Langer, and DeSnoo [5]. The general theory of symmetric linear relations as applied to Hamiltonian systems usually require the limit point hypothesis. It seems desirable to construct the operator theory for (1.1) directly to define a closed, densely defined operator in a Hilbert space determined by the weight matrix $A(x)$ and associated boundary conditions. For even order systems with arbitrary, but equal, deficiency indices this program was carried out in [11] for separated boundary conditions. Further in [13] for the self adjoint operator **A** defined in [11], the spectral resolution was obtained by using $M(\lambda)$ to obtain a spectral matrix $\rho(\lambda)$. The operator **A** was proved to be isometrically isomorphic to the operator of multiplication by $\lambda$ in the Hilbert space $L^2_\rho(\mathbb{R})$.

In a second paper we will consider the case of two singular endpoints generalizing the corresponding results for the case $k = 0$ and make the application to formally selfadjoint ordinary differential equations of odd order. The two singular endpoint case, with both endpoints limit point, gives rise to a self adjoint operator as contrasted with the case of one singular endpoint. Finally by considering the system of order $2(2t + k)$ in Section 8 we will indicate how the case of arbitrary deficiency index can be treated.

## 2. Assumptions and preliminary results

We make the following assumptions throughout the remainder of the paper (except in Section 3, $I$ is replaced by a compact interval and **III** is not used). Let $I := [a, b) := \{x \in \mathbb{R} | \, a \leqslant x < b\}$ be a half open interval of the real line $(-\infty < a < b \leqslant +\infty)$ and $A$, $B$ be locally integrable mappings from $I$ into the set of complex hermitian $n \times n$ matrices. We designate by $AC_{\text{loc}}(I)$ the set of mappings from $I$ into $\mathbb{C}^n$, which are locally absolutely continuous and by $\mathcal{M}(I)$ the space of measurable mappings defined almost everywhere in $I$ with values in $\mathbb{C}^n$.

With the invertible skew hermitian matrix (1.2), we define for $n = 2t + k$ the mappings

$$F : AC_{\text{loc}}(I) \to \mathcal{M}(I), \qquad G : \mathcal{M}(I) \to \mathcal{M}(I)$$

by

$$(Fy)(x) := Jy'(x) - B(x)y(x)$$
$$(Gy)(x) := A(x)y(x)$$

and consider the <u>Hamiltonian</u> system

$$Fy = \lambda Gy \tag{2.1}$$

under the following hypotheses:

**I**. For almost every $x \in I$ we assume

$$A(x) \geqslant 0.$$

**II**. If

$$N_\lambda := \{y \in AC_{\text{loc}}(I) | \, Fy = \lambda Gy\}$$

then we assume the existence of a $\lambda_0 \in \mathbb{C}$ with the following property: If $y \in N_{\lambda_0}$ with $\int_I y^*(x)A(x)y(x)\,dx = 0$, then $y = 0$.

**III**. The system (2.1) is in the limit point case at $b$ (see the Definition 4.1 below). In section 4 we will give some equivalent descriptions of this condition.

¿From these assumptions we get immediately some easy conclusions. The proofs may be found in [17] and [23], but we sketch how to prove them.

**1**. If $Y(x, \lambda)$ is a fundamental matrix of (2.1) with initial condition $Y(x_0, \lambda)$, then for all $(x, \lambda) \in I \times \mathbb{C}$ the identity

$$Y^*(x, \bar{\lambda})JY(x, \lambda) = Y^*(x_0, \bar{\lambda})JY(x_0, \lambda)$$

holds.

It follows by differentiation that $Y^*(x,\bar\lambda)JY(x,\lambda)$ is constant as thus has its value at $x = x_0$.

**2**. By means of the nonnegative matrix $A(x)$ we define the complex vector space

$$L^2_A(I) := \left\{ y \in \mathcal{M}(I) |\ \int_I y^*(x)A(x)y(x)\,dx < \infty \right\}.$$

Then for $u, v \in L^2_A(I)$ the integral

$$(u,v)_I := \int_I v^*(x)A(x)u(x)\,dx$$

exists and defines a positive semidefinite hermitian scalar product on $L^2_A(I)$ and the vector space $L^2_A(I)$ is complete with respect to the seminorm defined by $\|u\|_I := (u,u)_I^{1/2}$. Note also that $\|u\|_I = 0$ implies $A^{1/2}u = 0$ a.e. and hence $Au = 0$ a.e. Moreover for all $\lambda \in \mathbb{C}$ this scalar product is positive definite on the (finite dimensional) subspaces

$$E_\lambda(I) := N_\lambda \cap L^2_A(I)$$

since hypothesis **II** implies for all $\lambda$ and $y \in N_\lambda$ with $\|y\|_I = 0$ that $y = 0$.

This can be proved with the variation of constants formula. Suppose $y \in N_\lambda$ with $\|y\|_I = 0$. Then

$$Jy' = B(x)y + \lambda A(x)y = B(x)y + \lambda_0 A(x)y + (\lambda - \lambda_0)A(x)y,$$

so that by the variation of constants formula, with $Y(x,\lambda_0)$ as above,

$$y(x) = Y(x,\lambda_0)\left[c + (\lambda - \lambda_0)\int_{x_0}^x Y(s,\lambda_0)^{-1}A(s)y(s)\,ds\right],$$

where $c$ is a vector. Now $\|y\|_I = 0$ implies the above integral is 0 by application of Cauchy-Schwarz. This gives $y(x) = Y(x,\lambda_0)c$ which in turn implies $c = 0$ by hypothesis **II** so that $y = 0$.

**3**. For $u, v \in AC_{\text{loc}}(I)$ the functions $v^*(Fu)$ and $(Fv)^*u$ are locally integrable on $I$ and with

$$[u,v](x) := v^*(x)Ju(x)$$

the Lagrange formula

$$\int_\alpha^\beta v^*(x)(Fu)(x)\,dx - \int_\alpha^\beta (Fv)^*(x)u(x)\,dx = [u,v](\beta) - [u,v](\alpha) \tag{2.2}$$

holds for $a \leqslant \alpha < \beta < b$.

It follows by differentiation that

$$[u,v]'(x) = -(Jv'(x))^*u(x) + v^*(x)Ju'(x) = -(Fv+Bv)^*(x)u(x) + v^*(x)(Fu+Bu)(x),$$

from which (2.2) follows.

## 3. The regular case

We assume throughout this section conditions **I** and **II** of Section 2 hold with $I$ replaced by a compact interval $\Delta = [a,c] \subset [a,b)$. We consider the inhomogeneous problem of even order $2t$,

$$Jy' - B(x)y = \lambda A(x)y + A(x)f(x), \tag{3.1}$$

where $J$ is as in (1.2) with $k = 0$, and $f(x) \in L^2_A(\Delta)$. For (3.1) we impose the separated self adjoint boundary conditions, c.f. [1] or [18], with $\alpha = [\alpha_1, \alpha_2]$ and $\beta = [\beta_1, \beta_2]$,

$$[\alpha_1, \alpha_2]y(a) = 0 = [\beta_1, \beta_2]y(c), \tag{3.2}$$

where the $t \times t$ matrices $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfy the conditions

$$\text{rank } [\alpha_1, \alpha_2] = t = \text{rank } [\beta_1, \beta_2], \quad \alpha_1\alpha_2^* = \alpha_2\alpha_1^*, \quad \beta_1\beta_2^* = \beta_2\beta_1^*.$$

The problem is how to define an appropriate Hilbert space and selfadjoint operator in this space so the spectrum of the equations (3.1)-(3.2) is the same as the that of the selfadjoint operator. First we proceed to study (3.1)-(3.2) as in the development of scalar equations as in [16] or [27]. Here a minimal domain $D_0(\Delta)$ and a maximal domain $R(\Delta)$ are defined and their relationship to other spaces is determined. On $L^2_A(\Delta)$ we have the positive semidefinite inner product

$$(f, g)_\Delta = \int_\Delta g^*(x)A(x)f(x)\,dx,$$

and associated semi-norm $\|f\|^2_\Delta = (f, f)_\Delta$. Further we define the spaces

$$\begin{aligned}
R(\Delta) &= \{y \in AC_{loc}(\Delta) : Fy = Gf \text{ for some } f \in L^2_A(\Delta)\}. \\
N_0(\Delta) &= \{y \in R(\Delta) : Jy' - B(x)y = 0\} \\
N(\Delta) &= \{f \in L^2_A(\Delta) : \|f\|_\Delta = 0\} \\
D_0(\Delta) &= \{y \in R(\Delta) : y(a) = y(c) = 0\} \\
R_0(\Delta) &= \{f \in L^2_A(\Delta) : Fy = Gf \text{ for some } y \in D_0(\Delta)\} \\
S(\Delta) &= \{f \in L^2_A(\Delta) : \exists\, y \in R(\Delta) \cap N(\Delta) \text{ such that } Fy = Gf\} \\
S_{\alpha,\beta}(\Delta) &= \{f \in S(\Delta) : \exists\, y \in R(\Delta) \cap N(\Delta) \text{ such that (3.2) holds and } Fy = Gf\}
\end{aligned}$$

Now we form the Hilbert space $\mathbb{E}(\Delta) = L^2_A(\Delta)/N(\Delta)$, i.e., $\mathbb{E}(\Delta)$ is the Hilbert space of all equivalent classes ($f$ equivalent to $g$ means $\|f - g\|_\Delta = 0$) in $L^2_A(\Delta)$. Let $\pi$ denote the canonical homomorphism from $L^2_A(\Delta)$ onto $\mathbb{E}(\Delta)$. As noted in [13], the mapping that takes $\pi(y)$ in $\pi(R(\Delta))$ into $\pi(f)$ in $\mathbb{E}(\Delta)$ is in general a relation and not a function; hence it cannot be used to define an operator. The space $\mathbb{E}(\Delta)$ is in general too large to be taken for the Hilbert space setting for (3.1)-(3.2). As we shall see, it is the space $S(\Delta)$ that causes the difficulty. For $S(\Delta) = \{0\}$, the space $\mathbb{E}(\Delta)$ is the appropriate Hilbert space. In the case of the scalar equation (1.3), the space $S(\Delta) = \{0\}$. We also have $S(\Delta) = \{0\}$ if $A(x) > 0$ a.e. The assumption $S(\Delta) = \{0\}$ is assumed for the spectral resolution in Chapter 10, page 167, of Krall [15] and in a different context in Brown, Evans, and Plum [3].

To see the difficulty caused by the space $S(\Delta)$, suppose $f \in S(\Delta)$ and $Jy' - B(x)y = A(x)f(x)$ for $y \in R(\Delta) \cap N(\Delta)$ and (3.2) holds. Then if $\phi$ is an eigenfunction for (3.1)-(3.2) with eigenvalue $\lambda$,

$$(f, \phi)_\Delta = (y, \lambda\phi)_\Delta = 0,$$

where we have used the Lagrange formula (2.2) and the selfadjointness of the boundary conditions. This implies that if $\|f\|_\Delta \neq 0$, then $f$ cannot be expanded in a series of eigenfunctions. Hence the space $\mathbb{E}(\Delta)$ must be cut down ( i.e., $\mathbb{E}(\Delta)$ must be replaced by a subspace $\pi(S_{\alpha,\beta}(\Delta)^\perp)$ -see below) to obtain eigenfunction expansions.

To accomplish this, we now prove four short theorems which parallel the development for scalar equations. We note that when $S(\Delta) \neq \{0\}$, the Hilbert space for (3.1)-(3.2) depends on (3.2) as well as (3.1).

**Theorem 3.3.** *We have $N_0(\Delta)^\perp = R_0(\Delta)$ where $\perp$ is taken in the inner product $(\cdot,\cdot)_\Delta$.*

*Proof.* Let $z_1, ...z_n$ be the basis of $N_0(\Delta)$ defined by $Fz_i = 0$ and $z_i(c) = e_i$ where $e_i$ is the unit vector with 1 in the $ith$ component. If $y \in R(\Delta)$, $y(a) = 0$, and $Fy = Gf$, then by the Lagrange indentity (2.2),

$$y^* J z_i|_a^c = y^*(c) J z_i(c) = -\int_a^c f^*(x) A(x) z_i(x)\, dx. \tag{3.4}$$

Now suppose $f \in N_0(\Delta)^\perp$. Then by (3.4), we have that $\int_a^c f^* A z_i \, dx = 0$, $i = 1, ...n$. This gives $y(c) = 0$ which implies $y \in D_0(\Delta)$ for $y$ as in (3.4). Hence $f \in R_0(\Delta)$.

On the other hand, if $f \in R_0(\Delta)$, there exists a $y \in D_0(\Delta)$ so that $Fy = Gf$. Since then $y(c) = 0$, we have

$$0 = y^*(c) J z_i(c) = -\int_a^c f^*(x) A(x) z_i(x)\, dx,\ i = 1, ..., n.$$

Hence $f \in N_0(\Delta)^\perp$. □

**Theorem 3.5.** *We have $N_0(\Delta) \cap N(\Delta) = \{0\}$, and $R_0(\Delta)^\perp = N_0(\Delta) + N(\Delta)$.*

*Proof.* First we note $N_0(\Delta) \cap N(\Delta) = \{0\}$ by assumption **II**. Now by Theorem 3.3 we have $N_0(\Delta) \subset R_0(\Delta)^\perp$. Also $N(\Delta) \subset R_0(\Delta)^\perp$ since $f \in N(\Delta)$, $g \in L_A^2(\Delta)$ implies $\int_\Delta f^* A g \, dx = 0$.

Now suppose $f \in R_0(\Delta)^\perp$ and write $f = f_1 + f_2$ where $f_1 \in N_0(\Delta)$ and $f_2 \perp N_0(\Delta)$. This decomposition is possible since $N_0(\Delta)$ is finite dimensional. Then $f, f_1 \in R_0(\Delta)^\perp$ implies $f_2 \in R_0(\Delta)^\perp$. But $f_2 \in N_0(\Delta)^\perp$ implies $f_2 \in R_0(\Delta)$ by Theorem 3.3. Hence $f_2 \in R_0(\Delta)^\perp$, $f_2 \in R_0(\Delta)$ gives $\|f_2\|_\Delta = 0$. Thus $f_2 \in N(\Delta)$ and we have $R_0(\Delta)^\perp \subset N_0(\Delta) + N(\Delta)$. □

**Theorem 3.6.** *We have $D_0(\Delta)^\perp = S(\Delta)$,*

*Proof.* First suppose $f \in D_0(\Delta)^\perp$. Let $y$ be solution of $Fy = Gf$. If $g \in R_0(\Delta)$ and $z \in D_0(\Delta)$ satisfies $Fz = Gg$, then by the Lagrange identity (2.2),

$$0 = \int_a^c f^* A z \, dx = -y^* J z|_a^c - \int_a^c y^* A g \, dx.$$

Since $z(a) = z(c) = 0$, we conclude that $\int_a^c y^* A g \, ds = 0$ for all $g \in R_0(\Delta)$. Therefore $y \in R_0(\Delta)^\perp$. By 3.5 we can write $y = y_1 + y_2$ where $y_1 \in N_0(\Delta)$ and $y_2 \in N(\Delta)$. Since $Fy_1 = 0$, this gives $Fy_2 = Gf$, and using $\|y_2\|_\Delta = 0$, we get $f \in S(\Delta)$.

Now let $f \in S(\Delta)$ and let $y \in R(\Delta)$ be such that $\|y\|_\Delta = 0$ and $Fy = Gf$. Let $z \in D_0(\Delta)$ and $g$ be such that $Fz = Gg$. Then $\|y\|_\Delta = 0$ implies $\int_a^c y^* A g \, dx = 0$, and hence

$$0 = \int_a^c y^* A g \, dx = -y * J z|_a^c - \int_a^c f^* A z \, dx.$$

Thus $f \in D_0(\Delta)^\perp$. □

**Theorem 3.7.** *If $V$ is a subspace of $L_A^2(\Delta)$, then $\pi(V^\perp) = \pi(V)^\perp$ where the second $\perp$ is in the Hilbert space $\mathbb{E}(\Delta)$ while the first $\perp$ is in $L_A^2(\Delta)$.*

*Proof.* First suppose $f = \pi(g) \in \pi(V^\perp)$. Then for $h = \pi(k) \in \pi(V)$, and denoting the inner product in $\mathbb{E}$ by $(\cdot,\cdot)$,

$$0 = (g,k)_\Delta = (\pi(g), \pi(k)) \Rightarrow f = \pi(g) \in \pi(V)^\perp.$$

Hence $\pi(V^\perp) \subset \pi(V)^\perp$.

Conversely, suppose $h = \pi(k) \in \pi(V)^\perp$. Then for all $f = \pi(g) \in \pi(V)$,

$$0 = (h,f) = (k,g)_\Delta \Rightarrow k \in V^\perp \Rightarrow h \in \pi(V^\perp).$$

Hence $\pi(V)^\perp \subset \pi(V^\perp)$ which completes the proof. □

We see from Theorems 3.6 and 3.7 that $\overline{\pi(D_0(\Delta))^\perp} = \pi(S(\Delta))$ and hence $\pi(S(\Delta)^\perp) = \overline{\pi(D_0(\Delta)}$. Hence if $S(\Delta) = \{0\}$, we have $\overline{\pi(D_0(\Delta)} = \mathbb{E}(\Delta)$, and $\mathbb{E}(\Delta)$ is the appropriate Hilbert space. This is always the case for scalar equations as noted above. In general the appropriate Hilbert space is $\pi(S_{\alpha,\beta}(\Delta)^\perp)$. We will not work out the details for the regular case since the argument is the same as for the singular case which we prove in Section 7 below. The appropriate space is the orthogonal complement of the kernel of a resolvent operator. We will sketch the method.

We define

$$D(\Delta) = \{y \in R(\Delta) : (3.2) \text{ holds }\}.$$

Since the boundary conditions (3.2) are self adjoint, then for $Im\,\lambda \neq 0$, it is known, e.g., [1] or [18], that the problem $Fy = \lambda Gy + Gv$, with (3.2), is uniquely solvable yielding a solution $y(\cdot,\lambda,v)$. This then defines a resolvent operator, for $Im\,\lambda \neq 0$,

$$R_\lambda : L^2_A(\Delta) \longrightarrow D(\Delta), \quad R_\lambda v = y(\cdot,\lambda,v).$$

This leads to a mapping $\Gamma_\lambda$ on $\mathbb{E}(\Delta)$ defined by $\Gamma_\lambda(\pi(v)) = \pi(R_\lambda(v))$. If we define $D_A(\Delta) = \pi(D(\Delta))$, then the mapping $\Gamma_\lambda$ has the properties

$$\Gamma_\lambda(\mathbb{E}(\Delta)) = D_A(\Delta), \quad \Gamma^*_\lambda = \Gamma_{\bar\lambda}, \quad \ker \Gamma_\lambda = \pi(S_{\alpha,\beta}(\Delta)).$$

This last kernel relation (which is independent of $\lambda$) implies

$$\overline{D_A(\Delta)} = \overline{\pi(D(\Delta))} = \pi(S_{\alpha,\beta}(\Delta))^\perp.$$

One can now define an operator for $Im\,\lambda \neq 0$

$$\mathbf{A} : D_A(\Delta) \longrightarrow \overline{D_A(\Delta)} \text{ by } Ay = \lambda y + (\Gamma_\lambda)^{-1} y,$$

and prove the definition of $\mathbf{A}$ is independent of $\lambda$. Further one can prove (by the methods of Section 7) $\mathbf{A}=\mathbf{A}^*$ and for $y \in D(\Delta)$, $y$ nontrivial,

$$Fy = \lambda Gy, (3.2) \text{ holds } \Rightarrow \mathbf{A}(\pi(y)) = \lambda\pi(y),$$

and conversely if $A(\pi(y)) = \lambda\pi(y)$, $\|y\|_\Delta \neq 0$, then for some $g \in N(\Delta)$, we have $F(y+g) = \lambda G(y+g)$. Thus $\mathbf{A}$ is a self adjoint operator acting in the Hilbert space $\overline{D_A(\Delta)}$ with the same eigenvalues as (3.1)-(3.2).

We illustrate the spaces above with an example. Take $\Delta = [0,1]$, and

$$A = \begin{pmatrix} E_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_{12} \\ b_{12}^T & b_{22} \end{pmatrix}, \quad b_{12} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b_{22} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

First, to determine $S(\Delta)$, we suppose $y$ satisfies (3.1) with $\lambda = 0$ for some $f \in L^2_A(\Delta)$. This is equivalent to the equations

$$\begin{aligned} -y_3' &= -y_3 + f_1, \qquad & y_1' &= -y_1 + y_3 - y_4 \\ -y_4' &= -y_4 + f_2, \qquad & y_2' &= -y_2 - y_3 + y_4. \end{aligned} \tag{3.8}$$

If also $\|y\|_\Delta = 0$, then $y_1 = y_2 = 0$ and we conclude from (3.8) that $y_3 = y_4$ and $f_1 = f_2$. It follows that

$$S(\Delta) = \{f \in L^2_A(\Delta) : f_1 = f_2\}. \tag{3.9}$$

We consider boundary conditions for two cases.
Case (i):

$$\alpha_2 = \beta_2 = E_2, \qquad \alpha_1 = \beta_1 = 0. \tag{3.10}$$

¿From $y_1 = y_2 = 0$, $y_3 = y_4$, and $f_1 = f_2$, we calculate, using $y_3(0) = y_3(1) = 0$, that $y_3(x) = e^x \int_0^x e^{-s} f_1(s) ds$. Hence

$$S_{\alpha,\beta}(\Delta) = \{f \in S(\Delta) : \int_0^1 e^{-s} f_1(s) ds = 0\} \tag{3.11}$$

Case (ii):

$$\alpha_1 = \beta_1 = E_2, \qquad \alpha_2 = \beta_2 = 0 \tag{3.12}$$

In this case, since $\|y\|_\Delta = 0$ implies $y_1 = y_2 = 0$, the boundary conditions with (3.12) hold so that we have

$$S_{\alpha,\beta}(\Delta) = S(\Delta).$$

To determine $D_0(\Delta)$, we must solve (3.8) with the boundary conditions $y_i(0) = y_i(1) = 0$ for $i = 1, ..., 4$. Calculations yield that

$$y_3(x) = -e^x \int_0^x e^{-s} f_1(s)\, ds, \qquad y_4(x) = -e^x \int_0^x e^{-s} f_2(s)\, ds,$$

where

$$\int_0^1 e^{-s} f_1(s)\, ds = \int_0^1 e^{-s} f_2(s)\, ds = 0 \tag{3.13}$$

Also $y_2(x) = -y_1(x)$ and $y_1(x) = e^{-x} \int_0^x e^s [y_3(s) - y_4(s)]\, ds$ where

$$\int_0^1 e^s [y_3(s) - y_4(s)]\, ds = \int_0^1 \frac{1}{2}[e^{2-\tau} - e^\tau][f_2(s) - f_1(s)]\, ds = 0. \tag{3.14}$$

Hence $D_0(\Delta)$ consists of all $y$ given by the above equations where $f \in L^2_A(\Delta)$ satisfies (3.13) and (3.14).

For $z = e^{-x}[1, 1, 0, 0]^T$, it is clear (3.2) holds for case (i) above. Calculations show $Fz = 0$ which implies $\mathbf{A}(\pi(z)) = 0$. Thus $\pi(z) \in D_A(\Delta)$. Also for $y \in D_0(\Delta)$,

$$(z, y)_\Delta = \int_0^1 e^{-s}[y_1(s) + y_2(s)]\, ds = 0$$

since $y_1 = -y_2$ for $y \in D_0(\Delta)$. This yields by Theorems 3.6 and 3.7 that $z \in S(\Delta) = D_0(\Delta)^\perp$ and $\pi(z) \in \pi(D_0(\Delta))^\perp = \pi(S(\Delta)) = \overline{\pi(D_0(\Delta))^\perp}$. Thus $\overline{\pi(D_0(\Delta))}$ is a proper subset of $\overline{D_A(\Delta)}$. Hence $\overline{\pi(D_0(\Delta))}$ is not large enough for a Hilbert space formulation of (3.1)-(3.2) for the boundary conditions of case (i).

In the case of regular self adjoint problems, the theory of eigenfunctions can also be worked out from the theory of S-Hermitian boundary-eigenvalue problems as developed by Schäfke and Schneider in [20, 21, 22]. In these papers it is shown how in the right definite case these eigenvalue problems reduce to the eigenvalue problem for the associated self adjoint Wielandt operator $\mathbf{A}$. It has the same eigenfunctions and the reciprocal eigenvalues, and from the Parseval equation for $\mathbf{A}$ there follows the eigenfunction expansions for all elements in the closure of the range of $\mathbf{A}$, which is the appropriate pre-Hilbert space. To describe the problem by an unbounded differential operator, the role is played by the inverse $\mathbf{A}^{-1}$, existing on the orthogonal complement of the kernel of $\mathbf{A}$.

## 4. The limit point condition

If $Y(x,\lambda)$ is a fundamental matrix of (2.1), then for $f \in \mathbb{C}^n$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ we get by the Lagrange formula (2.2) for $a \leqslant \alpha < \beta < b$ the relation

$$\int_\alpha^\beta (Y(x,\lambda)f)^* A(x)(Y(x,\lambda)f)\,dx = f^*\mathcal{A}(\beta,\lambda)f - f^*\mathcal{A}(\alpha,\lambda)f$$

with

$$\mathcal{A}(x,\lambda) = (2i\,\mathrm{Im}\,\lambda)^{-1}Y^*(x,\lambda)JY(x,\lambda).$$

$\mathcal{A}(x,\lambda)$ is invertible, hermitian and monotone nondecreasing with respect to $x$. Since the hermitian matrix $i^{-1}J$ has $t+k$ positive and $t$ negative eigenvalues, then for the number $i^+(\lambda)$ of positive and $i^-(\lambda)$ of negative eigenvalues of $\mathcal{A}(x,\lambda)$ which are independent of $x$ we get

$$\big(i^+(\lambda), i^-(\lambda)\big) = \begin{cases} (t+k,t), & \lambda \in \mathbb{C}^+ \\ (t,t+k), & \lambda \in \mathbb{C}^- \,. \end{cases}$$

Therefore — compare with section 2 of [11] — we have

$$\dim E_\lambda(I) \geqslant i^-(\lambda) = \begin{cases} t, & \lambda \in \mathbb{C}^+ \\ t+k, & \lambda \in \mathbb{C}^- \,. \end{cases}$$

We introduce the "deficiency indices" $\tau_b(\lambda)$ by

$$\tau_b(\lambda) := \dim E_\lambda(I) - i^-(\lambda), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

$\tau_b(\lambda)$ is locally constant in $\mathbb{C} \setminus \mathbb{R}$ — compare with remark 2.11 of [11] — and then we make the following

**Definition 4.1.** The system (2.1) is said to be in the **limit point case** at $b$ if and only if $\tau_b(i) = \tau_b(-i) = 0$.

We define the vector space

$$R(I) := F^{-1}(G(L^2_A(I))) \cap L^2_A(I).$$

By Lagrange's formula (2.2) the existence of the limit

$$[u,v](b) := \lim_{\beta \to b} [u,v](\beta)$$

for all $u, v \in R(I)$ follows. Defining the subspace

$$R^b(I) := \{v \in R(I) \,|\, [u,v](b) = 0 \text{ for all } u \in R(I)\}$$

we know that the relation

$$\dim\big(R(I)\,/R^b(I)\,\big) = \tau_b(i) + \tau_b(-i) \tag{4.2}$$

is valid (see formula (2.9) in [11]). Hence we have

**Corollary 4.3.** *(2.1) is in the limit point case at $b$ if and only if $R(I) = R^b(I)$ and this is equivalent to the condition that $[u,v](b) = 0$ for all $u, v \in R(I)$. In this case the Lagrange formula reduces to*

$$\int_a^b v^*(x)(Fu)(x)\,dx - \int_a^b (Fv)^*(x)u(x)\,dx = -[u,v](a) \tag{4.4}$$

*for $u, v \in R(I)$.*

*Remark* . In the case $k = 0$ the system (2.1) is of even order $2t$ and the definition 4.1 of the limit point case at $b$ coincides with the classical definition known in this case.

## 5. The Titchmarsh-Weyl coefficient

In this section we show how a Titchmarsh-Weyl coefficient $M(\lambda)$ can be defined for the system (2.1) generalizing the known results for the case $k = 0$.

Let $A_1$, $A_2$ be $t \times t$ matrices satisfying the conditions

$$A_1 A_2^* = A_2 A_1^*; \qquad A_1 A_1^* + A_2 A_2^* = E_t \tag{5.1}$$

and then define the $(2t+k) \times (2t+k)$ matrix $\tilde{A}$ by

$$\tilde{A} := \begin{bmatrix} A_1^* & 0 & -A_2^* \\ 0 & E_k & 0 \\ A_2^* & 0 & A_1^* \end{bmatrix}.$$

This matrix fulfills the relation

$$\tilde{A}^* \cdot J \cdot \tilde{A} = J. \tag{5.2}$$

We denote by $W(x,\lambda)$ the fundamental matrix of (2.1) with the initial condition $W(a,\lambda) = \tilde{A}$. Then we prove the following

**Theorem 5.3.** *For $\operatorname{Im}\lambda \neq 0$ define the matrices*

$$\Theta(x,\lambda) := W(x,\lambda)\begin{pmatrix} E_{i^-(\lambda)} \\ 0 \end{pmatrix}; \qquad \Phi(x,\lambda) := W(x,\lambda)\begin{pmatrix} 0 \\ E_{i^+(\lambda)} \end{pmatrix}. \tag{5.4}$$

*Then for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists a $i^+(\lambda) \times i^-(\lambda)$ matrix $M(\lambda)$ with*

$$\Theta(\cdot,\lambda) + \Phi(\cdot,\lambda)M(\lambda) \in \left(L_A^2(I)\right)^{i^-(\lambda)}. \tag{5.5}$$

*$M(\lambda)$ is uniquely determined by (5.5).*

*Proof.* By assumption **III** the dimension of $E_\lambda(I)$ is equal to $i^-(\lambda)$. Hence there exist an $i^-(\lambda) \times i^-(\lambda)$ matrix $B_1(\lambda)$ and an $i^+(\lambda) \times i^-(\lambda)$ matrix $B_2(\lambda)$ such that

$$\operatorname{rank}\left(B_1(\lambda)^T, B_2(\lambda)^T\right) = i^-(\lambda) \tag{5.6}$$

$$\Theta(\cdot,\lambda)B_1(\lambda) + \Phi(\cdot,\lambda)B_2(\lambda) \in \left(L_A^2(I)\right)^{i^-(\lambda)} \tag{5.7}$$

$$\left[\Theta(\cdot,\lambda)B_1(\lambda) + \Phi(\cdot,\lambda)B_2(\lambda), u\right](b) = 0 \text{ for all } u \in R(I). \tag{5.8}$$

We will show that $B_1(\lambda)$ is invertible. To prove this let $f \in \mathbb{C}^{i^-(\lambda)}$ with $B_1(\lambda)f = 0$. Then

$$u_\lambda(\cdot) = \Phi(\cdot,\lambda)B_2(\lambda)f \in \left(L_A^2(I)\right)^{i^-(\lambda)}.$$

For elements $f, g \in \mathcal{M}(I)$ we introduce the bilinear form

$$[f,g]_I := \int_I g^*(x)f(x)\,dx$$

whenever the integral exists. By Lagrange's formula (4.4) we get

$$
\begin{aligned}
2i\,\mathrm{Im}\,\lambda\,[Gu_\lambda, u_\lambda]_I &= \lambda\,[Gu_\lambda, u_\lambda]_I - \bar{\lambda}\,[Gu_\lambda, u_\lambda]_I \\
&= [Fu_\lambda, u_\lambda]_I - [u_\lambda, Fu_\lambda]_I \\
&= -\,[u_\lambda, u_\lambda]\,(a).
\end{aligned} \tag{5.9}
$$

By definition of $\Phi(x,\lambda)$ we have

$$
\begin{aligned}
\Phi^*(a,\lambda)J\Phi(a,\lambda) &= \big(0, E_{i^+(\lambda)}\big)\, W^*(a,\lambda)JW(a,\lambda)\begin{pmatrix} 0 \\ E_{i^+(\lambda)} \end{pmatrix} \\
&= \big(0, E_{i^+(\lambda)}\big)\, J\begin{pmatrix} 0 \\ E_{i^+(\lambda)} \end{pmatrix} \\
&= \begin{cases} \begin{pmatrix} iE_k & 0 \\ 0 & 0 \end{pmatrix}, & \lambda \in \mathbb{C}^+ \\ 0, & \lambda \in \mathbb{C}^- \end{cases}.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
(u_\lambda, u_\lambda)_I &= -\frac{1}{2i\,\mathrm{Im}\,\lambda}\cdot [u_\lambda, u_\lambda]\,(a) \\
&= -\frac{1}{2i\,\mathrm{Im}\,\lambda}\cdot (B_2(\lambda)f)^*\,\Phi^*(a,\lambda)J\Phi(a,\lambda)(B_2(\lambda)f) \\
&= \begin{cases} -\frac{1}{2\,\mathrm{Im}\,\lambda}(B_2(\lambda)f)^*\begin{pmatrix} E_k & 0 \\ 0 & 0 \end{pmatrix}(B_2(\lambda)f), & \lambda \in \mathbb{C}^+ \\ 0, & \lambda \in \mathbb{C}^- \end{cases}
\end{aligned}
$$

from which we see, that

$$(u_\lambda, u_\lambda)_I = \|u_\lambda\|_I^2 = 0.$$

Therefore by assumption **II**,

$$u_\lambda(x) = \Phi(x,\lambda)(B_2(\lambda)f) \equiv 0,$$

so that $B_2(\lambda)f = 0 = B_1(\lambda)f$. But then $f = 0$ proving that $B_1(\lambda)$ is invertible. Now

$$\Theta(\cdot,\lambda)B_1(\lambda) + \Phi(\cdot,\lambda)B_2(\lambda) = \left[\Theta(\cdot,\lambda) + \Phi(\cdot,\lambda)B_2(\lambda)B_1(\lambda)^{-1}\right]B_1(\lambda)$$

and defining the $i^+(\lambda) \times i^-(\lambda)$ matrix

$$M(\lambda) := B_2(\lambda)B_1(\lambda)^{-1}$$

the assertion (5.5) follows.

Now we prove that $M(\lambda)$ is uniquely determined by (5.5). For this reason let $\hat{M}(\lambda)$ be a $i^+(\lambda) \times i^-(\lambda)$ matrix fulfilling (5.5) for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Let $f \in \mathbb{C}^{i^-(\lambda)}$ be arbitrary and consider

$$
\begin{aligned}
v_\lambda(\cdot) &:= \big(\Theta(\cdot,\lambda) + \Phi(\cdot,\lambda)M(\lambda)\big)f - \big(\Theta(\cdot,\lambda) + \Phi(\cdot,\lambda)\hat{M}(\lambda)\big)f \\
&= \Phi(\cdot,\lambda)\big(M(\lambda) - \hat{M}(\lambda)\big)f.
\end{aligned}
$$

$v_\lambda \in E_\lambda(I)$ and therefore $[v_\lambda, v_\lambda](b) = 0$. Then by the Lagrange formula (4.4) we get

$$\begin{aligned} 2i \operatorname{Im} \lambda \, (v_\lambda, v_\lambda)_I &= \lambda \, [G v_\lambda, v_\lambda]_I - \bar{\lambda} \, [v_\lambda, G v_\lambda]_I \\ &= [F v_\lambda, v_\lambda]_I - [v_\lambda, F v_\lambda]_I \\ &= - \, [v_\lambda, v_\lambda] \, (a) \\ &= - \Big( \big(M(\lambda) - \hat{M}(\lambda)\big) f \Big)^* \, \Phi^*(a, \lambda) J \Phi(a, \lambda) \, \Big( \big(M(\lambda) - \hat{M}(\lambda)\big) f \Big). \end{aligned}$$

Defining $h(\lambda) := \big(M(\lambda) - \hat{M}(\lambda)\big) f$ the relation

$$\begin{aligned} (v_\lambda, v_\lambda)_I &= - \frac{1}{2i \operatorname{Im} \lambda} \cdot [v_\lambda, v_\lambda] \, (a) \\ &= \begin{cases} -\frac{1}{2 \operatorname{Im} \lambda} h^*(\lambda) \begin{pmatrix} E_k & 0 \\ 0 & 0 \end{pmatrix} h(\lambda), & \lambda \in \mathbb{C}^+ \\ 0, & \lambda \in \mathbb{C}^- \end{cases} \end{aligned}$$

follows. Thus

$$(v_\lambda, v_\lambda)_I = \|v_\lambda\|_I^2 = 0$$

and with respect to assumption **II** again we conclude that

$$v_\lambda(x) = \Phi(x, \lambda) \big(M(\lambda) - \hat{M}(\lambda)\big) f \equiv 0,$$

yielding $\big(M(\lambda) - \hat{M}(\lambda)\big) f = 0$ and since $f$ is arbitrary $M(\lambda) = \hat{M}(\lambda)$ follows and the proof of Theorem 5.3 is complete. □

Next we state a generalization of Theorem 4.16 in [11].

**Theorem 5.10.** *If* $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, *then*

$$\begin{aligned} (5.11) \quad (\lambda - \mu) \big(\Theta(\cdot, \lambda) + \Phi(\cdot, \lambda) M(\lambda), \Theta(\cdot, \bar{\mu}) + \Phi(\cdot, \bar{\mu}) M(\bar{\mu})\big)_I \\ = - \big(E_{i^-(\bar{\mu})}, M^*(\bar{\mu})\big) \cdot J \begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix}. \end{aligned}$$

*Proof.* We set

$$z_\alpha := \Theta(\cdot, \alpha) + \Phi(\cdot, \alpha) M(\alpha) = W(\cdot, \alpha) \begin{pmatrix} E_{i^-(\alpha)} \\ M(\alpha) \end{pmatrix} \tag{5.12}$$

and using the Lagrange formula (4.4) we get

$$\begin{aligned} (\lambda - \mu)(z_\lambda, z_{\bar{\mu}})_I &= \lambda \, [G z_\lambda, z_{\bar{\mu}}]_I - \mu \, [z_\lambda, G z_{\bar{\mu}}]_I \\ &= [F z_\lambda, z_{\bar{\mu}}]_I - [z_\lambda, F z_{\bar{\mu}}]_I \\ &= - \, [z_\lambda, z_{\bar{\mu}}] \, (a) \\ &= - \big(E_{i^-(\bar{\mu})}, M^*(\bar{\mu})\big) W^*(a, \bar{\mu}) J W(a, \lambda) \begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix} \end{aligned}$$

and (5.11) follows with respect to (5.2). □

*Remark* . If $k = 0$, then we have $i^-(\bar{\mu}) = i^-(\lambda) = t$ and the right-hand side of (5.11) is equal to

$$- \big(E_t, M^*(\bar{\mu})\big) \begin{pmatrix} 0 & -E_t \\ E_t & 0 \end{pmatrix} \begin{pmatrix} E_t \\ M(\lambda) \end{pmatrix} = M(\lambda) - M^*(\bar{\mu}).$$

Hence (4.17) of [11] is a special case of (5.11).

¿From Theorem 5.10 we get as corollaries generalizations of the corresponding corollaries 4.18 and 4.20 in [11].

**Corollary 5.13.** *If* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*, then*

$$M^*(\bar\lambda) = \begin{cases} CM(\lambda), & \lambda \in \mathbb{C}^+ \\ M(\lambda)C, & \lambda \in \mathbb{C}^- \end{cases}. \tag{5.14}$$

*Here* $C$ *is the* $(t+k) \times (t+k)$ *matrix:*

$$C = \begin{pmatrix} 0 & E_t \\ -iE_k, & 0 \end{pmatrix}.$$

*Proof.* In (5.11) we choose $\mu := \lambda$ and consider first the case $\lambda \in \mathbb{C}^+$. Then $i^-(\lambda) = t$ and $i^-(\bar\lambda) = t + k$ and (5.11) yields

$$\begin{aligned} 0 &= \left(E_{t+k}, M^*(\bar\lambda)\right) \begin{pmatrix} 0 & -C \\ E_t & 0 \end{pmatrix} \begin{pmatrix} E_t \\ M(\lambda) \end{pmatrix} \\ &= \left(E_{t+k}, M^*(\bar\lambda)\right) \begin{pmatrix} -CM(\lambda) \\ E_t \end{pmatrix} \\ &= -CM(\lambda) + M^*(\bar\lambda) \end{aligned}$$

and the first part of (5.14) follows. If $\lambda \in \mathbb{C}^-$, set $\mu := \bar\lambda$. Then $\mu \in \mathbb{C}^+$ and from the first part of (5.14) we get

$$CM(\mu) = CM(\bar\lambda) = M^*(\bar\mu) = M^*(\lambda).$$

Since $C^* = C^{-1}$ we receive by taking the adjoint

$$M(\lambda) = M^*(\bar\lambda)C^* = M^*(\bar\lambda)C^{-1}$$

and thus the equation

$$M^*(\bar\lambda) = M(\lambda)C$$

follows. □

*Remark* . In the case $k = 0$ we have $C = E_t$ and (5.14) reduces to the known relation $M^*(\bar\lambda) = M(\lambda)$.

If we choose $\mu := \bar\lambda$ in Theorem 5.10 then have

**Corollary 5.15.** *For* $\lambda \in \mathbb{C} \setminus \mathbb{R}$ *the relations*

$$\begin{aligned} &2i \operatorname{Im} \lambda \left(\Theta(\cdot,\lambda) + \Phi(\cdot,\lambda)M(\lambda), \Theta(\cdot,\lambda) + \Phi(\cdot,\lambda)M(\lambda)\right)_I \\ &= \begin{cases} \left(-M(\bar\lambda), E_{i^-(\lambda)}\right) \begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix}, & \lambda \in \mathbb{C}^+ \\ C\left(-M(\bar\lambda), E_{i^-(\lambda)}\right) \begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix}, & \lambda \in \mathbb{C}^- \end{cases} \end{aligned} \tag{5.16}$$

*are valid.*

We omit the proof since it follows by straightforward calculations using Corollary 5.13.

*Remark* . In case $k=0$ we have $C=E_t$ and $i^-(\lambda)=i^+(\lambda)=t$. Then (5.16) yields for $\lambda\in\mathbb{C}\setminus\mathbb{R}$:

$$\begin{aligned}
&2i\,\mathrm{Im}\,\lambda\big(\Theta(\cdot,\lambda)+\Phi(\cdot,\lambda)M(\lambda),\Theta(\cdot,\lambda)+\Phi(\cdot,\lambda)M(\lambda)\big)_I\\
&\qquad=\big(-M(\bar\lambda),E_t\big)\begin{pmatrix}E_t\\ M(\lambda)\end{pmatrix}\\
&\qquad=-M(\bar\lambda)+M(\lambda)\\
&\qquad=M(\lambda)-M^*(\lambda)=2i\,\mathrm{Im}\,M(\lambda).
\end{aligned}\tag{5.17}$$

Hence (4.21) of [11] is a special case of (5.16).

As a special case of Theorem 5.10 we have

**Corollary 5.18.** *The matrix* $M(\lambda)$ *satisfies the relations*

$$M(\lambda)-M(\mu)=\begin{cases}(\lambda-\mu)C^*(z_\lambda,z_{\bar\mu})_I, & \lambda,\mu\in\mathbb{C}^+\\ (\lambda-\mu)(z_\lambda,z_{\bar\mu})_I, & \lambda,\mu\in\mathbb{C}^-\end{cases}\tag{5.19}$$

*where* $z_\alpha$ *is defined by* (5.12).

*Proof.* For $\lambda,\mu\in\mathbb{C}^+$ we have $i^-(\bar\mu)=t+k$ and $i^-(\lambda)=t$. Thus from (5.11) and (5.14) we get

$$\begin{aligned}
(\lambda-\mu)(z_\lambda,z_{\bar\mu})_I&=-\big(E_{t+k},M^*(\bar\mu)\big)\begin{pmatrix}0&-C\\E_t&0\end{pmatrix}\begin{pmatrix}E_t\\M(\lambda)\end{pmatrix}\\
&=-\big(E_{t+k},M^*(\bar\mu)\big)\begin{pmatrix}-CM(\lambda)\\E_t\end{pmatrix}\\
&=-\big(E_{t+k},CM(\mu)\big)\begin{pmatrix}-CM(\lambda)\\E_t\end{pmatrix}\\
&=CM(\lambda)-CM(\mu)=C\big(M(\lambda)-M(\mu)\big).
\end{aligned}$$

Hence

$$(\lambda-\mu)C^*(z_\lambda,z_{\bar\mu})_I=M(\lambda)-M(\mu).$$

If $\lambda,\mu\in\mathbb{C}^-$, then $i^-(\bar\mu)=t$ and $i^-(\lambda)=t+k$. Now we get respecting (5.14)

$$\begin{aligned}
(\lambda-\mu)(z_\lambda,z_{\bar\mu})_I&=-\big(E_t,M^*(\bar\mu)\big)\begin{pmatrix}0&-E_t\\C^*&0\end{pmatrix}\begin{pmatrix}E_{t+k}\\M(\lambda)\end{pmatrix}\\
&=-\big(E_t,M^*(\bar\mu)\big)\begin{pmatrix}-M(\lambda)\\C^*\end{pmatrix}\\
&=M(\lambda)-M^*(\bar\mu)C^*\\
&=M(\lambda)-(M(\mu)C)C^*\\
&=M(\lambda)-M(\mu)
\end{aligned}$$

and the proof of (5.19) is complete. □

## 6. The inhomogeneous equation

By means of the $M(\lambda)$ matrix we will construct a resolvent representing the uniquely determined solution of a singular inhomogeneous boundary value problem defined by the Hamiltonian system (2.1) and suitable boundary conditions. Let

$$D^+:=\big\{y\in AC_{\mathrm{loc}}(I)\cap L^2_A(I)\,|\,(A_1,0,A_2)y(a)=0\big\}.$$

Then we get

**Lemma 6.1.** *The system (2.1) has only the trivial solution in $D^+$ for $\lambda \in \mathbb{C}^+$.*

*Proof.* Let $w \in E_\lambda(I) \cap D^+$. Since the columns of

$$W(\cdot,\lambda)\begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix}$$

form a basis of $E_\lambda(I)$ the function $w$ is of the form

$$w = W(\cdot,\lambda)\begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix} c$$

with some $c \in \mathbb{C}^{i^-(\lambda)}$ and then $w \in D^+$ if and only if

$$\begin{aligned}(A_1, 0, A_2)w(a) &= (A_1, 0, A_2)\begin{pmatrix} A_1^* & 0 & -A_2^* \\ 0 & E_k & 0 \\ A_2^* & 0 & A_1^* \end{pmatrix}\begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix} c \\ &= (E_t, 0, 0)\begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix} c = 0.\end{aligned}$$

and for $\lambda \in \mathbb{C}^+$ this is equivalent to $c = 0$ and the assertion follows. □

*Remark* 6.2. ¿From the preceeding proof we conclude that

$$\dim\left(E_\lambda(I) \cap D^+\right) = i^-(\lambda) - t.$$

Hence in the case $k > 0$ equation (2.1) has a nontrivial solution in $D^+$ for $\lambda \in \mathbb{C}^-$.

Now we consider the inhomogeneous boundary value problem

$$(F - \lambda G)y = Gv$$

for $v \in L^2_A(I)$ and we show

**Theorem 6.3.** *For $\lambda \in \mathbb{C}^+$ the inhomogeneous equation*

$$(F - \lambda G)y = Gv \tag{6.4}$$

*has a uniquely determined solution $u(\cdot,\lambda,v) \in D^+$ for every $v \in L^2_A(I)$. This solution is given by the formula*

$$\begin{aligned}u(x,\lambda,v) = W(x,\lambda)&\begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix} \cdot \int_a^x \left(W(t,\bar\lambda)\begin{pmatrix} 0 \\ E_{i^+(\bar\lambda)} \end{pmatrix}\right)^* A(t)v(t)\,dt \\ &+ W(x,\lambda)\begin{pmatrix} 0 \\ E_{i^+(\lambda)} \end{pmatrix} \cdot C^* \cdot \int_x^b \left(W(t,\bar\lambda)\begin{pmatrix} E_{i^-(\bar\lambda)} \\ M(\bar\lambda) \end{pmatrix}\right)^* A(t)v(t)\,dt\end{aligned} \tag{6.5}$$

*Proof.* The uniqueness of the solution is obvious since the homogeneous equation has only the trivial solution. Since $u(\cdot,\lambda,v)\in AC_{\mathrm{loc}}(I)$ and for almost every $x\in I$

$$\begin{aligned}
u'(x,\lambda,v) &= W'(x,\lambda)W(x,\lambda)^{-1}u(x,\lambda,v)\\
&+ W(x,\lambda)\begin{pmatrix}E_t\\ M(\lambda)\end{pmatrix}(0,E_t)W(x,\bar\lambda)^*A(x)v(x)\\
&- W(x,\lambda)\begin{pmatrix}0\\ C^*\end{pmatrix}\big(E_{t+k},M^*(\bar\lambda)\big)\,W(x,\bar\lambda)^*A(x)v(x)\\
&= W'(x,\lambda)W(x,\lambda)^{-1}u(x,\lambda,v)\\
&+ W(x,\lambda)\left\{\begin{pmatrix}0 & E_t\\ 0 & M(\lambda)\end{pmatrix}-\begin{pmatrix}0 & 0\\ C^* & C^*M^*(\bar\lambda)\end{pmatrix}\right\}W(x,\bar\lambda)^*A(x)v(x)\\
&= W'(x,\lambda)W(x,\lambda)^{-1}u(x,\lambda,v)\\
&+ W(x,\lambda)\begin{pmatrix}0 & 0\\ -C^*, & M(\lambda)-C^*M^*(\bar\lambda)\end{pmatrix}W(x,\bar\lambda)^*A(x)v(x)\\
&= W'(x,\lambda)W(x,\lambda)^{-1}u(x,\lambda,v) - W(x,\lambda)JW(x,\bar\lambda)^*A(x)v(x)
\end{aligned}$$

by (5.14) and thus

$$\begin{aligned}
&Ju'(x,\lambda,v)-\big(B(x)-\lambda A(x)\big)u(x,\lambda,v)\\
&\qquad=\big[JW'(x,\lambda)-\big(B(x)-\lambda A(x)\big)W(x,\lambda)\big]\,W(x,\lambda)^{-1}u(x,\lambda,v)\\
&\qquad\qquad- JW(x,\lambda)JW(x,\bar\lambda)^*A(x)v(x)=A(x)v(x).
\end{aligned}$$

Since from $W(x,\bar\lambda)^*JW(x,\lambda)=J$ we get

$$\big(JW(x,\bar\lambda)^*\big)\,\big(JW(x,\lambda)\big)=-E_{2t+k}$$

hence

$$-\big(JW(x,\lambda)\big)\,\big(JW(x,\bar\lambda)^*\big)=E_{2t+k}.$$

By definition

$$u(a,\lambda,v)=\tilde A\begin{pmatrix}0\\ C^*\end{pmatrix}\cdot f=\begin{pmatrix}-A_2^* & 0\\ 0 & iE_k\\ A_1^* & 0\end{pmatrix}f \tag{6.6}$$

with some $f\in\mathbb{C}^{t+k}$. Therefore

$$(A_1,0,A_2)u(a,\lambda,v)=(-A_1A_2^*+A_2A_1^*,0)\,f=0.$$

Thus the mapping $u(\cdot,\lambda,v)$ is a solution of the inhomogeneous equation fulfilling the boundary conditions in $a$.

Now we prove that $u(\cdot,\lambda,v)\in L^2_A(I)$. First we consider the case $v\in L^2_A(I)$ with compact support in $I$. Then from (6.5) we have — compare (6.6) —

$$u(a,\lambda,v)=\begin{pmatrix}-A_2^* & 0\\ 0 & iE_k\\ A_1^* & 0\end{pmatrix}f$$

with some $f\in\mathbb{C}^{t+k}$ and

$$u(x,\lambda,v)=W(x,\lambda)\begin{pmatrix}E_{i^-(\lambda)}\\ M(\lambda)\end{pmatrix}h$$

with $h \in \mathbb{C}^t$ locally at $b$. Therefore by (5.5) $u(\cdot,\lambda,v) \in L^2_A(I)$ and hence the assertion is shown in this case. Since

$$u(a,\lambda,v) = W(a,\lambda)\begin{pmatrix}0\\C^*\end{pmatrix} f$$

we get

$$\begin{aligned}[u(\cdot,\lambda,v),u(\cdot,\lambda,v)]\,(a) &= f^*(0,C)W(a,\lambda)^*JW(a,\lambda)\begin{pmatrix}0\\C^*\end{pmatrix} f\\ &= f^*(0,C)J\begin{pmatrix}0\\C^*\end{pmatrix} f\\ &= f^*(0,C)\begin{pmatrix}0 & -C\\E_t & 0\end{pmatrix}\begin{pmatrix}0\\C^*\end{pmatrix} f \qquad (6.7)\\ &= f^*(0,C)\begin{pmatrix}-E_{t+k}\\0\end{pmatrix} f\\ &= f^*\begin{pmatrix}0 & 0\\0 & iE_k\end{pmatrix} f.\end{aligned}$$

By the Lagrange formula (4.4) we have

$$\begin{aligned}-[u(\cdot,\lambda,v)\ ,u(\cdot,\lambda,v)]\,(a) &= [Fu(\cdot,\lambda,v),u(\cdot,\lambda,v)]_I - [u(\cdot,\lambda,v),Fu(\cdot,\lambda,v)]_I\\ &= [\lambda Gu(\cdot,\lambda,v)+Gv,u(\cdot,\lambda,v)]_I - [u(\cdot,\lambda,v),\lambda Gu(\cdot,\lambda,v)+Gv]_I\\ &= (\lambda-\bar\lambda)\big(u(\cdot,\lambda,v),u(\cdot,\lambda,v)\big)_I + \big(v,u(\cdot,\lambda,v)\big)_I - \big(u(\cdot,\lambda,v),v\big)_I\\ &= 2i\operatorname{Im}\lambda\|u(\cdot,\lambda,v)\|_I^2 - 2i\operatorname{Im}\big(u(\cdot,\lambda,v),v\big)_I\end{aligned}$$

and now from (6.7) the equation

$$(\operatorname{Im}\lambda)\|u(\cdot,\lambda,v)\|_I^2 - \operatorname{Im}\big(u(\cdot,\lambda,v),v\big)_I = -\frac{1}{2}f^*\begin{pmatrix}0 & 0\\0 & E_k\end{pmatrix} f \qquad (6.8)$$

follows. But then

$$\begin{aligned}\|u(\cdot,\lambda,v)\|_I^2 &\leqslant \frac{1}{\operatorname{Im}\lambda}\operatorname{Im}(u(\cdot,\lambda,v),v)_I\\ &\leqslant \frac{1}{\operatorname{Im}\lambda}\|u(\cdot,\lambda,v)\|_I\cdot\|v\|_I\end{aligned}$$

from which the inequality

$$\|u(\cdot,\lambda,v)\|_I \leqslant \frac{1}{\operatorname{Im}\lambda}\|v\|_I \qquad (6.9)$$

follows.

Now let $v \in L^2_A(I)$ be arbitrary. Then we choose $c_n \in I$ with $c_n \to b$ for $n \to \infty$ and define $v_n := \chi_{[a,c_n]}v$ where $\chi_{[a,c_n]}$ is the characteristic function of the interval $[a,c_n]$. Then $v_n \in L^2_A(I)$ with support $(v_n)$ compact in $I$. Obviously $\|v_n - v\|_I \to 0$ for $n \to +\infty$ and thus we get

$$\|v_n\|_I \to \|v\|_I \qquad (6.10)$$

$$u(x,\lambda,v_n) \to u(x,\lambda,v) \text{ for } x \in I. \qquad (6.11)$$

Consider the functions $f_n$ and $f$ defined by

$$\begin{aligned}f_n(x) &:= u^*(x,\lambda,v_n)A(x)u(x,\lambda,v_n)\\ f(x) &:= u^*(x,\lambda,v)A(x)u(x,\lambda,v).\end{aligned}$$

Then we have using (6.9)

- $f_n$ is integrable and $f_n(x) \geqslant 0$
- $f_n(x) \to f(x)$ for $n \to \infty$ in $I$
- $\displaystyle\int_I f_n(x)\,dx = \|u(\cdot,\lambda,v_n)\|_I^2 \leqslant \frac{1}{(\operatorname{Im}\lambda)^2}\|v\|_I^2$

and therefore $f$ is integrable by Fatou's lemma, which gives that $u(\cdot,\lambda,v) \in L_A^2(I)$ and further

$$\begin{aligned}\|u(\cdot,\lambda,v)\|_I^2 = \int_I f(x)\,dx &\leqslant \liminf_{n\to\infty}\left(\int_I f_n(x)\,dx\right)\\ &\leqslant \frac{1}{(\operatorname{Im}\lambda)^2}\|v\|_I^2\end{aligned}$$

□

*Remark .* Observe that for $k = 0$, $i^+(\lambda) = i^-(\lambda) = t$ for $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and $C = E_t$ and therefore formula (6.5) coincides with (5.4) in [11].

As a result from the forgoing considerations we get

**Theorem 6.12.** *Define for $\lambda \in \mathbb{C}^+$ the mapping*

$$R_\lambda^+ : L_A^2(I) \to D^+ \subset L_A^2(I)$$

*by*

$$R_\lambda^+ v := u(\cdot,\lambda,v).$$

*Then $R_\lambda^+$ is a linear mapping with the properties:*

$$(F - \lambda G)R_\lambda^+ = G \tag{6.13}$$

$$\|R_\lambda^+ v\|_I \leqslant \frac{1}{\operatorname{Im}\lambda}\|v\|_I \tag{6.14}$$

$$R_\lambda^+ = R_\mu^+ + (\lambda-\mu)R_\lambda^+ R_\mu^+; \quad \lambda,\mu \in \mathbb{C}^+ \tag{6.15}$$

*Proof.* (6.13) and (6.14) are valid by the properties of $u(\cdot,\lambda,v)$. Let $v \in L_A^2(I)$ and consider the mapping $\omega$ defined by

$$\omega := R_\lambda^+ v - R_\mu^+ v - (\lambda-\mu)R_\lambda^+ R_\mu^+.$$

Then $\omega \in D^+$ and

$$\begin{aligned}(F-\lambda G)\omega &= Gv - (F-\lambda G)R_\mu^+ v - (\lambda-\mu)GR_\mu^+ v\\ &= Gv - (F-\mu G)R_\mu^+ v\\ &= Gv - Gv = 0,\end{aligned}$$

but then $\omega = 0$ by Lemma 6.1. □

**Corollary 6.16.** *$R_\lambda^+$ is continuous in $\mathbb{C}^+$.*

*Proof.* Fix $\mu_0 \in \mathbb{C}^+$ and let $v \in L_A^2(I)$ be arbitrary. Then for $\lambda$ near $\mu_0$ we get

$$\begin{aligned}\left\|R_\lambda^+ v - R_{\mu_0}^+ v\right\|_I &= |\lambda-\mu_0|\cdot\left\|R_\lambda^+ \cdot R_{\mu_0}^+ v\right\|_I\\ &\leqslant |\lambda-\mu_0|\frac{1}{(\operatorname{Im}\lambda)(\operatorname{Im}\mu_0)}\|v\|_I.\end{aligned}$$

Hence

$$\left\|R_\lambda^+ - R_{\mu_0}^+\right\| \leqslant \frac{1}{(\operatorname{Im}\lambda)(\operatorname{Im}\mu_0)}|\lambda-\mu_0|$$

proving the corollary. □

¿From formula (6.5) we see that $(R_\lambda^+ v)(x)$ can be represented as

$$(R_\lambda^+ v)(x) = \big(v, \mathcal{A}^+(\cdot, x, \lambda)\big)_I$$

with the Green's matrix

$$\mathcal{A}^+(t,x,\lambda) := \begin{cases} W(t,\bar\lambda)\begin{pmatrix} 0 \\ E_{i^+(\bar\lambda)} \end{pmatrix}\big(E_{i^-(\lambda)}, M^*(\lambda)\big)W^*(x,\lambda); & t < x \\ W(t,\bar\lambda)\begin{pmatrix} E_{i^-(\bar\lambda)} \\ M(\bar\lambda) \end{pmatrix} C\big(0, E_{i^+(\lambda)}\big)W^*(x,\lambda); & t > x. \end{cases} \tag{6.17}$$

Then from (6.17) we get

**Lemma 6.18.** *For $x \in I$ and $v \in L^2_A(I)$, $(R_\lambda^+ v)(x)$ is continuous as a function of $\lambda$ in $\mathbb{C}^+$.*

*Proof.* Let $\lambda_0 \in \mathbb{C}^+$. Then for $\lambda$ locally at $\lambda_0$ we get

$$\begin{aligned}(R_\lambda^+ v)(x) &= (R_{\lambda_0}^+ v)(x) + (\lambda - \lambda_0)R_{\lambda_0}^+(R_\lambda^+ v)(x) \\ &= (R_{\lambda_0}^+ v)(x) + (\lambda - \lambda_0)\big(R_\lambda^+ v, \mathcal{A}^+(\cdot, x, \lambda_0)\big)_I.\end{aligned}$$

The right side is continuous in $\lambda$ and thus the proof is complete. □

**Corollary 6.19.** *For $\lambda_0 \in \mathbb{C}^+$ and $x \in I$ the matrix $R_\lambda^+\big(\mathcal{A}^+(\cdot, a, \lambda_0)\big)(x)$ is continuous in $\lambda$ on $\mathbb{C}^+$.*

Combining these results we get

**Theorem 6.20.** *The Titchmarsh-Weyl matrix $M(\lambda)$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}$.*

*Proof.* First we consider the upper halfplane $\mathbb{C}^+$. With the definition of $z_\alpha$ in (5.12) we get from (6.17)

$$\begin{aligned}\mathcal{A}^+(t,a,\lambda) &= W(t,\bar\lambda)\begin{pmatrix} E_{i^-(\bar\lambda)} \\ M(\bar\lambda) \end{pmatrix}(0, C)\tilde{A}^* \\ &= z_{\bar\lambda}(t)(0,C)\tilde{A}^*.\end{aligned}$$

Then for $\lambda, \mu \in \mathbb{C}^+$ we have by Theorem 5.10

$$\begin{aligned}R_\lambda^+(z_\mu)(a) &= \big(z_\mu, \mathcal{A}^+(\cdot, a, \lambda)\big)_I \\ &= \tilde{A}\begin{pmatrix} 0 \\ C^* \end{pmatrix}(z_\mu, z_{\bar\lambda})_I \\ &= \tilde{A}\begin{pmatrix} 0 \\ C^* \end{pmatrix} C \cdot \frac{M(\lambda) - M(\mu)}{\lambda - \mu}\end{aligned}$$

and therefore

$$\begin{aligned}C^*(0,C)\tilde{A}^{-1}R_\lambda^+(z_\mu)(a) &= C^*(0,C)\tilde{A}^{-1}\tilde{A}\begin{pmatrix} 0 \\ C^* \end{pmatrix} C\frac{M(\lambda) - M(\mu)}{\lambda - \mu} \\ &= C^*(0,C)\begin{pmatrix} 0 \\ C^* \end{pmatrix} C\frac{M(\lambda) - M(\mu)}{\lambda - \mu} \\ &= \frac{M(\lambda) - M(\mu)}{\lambda - \mu}.\end{aligned}$$

Due to Lemma 6.18 the left-hand side is convergent for $\lambda \to \mu$ and so is the right-hand side and this proves that $M(\lambda)$ is holomorphic in $\mathbb{C}^+$. For $\lambda \in \mathbb{C}^-$ we have from Corollary 5.13

$$M(\lambda) = M^*(\bar{\lambda})C^*.$$

Since $M(\lambda)$ is holomorphic in $\mathbb{C}^+$, $M^*(\bar{\lambda})$ is holomorphic in $\mathbb{C}^-$ and this completes the proof of Theorem 6.20. □

¿From Remark 6.2 it follows that for $\lambda \in \mathbb{C}^-$ the equation (2.1) has a nontrivial solution in $E_\lambda(I) \cap D^+$ if $k > 0$. To achieve that (2.1) shall only have the trivial solution on a suitable subspace of $E_\lambda(I)$ we will take $i^-(\lambda) = t + k$ boundary conditions. For this reason we define the subspace

$$D^- := \left\{ y \in AC_{\text{loc}}(I) \cap L^2_A(I) | \begin{pmatrix} 0 & E_k & 0 \\ A_1 & 0 & A_2 \end{pmatrix} y(a) = 0 \right\}$$

which turns out to be adjoint to $D^+$ in a sense which will become evident at the end of this section. Then we have

**Lemma 6.21.** *The system (2.1) has only the trivial solution in $D^-$ for $\lambda \in \mathbb{C}^-$.*

*Proof.* For $w \in E_\lambda(I)$ we have the representation

$$w = W(\cdot, \lambda) \begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix} c$$

with some $c \in \mathbb{C}^{i^-(\lambda)}$ and then $w \in E_\lambda(I) \cap D^-$ if and only if

$$\begin{aligned} \begin{pmatrix} 0 & E_k & 0 \\ A_1 & 0 & A_2 \end{pmatrix} w(a) &= \begin{pmatrix} 0 & E_k & 0 \\ A_1 & 0 & A_2 \end{pmatrix} \begin{pmatrix} A_1^* & 0 & -A_2^* \\ 0 & E_k & 0 \\ A_2^* & 0 & A_1^* \end{pmatrix} \begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix} c \\ &= \begin{pmatrix} 0 & E_k & 0 \\ E_t & 0 & 0 \end{pmatrix} c = 0, \end{aligned}$$

and this condition is true if and only if $c = 0$ thus proving the assertion. □

Next we study the inhomogeneous equation. For this equation we get in correspondence to Theorem 6.3

**Theorem 6.22.** *For $\lambda \in \mathbb{C}^-$ the inhomogeneous equation*

$$(F - \lambda G)y = Gv \tag{6.23}$$

*has a uniquely determined solution $w(\cdot, \lambda, v) \in D^-$ for every $v \in L^2_A(I)$. This solution is given by the formula*

$$\begin{aligned} w(x, \lambda, v) = W(x, \lambda) &\begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix} \int_a^x \left( W(t, \bar{\lambda}) \begin{pmatrix} 0 \\ E_{i^+(\bar{\lambda})} \end{pmatrix} C^* \right)^* A(t)v(t)\, dt \\ &+ W(x, \lambda) \begin{pmatrix} 0 \\ E_{i^+(\lambda)} \end{pmatrix} \int_x^b \left( W(t, \bar{\lambda}) \begin{pmatrix} E_{i^-(\bar{\lambda})} \\ M(\bar{\lambda}) \end{pmatrix} \right)^* A(t)v(t)\, dt. \end{aligned} \tag{6.24}$$

*Proof.* Again the uniqueness of the solution is obvious since the homogeneous equation has only the trivial solution now due to Lemma 6.21. Also $w(\cdot, \lambda, v) \in AC_{\text{loc}}(I)$

is evident and for almost every $x \in I$ we have

$$\begin{aligned} w'(x,\lambda,v) &= W'(x,\lambda)W(x,\lambda)^{-1}w(x,\lambda,v) \\ &+ W(x,\lambda)\begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix} C\big(0, E_{i^+(\bar\lambda)}\big) W(x,\bar\lambda)^* A(x)v(x) \\ &- W(x,\lambda)\begin{pmatrix} 0 \\ E_{i^+(\lambda)} \end{pmatrix} \big(E_{i^-(\bar\lambda)}, M^*(\bar\lambda)\big) W(x,\bar\lambda)^* A(x)v(x) \\ &= W'(x,\lambda)W(x,\lambda)^{-1}w(x,\lambda,v) \\ &+ W(x,\lambda)\left\{\begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix}(0,C) - \begin{pmatrix} 0, & 0 \\ E_{i^+(\lambda)}, & M^*(\bar\lambda) \end{pmatrix}\right\} W(x,\bar\lambda)^* A(x)v(x) \\ &= W'(x,\lambda)W(x,\lambda)^{-1}w(x,\lambda,v) \\ &+ W(x,\lambda)\begin{pmatrix} 0, & C \\ -E_{i^+(\lambda)} & M(\lambda)C - M^*(\bar\lambda) \end{pmatrix} W(x,\bar\lambda)^* A(x)v(x) \\ &= W'(x,\lambda)W(x,\lambda)^{-1}w(x,\lambda,v) - W(x,\lambda)JW(x,\bar\lambda)^* A(x)v(x) \end{aligned}$$

with respect to Corollary 5.13 and hence $w(\cdot,\lambda,v)$ is a solution of (6.23) as shown in the proof of Theorem 6.3.

¿From (6.24) we see that

$$w(a,\lambda,v) = W(a,\lambda)\begin{pmatrix} 0 \\ E_{i^+(\lambda)} \end{pmatrix} c = \begin{pmatrix} -A_2^* \\ 0 \\ A_1^* \end{pmatrix} c \tag{6.25}$$

with some $c \in \mathbb{C}^{i^+(\lambda)}$. But then

$$\begin{pmatrix} 0 & E_k & 0 \\ A_1 & 0 & A_2 \end{pmatrix} w(a,\lambda,v) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} c = 0.$$

Hence the mapping $w(\cdot,\lambda,v)$ is a solution of the inhomogeneous equation fulfilling the boundary condition in $a$.

Next we prove that $w(\cdot,\lambda,v) \in L_A^2(I)$. Again we consider first the case $v \in L_A^2(I)$ with compact support in $I$. Then for $x$ near $b$ we get

$$w(x,\lambda,v) = W(x,\lambda)\begin{pmatrix} E_{i^-(\lambda)} \\ M(\lambda) \end{pmatrix} h$$

with some vector $h \in \mathbb{C}^{i^-(\lambda)}$. Therefore $w(\cdot,\lambda,v) \in L_A^2(I)$ and hence the assertion is shown in this case.

¿From (6.25) we see that

$$\begin{aligned} [w(\cdot,\lambda,v), w(\cdot,\lambda,v)]\,(a) &= c^*\big(-A_2, 0, A_1\big) J \begin{pmatrix} -A_2^* \\ 0 \\ A_1^* \end{pmatrix} c \\ &= c^*\big(-A_2, 0, A_1\big) \begin{pmatrix} -A_1^* \\ 0 \\ -A_2^* \end{pmatrix} c \\ &= c^*\big(A_2 A_1^* - A_1 A_2^*\big) c = 0 \end{aligned}$$

and therefore by the Lagrange formula (4.4) the relation

$$\begin{aligned}0 &= [Fw(\cdot,\lambda,v),w(\cdot,\lambda,v)]_I - [w(\cdot,\lambda,v),Fw(\cdot,\lambda,v)]_I \\ &= [\lambda Gw(\cdot,\lambda,v)+Gv,w(\cdot,\lambda,v)]_I - [w(\cdot,\lambda,v),\lambda Gw(\cdot,\lambda,v)+Gv]_I \\ &= 2i\operatorname{Im}\lambda\cdot\|w(\cdot,\lambda,v)\|_I^2 + 2i\big(v,w(\cdot,\lambda,v)\big)_I\end{aligned}$$

is valid from which the inequality

$$\|w(\cdot,\lambda,v)\|_I \leqslant \frac{1}{|\operatorname{Im}\lambda|}\cdot\|v\|_I \tag{6.26}$$

follows.

Now for $v \in L_A^2(I)$ arbitrary the proof follows by approximating $v$ by functions $v_n := \chi_{[a,c_n)}v$ analogously as in the proof of Theorem 6.3 and the inequality (6.26) remains true for arbitrary $v \in L_A^2(I)$. □

As a result from Theorem 6.22 we get

**Theorem 6.27.** *For $\lambda \in \mathbb{C}^-$ let $R_\lambda^-$ be the mapping*

$$R_\lambda^- : L_A^2(I) \to D^- \subset L_A^2(I)$$

*defined by*

$$R_\lambda^- v := w(\cdot,\lambda,v).$$

*Then $R_\lambda^-$ is a linear mapping satisfying*

$$(F-\lambda G)R_\lambda^- = G \tag{6.28}$$

$$\|R_\lambda^- v\|_I \leqslant |\operatorname{Im}\lambda|^{-1}\cdot\|v\|_I \tag{6.29}$$

$$R_\lambda^- = R_\mu^- + (\lambda-\mu)R_\lambda^- R_\mu^-, \qquad \lambda,\mu\in\mathbb{C}^- \tag{6.30}$$

*Proof.* The properties of $R_\lambda^-$ described by (6.28) and (6.29) are clear by Theorem 6.22. Then let $v \in L_A^2(I)$ and consider the function

$$z := R_\lambda^- v - R_\mu^- v - (\lambda-\mu)R_\lambda^- R_\mu^- v.$$

Then $z \in D^-$ and

$$\begin{aligned}(F-\lambda G)z &= Gv - (F-\lambda G)R_\mu^- v - (\lambda-\mu)GR_\mu^- v \\ &= Gv - (F-\mu G)R_\mu^- v \\ &= Gv - Gv = 0.\end{aligned}$$

But then $z = 0$ by Lemma 6.21 and hence (6.30) is proved.

¿From formula (6.24) we see that also $(R_\lambda^- v)(x)$ has a representation as

$$(R_\lambda^- v)(x) := \big(v,\mathcal{A}^-(\cdot,x,\lambda)\big)_I$$

with the Green's matrix

$$\mathcal{A}^-(t,x,\lambda) := \begin{cases} W(t,\bar\lambda)\begin{pmatrix}0\\ E_{i+(\bar\lambda)}\end{pmatrix} C^*\big(E_{i-(\lambda)},M^*(\lambda)\big)W(x,\lambda)^*; & t<x \\ W(t,\bar\lambda)\begin{pmatrix}E_{i-(\bar\lambda)}\\ M(\bar\lambda)\end{pmatrix}\big(0,E_{i+(\lambda)}\big)W(x,\lambda)^*; & t>x\end{cases} \tag{6.31}$$

so that the corresponding lemmata and corollaries as for $R_\lambda^+$ can be proved. □

The resolvents $R_\lambda^+$ and $R_\lambda^-$ are adjoint to each other in the sense described by

**Theorem 6.32.** *For $u, v \in L^2_A(I)$ the relations*

$$\big(R^+_\lambda u, v\big)_I = \big(u, R^-_{\bar\lambda} v\big)_I; \qquad \lambda \in \mathbb{C}^+ \tag{6.33}$$

$$\big(R^-_\lambda u, v\big)_I = \big(u, R^+_{\bar\lambda} v\big)_I; \qquad \lambda \in \mathbb{C}^- \tag{6.34}$$

*hold.*

*Proof.* ¿From formula (6.5) and (6.24) we have for $\lambda \in \mathbb{C}^+$

$$(R^+_\lambda u)(a) = W(a,\lambda)\begin{pmatrix} 0 \\ E_{t+k}\end{pmatrix} f = \tilde{A}\begin{pmatrix} 0 \\ E_{t+k}\end{pmatrix} f$$

and

$$(R^-_{\bar\lambda} v)(a) = W(a,\bar\lambda)\begin{pmatrix} 0 \\ E_{t}\end{pmatrix} h = \tilde{A}\begin{pmatrix} 0 \\ E_{t}\end{pmatrix} h$$

with $f \in \mathbb{C}^{t+k}$ and $h \in \mathbb{C}^t$. Then

$$\begin{aligned}\left[R^+_\lambda u, R^-_{\bar\lambda} v\right](a) &= h^*(0, E_t)\tilde{A}^* J \tilde{A}\begin{pmatrix} 0 \\ E_{t+k}\end{pmatrix} f \\ &= h^*(0, E_t) J\begin{pmatrix} 0 \\ E_{t+k}\end{pmatrix} f \\ &= h^*(0, E_t)\begin{pmatrix} -C \\ 0\end{pmatrix} f = 0\end{aligned}$$

and hence by the Lagrange formula (4.4) we get

$$\begin{aligned} 0 &= \left[F R^+_\lambda u, R^-_{\bar\lambda} v\right]_I - \left[R^+_\lambda u, F R^-_{\bar\lambda} v\right]_I \\ &= \lambda\big(R^+_\lambda u, R^-_{\bar\lambda} v\big)_I + \big(u, R^-_{\bar\lambda} v\big)_I - \lambda\big(R^+_\lambda u, R^-_{\bar\lambda} v\big)_I - \big(R^+_\lambda u, v\big)_I \\ &= \big(u, R^-_{\bar\lambda} v\big)_I - \big(R^+_\lambda u, v\big)_I \end{aligned}$$

and this is the relation (6.33). (6.34) follows from (6.33) replacing $\lambda$ by $\bar\lambda$. □

*Remark* 6.35. In Hilbert spaces the equation (6.33) is just the relation

$$(R^+_\lambda)^* = R^-_{\bar\lambda}; \qquad \lambda \in \mathbb{C}^+$$

and (6.34)

$$(R^-_\lambda)^* = R^+_{\bar\lambda}; \qquad \lambda \in \mathbb{C}^-.$$

*Remark* 6.36. ¿From the definition of the matrices $\mathcal{A}^+(t,x,\lambda)$ and $\mathcal{A}^-(t,x,\lambda)$ we have immediately

$$\begin{aligned}\mathcal{A}^-(t,x,\bar\lambda) &= \mathcal{A}^+(x,t,\lambda)^* \qquad \lambda \in \mathbb{C}^+ \\ \mathcal{A}^-(t,x,\lambda) &= \mathcal{A}^+(x,t,\bar\lambda)^* \qquad \lambda \in \mathbb{C}^-\end{aligned}$$

and these relations are equivalent to (6.33) or (6.34) respectively.

## 7. The associated differential operators

In this section we introduce two differential operators **A** and **B** by means of the resolvents $R_\lambda^+$ and $R_\lambda^-$ which turn out to have the same eigenvalues as the singular boundary value problems

$$Fy = \lambda G y; \qquad y \in D^+ \tag{7.1}$$

and

$$Fy = \lambda G y; \qquad y \in D^- \tag{7.2}$$

In section 4 we had defined the subspace $R(I) = F^{-1}\big(G\big(L_A^2(I)\big)\big) \cap L_A^2(I)$ and for the ranges and the kernels of the resolvents we get

**Lemma 7.3.** *The range of $R_\lambda^+$ is given by*

$$R_\lambda^+\big(L_A^2(I)\big) = D^+ \cap R(I) \tag{7.4}$$

*and for all $\lambda, \mu \in \mathbb{C}^+$*

$$\ker(R_\lambda^+) = \ker(R_\mu^+). \tag{7.5}$$

*Analogously we have for $\lambda \in \mathbb{C}^-$*

$$R_\lambda^-\big(L_A^2(I)\big) = D^- \cap R(I) \tag{7.6}$$

*and for all $\lambda, \mu \in \mathbb{C}^-$*

$$\ker(R_\lambda^-) = \ker(R_\mu^-). \tag{7.7}$$

*Proof.* The inclusion $R_\lambda^+\big(L_A^2(I)\big) \subset D^+ \cap R(I)$ is obvious by the definition of $R_\lambda^+$ and $R(I)$. Let conversely $w \in D^+ \cap R(I)$. Then we have for some $v \in L_A^2(I)$ the equation $Fw = Gv$ and thus

$$(F - \lambda G)w = G(v - \lambda w) = Gz$$

with some $z \in L_A^2(I)$. But from the uniqueness of the solution of the inhomogeneous equation according to Theorem 6.3 we get $w = R_\lambda^+ z$ and thus the relation (7.4) holds. The proof of (7.6) follows in the same way using now the uniqueness of the solution of the inhomogeneous equation by Theorem 6.22. Now let $\lambda, \mu \in \mathbb{C}^+$ and $v \in \ker(R_\lambda^+)$. By (6.15) we have

$$0 = R_\lambda^+ v = R_\mu^+ v + (\lambda - \mu) R_\mu^+ R_\lambda^+ v$$

and we get $R_\mu^+ v = 0$. Hence

$$\ker(R_\lambda^+) \subset \ker(R_\mu^+).$$

Since $\lambda, \mu$ are arbitrary we also have

$$\ker(R_\mu^+) \subset \ker(R_\lambda^+)$$

and thus (7.5) is shown. (7.7) follows from (6.30) in the same way. □

With the next step we will define a suitable Hilbert space appropriate to the singular boundary value problems. We start by considering the subspace $N := \big\{u \in L_A^2(I) \,|\, \|u\|_I = 0\big\}$ and define the quotient space $\mathbb{E} := L_A^2(I)\big/ N$ and denote by $\pi$ the canonical homomorphism from $L_A^2(I)$ onto $\mathbb{E}$. Next we define on $\mathbb{E}$ the positive definite hermitian scalar product

$$\big(\pi(u), \pi(v)\big) := (u, v)_I$$

and we obtain the Hilbert space $\big(\mathbb{E},(\,,\,)\big)$.

Now we define for $\lambda \in \mathbb{C}^+$ on $\mathbb{E}$ the mapping

$$\Gamma_\lambda^+\big(\pi(u)\big) := \pi\big(R_\lambda^+ u\big) \tag{7.8}$$

and the definition is independent of the choice of the representative $u$ due to (6.14). Analogously we define for $\lambda \in \mathbb{C}^-$ the mapping

$$\Gamma_\lambda^-\big(\pi(u)\big) := \pi(R_\lambda^- u) \tag{7.9}$$

which is well defined with respect to (6.29). The properties of these mappings are summarized in

**Lemma 7.10.** *The linear mappings $\Gamma_\lambda^+$ and $\Gamma_\lambda^-$ are bounded with*

$$\|\Gamma_\lambda^+\| \leqslant \frac{1}{|\operatorname{Im}\lambda|}; \qquad \|\Gamma_\lambda^-\| \leqslant \frac{1}{|\operatorname{Im}\lambda|}.$$

*For the adjoint operators we have*

$$(\Gamma_\lambda^+)^* = \Gamma_{\bar\lambda}^-; \qquad \lambda \in \mathbb{C}^+ \tag{7.11}$$

$$(\Gamma_\lambda^-)^* = \Gamma_{\bar\lambda}^+; \qquad \lambda \in \mathbb{C}^- \tag{7.12}$$

*Further the Hilbert relations*

$$\Gamma_\lambda^+ = \Gamma_\mu^+ + (\lambda-\mu)\Gamma_\lambda^+\Gamma_\mu^+; \qquad \lambda,\mu \in \mathbb{C}^+ \tag{7.13}$$

$$\Gamma_\lambda^- = \Gamma_\mu^- + (\lambda-\mu)\Gamma_\lambda^-\Gamma_\mu^-; \qquad \lambda,\mu \in \mathbb{C}^- \tag{7.14}$$

*are valid.*

*Proof.* For $\lambda \in \mathbb{C}^+$ and $u, v \in L_A^2(I)$ we get from (6.33)

$$\big(\Gamma_\lambda^+\pi(u),\pi(v)\big) = \big(R_\lambda^+u, v\big)_I = \big(u, R_{\bar\lambda}^- v\big)_I = \big(\pi(u), \Gamma_{\bar\lambda}^-\pi(v)\big).$$

This proves (7.11) and for $\lambda \in \mathbb{C}^-$ we get from (6.34)

$$\big(\Gamma_\lambda^-\pi(u),\pi(v)\big) = \big(R_\lambda^-u, v\big)_I = \big(u, R_{\bar\lambda}^+ v\big)_I = \big(\pi(u), \Gamma_{\bar\lambda}^+\pi(v)\big)$$

which is identical with (7.12). (7.13) and (7.14) are immediate consequences of the Hilbert relations (6.15) and (6.30) for $R_\lambda^+$ and $R_\lambda^-$ respectively. □

Now we introduce the linear manifolds $D_A$ and $D_B$ in the Hilbert space $\mathbb{E}$ by

$$D_A := \pi\big(D^+ \cap R(I)\big); \qquad D_B := \pi\big(D^- \cap R(I)\big).$$

For these manifolds we have

$$D_B \subset D_A \tag{7.15}$$

$$D_A = \Gamma_\lambda^+(\mathbb{E}) \qquad \lambda \in \mathbb{C}^+ \tag{7.16}$$

$$D_B = \Gamma_\lambda^-(\mathbb{E}) \qquad \lambda \in \mathbb{C}^-. \tag{7.17}$$

Clearly (7.15) is obvious since $D^- \subset D^+$. Further by (7.4) we have for $\lambda \in \mathbb{C}^+$,

$$D_A = \pi\big(D^+ \cap R(I)\big) = \pi\big(R_\lambda^+\big(L_A^2(I)\big)\big) = \Gamma_\lambda^+\big(\pi\big(L_A^2(I)\big)\big) = \Gamma_\lambda^+(\mathbb{E})$$

and for $\lambda \in \mathbb{C}^-$,

$$D_B = \pi\big(D^- \cap R(I)\big) = \pi\big(R_\lambda^-\big(L_A^2(I)\big)\big) = \Gamma_\lambda^-\big(\pi\big(L_A^2(I)\big)\big) = \Gamma_\lambda^-(\mathbb{E}).$$

Therefore for the kernels of the mappings $\Gamma_\lambda^+$ and $\Gamma_\lambda^-$ we get

$$\ker(\Gamma_\lambda^+) = \big((\Gamma_\lambda^+)^*(\mathbb{E})\big)^\perp = \big(\Gamma_{\bar\lambda}^-(\mathbb{E})\big)^\perp = D_B^\perp; \quad \lambda \in \mathbb{C}^+,$$

and

$$\ker(\Gamma_\lambda^-) = \big((\Gamma_\lambda^-)^*(\mathbb{E})\big)^\perp = \big(\Gamma_{\bar\lambda}^+(\mathbb{E})\big)^\perp = D_A^\perp; \quad \lambda \in \mathbb{C}^-.$$

For the kernels of these mappings we prove

**Theorem 7.18.** *Let $S(I) := G^{-1}\big(F(R(I) \cap N)\big) \cap L_A^2(I)$, and let*

$$S^+(I) := \{f \in S(I) : Fv = Gf \text{ for some } v \in D^+ \cap N\},$$
$$S^-(I) := \{f \in S(I) : Fv = Gf \text{ for some } v \in D^- \cap N\}.$$

*Then for $\lambda \in \mathbb{C}^+$,*

$$\ker(\Gamma_\lambda^+) = \pi(S^+(I)), \qquad \ker(\Gamma_{\bar\lambda}^-) = \pi(S^-(I)). \tag{7.19}$$

*Proof.* If $x \in \ker(\Gamma_\lambda^+)$ and $x = \pi(u)$ with $u \in L_A^2(I)$ then $0 = \Gamma_\lambda^+ x = \pi(R_\lambda^+ u)$ and hence $R_\lambda^+ u \in R(I) \cap N$ and $R_\lambda^+ u \in D^+$ by (7.4). This implies $G(R_\lambda^+ u) = 0$ and therefore the equation $F(R_\lambda^+ u) = G(\lambda R_\lambda^+ u + u) = Gu$ is valid. Thus $u \in G^{-1}\big(F(R(I) \cap N)\big) \cap L_A^2(I) = S(I)$ and since $R_\lambda^+ u \in D^+$, we have $u \in S^+(I)$ and $x = \pi(u) \in \pi(S^+(I))$. This proves that $\ker(\Gamma_\lambda^+) \subset \pi(S^+(I))$.

Let conversely $x = \pi(u) \in \pi(S^+(I))$. By the definition of $S^+(I)$ there exists an element $v \in D^+ \cap N$ with $Fv = Gu$. Since $\|v\|_I = 0$ we have $Gv = 0$ and therefore

$$Fv = (F - \lambda G)v = Gu.$$

On the other hand the equation

$$(F - \lambda G) R_\lambda^+ u = Gu$$

is valid and hence by the uniqueness of the solution of the inhomogeneous equation due to Theorem 6.3 we get $v = R_\lambda^+ u$ and then

$$0 = \pi(v) = \pi(R_\lambda^+ u) = \Gamma_\lambda^+(\pi(u)) = \Gamma_\lambda^+ x.$$

Now we also have the inclusion $\pi(S^+(I)) \subset \ker(\Gamma_\lambda^+)$, proving the first equality. The second equality in (7.19) follows with the same arguments and will be omitted. □

Now we split the Hilbert space $\mathbb{E}$ into the orthogonal sum

$$\begin{aligned}\mathbb{E} &= \ker(\Gamma_\lambda^+) \oplus \big(\ker(\Gamma_\lambda^+)\big)^\perp \\ &= \ker(\Gamma_\lambda^+) \oplus (D_B)^{\perp\perp} \\ &= \ker(\Gamma_\lambda^+) \oplus \bar D_B; \qquad \lambda \in \mathbb{C}^+\end{aligned}$$

and

$$\begin{aligned}\mathbb{E} &= \ker(\Gamma_\lambda^-) \oplus \big(\ker(\Gamma_\lambda^-)\big)^\perp \\ &= \ker(\Gamma_\lambda^-) \oplus (D_A)^{\perp\perp} \\ &= \ker(\Gamma_\lambda^-) \oplus \bar D_A; \qquad \lambda \in \mathbb{C}^-.\end{aligned}$$

Then restricting the mappings $\Gamma_\lambda^+$ and $\Gamma_\lambda^-$ we get

**Lemma 7.20.** *The mappings*

$$\Gamma_\lambda^+ : \bar D_B \to D_A; \qquad \lambda \in \mathbb{C}^+$$

*and*

$$\Gamma_\lambda^- : \bar D_A \to D_B; \qquad \lambda \in \mathbb{C}^-$$

*are bijective and continuous.*

Now we define the operators **A** and **B**

$$\mathbf{B} : D_B \to \bar{D}_A \quad \text{by} \quad \mathbf{B}y := \lambda y + (\Gamma_\lambda^-)^{-1}y; \quad \lambda \in \mathbb{C}^-.$$

and

$$\mathbf{A} : D_A \to \bar{D}_B \quad \text{by} \quad \mathbf{A}y := \mathbf{B}^* y.$$

We must show the definition of **B** is independent of the parameter and derive an expression for **A**. We will consider first the operator **B**. Let $u := \lambda y + (\Gamma_\lambda^-)^{-1}y$ and $w := \mu y + (\Gamma_\mu^-)^{-1}y$ with $\lambda, \mu \in \mathbb{C}^-$ then

$$(u - \lambda y) = (\Gamma_\lambda^-)^{-1}y; \qquad w - \mu y = (\Gamma_\mu^-)^{-1}y$$

and then

$$\Gamma_\lambda^-(u - \lambda y) = y = \Gamma_\mu^-(w - \mu y).$$

Now the Hilbert relation (7.14) yields

$$\begin{aligned}\Gamma_\lambda^-(u - \lambda y) &= \Gamma_\mu^-(w - \mu y) \\ &= \Gamma_\lambda^-(w - \mu y) + (\mu - \lambda)\Gamma_\lambda^-\Gamma_\mu^-(w - \mu y) \\ &= \Gamma_\lambda^-(w - \mu y) + (\mu - \lambda)\Gamma_\lambda^- y \\ &= \Gamma_\lambda^-(w - \lambda y).\end{aligned}$$

and from the injectivity of $\Gamma_\lambda^-$ we get $u - \lambda y = w - \lambda y$ hence $u = w$. Thus the definition of **B** is independent of the parameter $\lambda$.

For the operator **A**, we use adjoint theory from [26], sections 4.4 and 5.1. Let $\lambda \in \mathbb{C}^+$ and define the Hilbert spaces $\mathbb{H}_1 = \bar{D}_B$, $\mathbb{H}_2 = \bar{D}_A$. Further define $\mathbf{C} = \Gamma_\lambda^+|\bar{D}_B$. Then $\mathbf{C} : \bar{D}_B \to D_A$, and since $(\Gamma_\lambda^+)^* = (\Gamma_{\bar{\lambda}}^-)$, we have that $\mathbf{C}^* = \Gamma_{\bar{\lambda}}^-|\bar{D}_A$ so that $\mathbf{C}^* : \bar{D}_A \to D_B$. Thus $\mathbf{C}$, $\mathbf{C}^*$ are bijective and applying Theorem 4.17 from [26], we get that $(\mathbf{C}^*)^{-1} = (\mathbf{C}^{-1})^*$. Further $\mathbf{C}$, $\mathbf{C}^*$ are closed since they are bounded; also by [26], pages 89-90, $\mathbf{C}^{-1}$ is closed and $\mathbf{C}^{**} = \mathbf{C}$ and $(\mathbf{C}^{-1})^{**} = \mathbf{C}^{-1}$. The definition of **B** is then

$$\mathbf{B} = \bar{\lambda}\mathbf{I} + (\mathbf{C}^*)^{-1} = \bar{\lambda}I + (\mathbf{C}^{-1})^*,$$

where **I** is the identity operator from $\bar{D}_B$ into $\bar{D}_A$. . Thus by Theorem 4.20 of [26],

$$\mathbf{B}^* = (\bar{\lambda}\mathbf{I})^* + (\mathbf{C}^{-1})^{**} = \lambda\mathbf{I}^* + \mathbf{C}^{-1} = \mathbf{A}.$$

Calculations show that $\mathbf{I}^*$ satisfies $\mathbf{I}^* y = y$ if $y \in \bar{D}_B$, and $\mathbf{I}^* y = 0$ if $y \in \bar{D}_B^\perp \cap \bar{D}_A$. Note also $\mathbf{A} : D_A \to \bar{D}_B$. We also have from adjoint theory that

$$\mathbf{A}^* = \mathbf{B}^{**} = (\bar{\lambda}\mathbf{I})^{**} + (\mathbf{C}^{-1})^* = \bar{\lambda}\mathbf{I} + (\mathbf{C}^{-1})^* = \mathbf{B}.$$

**Theorem 7.21.** *If $x \in D_B$, then $\mathbf{A}x - \mathbf{B}x \in \bar{D}_B^\perp \cap \bar{D}_A$.*

*Proof.* First observe that $D_B \subset D_A$. Then let $x \in D_B$. By definition of $D_A$ we have

$$x = \Gamma_{-i}^- \pi(u) = \pi(R_{-i}^- u)$$

with some $\pi(u) \in \bar{D}_A$. But then by definition of **B**,

$$\mathbf{B}x = -ix + \pi(u). \tag{7.22}$$

On the other hand $(F + iG)R_{-i}^- u = Gu$ and hence

$$(F - iG)R_{-i}^- u = G(u - 2iR_{-i}^- u)$$

and now by the uniqueness property of Theorem 6.3 (recall that $D^- \subset D^+$),

$$R_{-i}^- u = R_i^+(u - 2iR_{-i}^- u) \tag{7.23}$$

follows. From (7.23) we get

$$\begin{aligned} x = \pi(R_{-i}^- u) &= \Gamma_i^+\left(\pi(u) - 2i\pi(R_{-i}^- u)\right) \\ &= \Gamma_i^+(\pi(u) - 2ix). \end{aligned}$$

Now decompose $\pi(u) = y_1 + y_2$, $y_1 \in \bar{D}_B$, $y_2 \in \bar{D}_B^\perp$. Since $\ker \Gamma_i^+ = \bar{D}_B^\perp$, we have $\Gamma_i^+ y_2 = 0$. Since $x \in D_A$ we finally get, since $\Gamma_i^+$ is injective on $\bar{D}_B$,

$$\mathbf{A}x - ix = y_1 - 2ix$$

and hence with respect to (7.22)

$$\mathbf{A}x = y_1 - ix = \mathbf{B}x - y_2$$

thus completing the proof. □

When $\bar{D}_B = \bar{D}_A$, we define the Hilbert space $\mathbb{H}$ by

$$\mathbb{H} = \bar{D}_A = \bar{D}_B.$$

Hence $\mathbf{A}$ and $\mathbf{B}$ are closed densely defined operators acting in the Hilbert space $\mathbb{H}$, and $\mathbf{B} \subset \mathbf{A}$. Let $id_{\mathbb{H}}$ be the identity map on $\mathbb{H}$. We see from Theorem 7.18 that

$$\bar{D}_B = \bar{D}_A \Leftrightarrow \ker \Gamma_\lambda^+ = \ker \Gamma_{\bar{\lambda}}^- \Leftrightarrow \pi(S^+(I)) = \pi(S^-(I)).$$

In particular we will have $\bar{D}_B = \bar{D}_A$ if $k = 0$ or $S(I) = \{0\}$.

We now give an example where $\bar{D}_B \neq \bar{D}_A$. In (1.1) take $k = t = 1$ and $I = [0, \infty)$, and set

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 1 & \alpha \\ 0 & \alpha & 0 \end{pmatrix},$$

where $\alpha(t) = \frac{1}{t+1}$. We take the boundary matrices in (5.1) to be $A_1 = 1$ and $A_2 = 0$. Then the equation $Jy' = By + Af$ is equivalent to the system

$$-y_3' = \alpha y_2 + f_1, \quad iy_2' = \alpha y_1 + y_2 + \alpha y_3, \quad y_1' = \alpha y_2 + f_3.$$

First we compute $S^-(I)$. Here the boundary conditions are $y_1(0) = y_2(0) = 0$. Since $\|y\|_I = 0$ implies $y_1 = y_3 = 0$, we get as solution to the above system that $f_1 = f_3 = -\alpha y_2$, and $y_2(t) = y_2(0)e^{-it}$. But $y_2(0) = 0$ implies $f(t) = 0$; hence $S^-(I) = \{0\}$. For $S^+(I)$, the boundary conditions are just $y_1(0) = 0$. Thus with $y_1 = y_3 = 0, f_1(t) = f_3(t) = c\alpha(t)e^{-it}$, and $y_2(t) = -ce^{-it}$, we have an $f \in S^+(I)$. For $c \neq 0$, it is clear that $\|f\|_I \neq 0$, so that $\pi(S^+(I) \neq \{0\}$. Thus Theorem 7.18 yields that $\bar{D}_B \neq \bar{D}_A$.

**Corollary 7.24.** *Assume $\bar{D}_B = \bar{D}_A$. Then operator $\mathbf{B}$ is symmetric.*

*Proof.* Since $\mathbf{A}$=$\mathbf{B}^*$ we have by Theorem 7.21 that $\mathbf{B} \subset \mathbf{A} = \mathbf{B}^*$. □

**Theorem 7.25.** *Assume $\bar{D}_B = \bar{D}_A$. If $\rho(\mathbf{A})$ denotes the resolvent set of $\mathbf{A}$ and $\sigma_p(\mathbf{A})$ the point spectrum of $\mathbf{A}$, then we have in case $k > 0$*

$$\mathbb{C}^+ \subset \rho(\mathbf{A}) \tag{7.26}$$

$$\mathbb{C}^- \subset \sigma_p(\mathbf{A}). \tag{7.27}$$

*Hence we get the spectrum $\sigma(\mathbf{A}) = \mathbb{C} \setminus \mathbb{C}^+$.*

*Proof.* If $\lambda \in \mathbb{C}^+$ then by definition of $\mathbf{A}$ we have $(\mathbf{A} - \lambda\,\mathrm{id}_{\mathbb{H}})^{-1} = \Gamma_\lambda^+$ and therefore $\lambda \in \rho(\mathbf{A})$. Thus (7.26) holds. To prove (7.27) observe that for $\lambda \in \mathbb{C}^-$, $\dim E_\lambda(I) = i^-(\lambda) = t + k > t$. Let $e_1, \dots, e_{t+k}$ be a basis for $E_\lambda(I)$. Then every $y \in E_\lambda(I)$ has a unique representation

$$y = \sum_{v=1}^{t+k} \alpha_v e_v$$

and $y$ is a nontrivial element in $D^+ \cap R(I)$ if and only if the equation

$$\sum_{v=1}^{t+k} \alpha_v (A_1, 0, A_2) e_v(a) = 0 \tag{7.28}$$

has a solution $(\alpha_1, \dots, \alpha_{t+k}) \neq (0, \dots, 0)$. Since the rank of the matrix of the coefficients of the system (7.28) is at most $t$, (7.28) has a nontrivial solution. Now let $y \in E_\lambda(I) \setminus \{0\}$. Then $(F - iG)y = G(\lambda - i)y$ and hence by Theorem 6.3 we get

$$y = R_i^+((\lambda - i)y) = (\lambda - i)R_i^+ y$$

and then

$$\pi(y) = (\lambda - i)\pi(R_i^+ y) = (\lambda - i)\Gamma_i^+ \pi(y).$$

Applying the operator $\mathbf{A} - i\,\mathrm{id}_{\mathbb{H}}$ we obtain the equation

$$(\mathbf{A} - i\,\mathrm{id}_{\mathbb{H}})\pi(y) = (\lambda - i)\pi(y) \quad \text{or} \quad \mathbf{A}\pi(y) = \lambda\pi(y) \text{ with } \pi(y) \neq 0$$

since $\pi$ is injective on $E_\lambda(I)$. Thus $\lambda$ is an eigenvalue of $\mathbf{A}$. Observe that $\mathbf{B}$ is symmetric and thus $\lambda \in \mathbb{C}^-$ is not an eigenvalue of $\mathbf{B}$. Further $\mathbf{A} = \mathbf{B}^*$ and $\dim(D_A / D_B) \leqslant k$. Therefore the values $\lambda \in \mathbb{C}^-$ are eigenvalues of finite multiplicity and this completes the proof of (7.27). □

For the operator $\mathbf{B}$ the following theorem is true.

**Theorem 7.29.** *Assume $\bar{D}_B = \bar{D}_A$. If $\sigma_{\mathrm{res}}(\mathbf{B})$ denotes the residual spectrum of $\mathbf{B}$ we get*

$$\mathbb{C}^- \subset \rho(\mathbf{B}) \tag{7.30}$$

$$\mathbb{C}^+ \subset \sigma_{\mathrm{res}}(\mathbf{B}). \tag{7.31}$$

*Hence $\sigma(\mathbf{B}) = \mathbb{C} \setminus \mathbb{C}^-$.*

*Proof.* For $\lambda \in \mathbb{C}^-$ we have by definition $\mathbf{B} = \lambda\,\mathrm{id}_{\mathbb{H}} + (\Gamma_\lambda^-)^{-1}$. Therefore $(\mathbf{B} - \lambda\,\mathrm{id}_{\mathbb{H}}) = (\Gamma_\lambda^-)^{-1}$ and thus (7.30) is obvious. Now let $\lambda \in \mathbb{C}^+$. Then $\bar{\lambda} \in \mathbb{C}^-$ and for the range $R(\mathbf{B} - \lambda\,\mathrm{id}_{\mathbb{H}})$ we obtain the relation

$$R(\mathbf{B} - \lambda\,\mathrm{id}_{\mathbb{H}})^\perp = \ker(\mathbf{B}^* - \bar{\lambda}\,\mathrm{id}_{\mathbb{H}}) = \ker(\mathbf{A} - \bar{\lambda}\,\mathrm{id}_{\mathbb{H}}) \neq \{0\}$$

and hence (7.31) is true. □

We conclude this section by proving that the operator $\mathbf{B}$ and the singular boundary eigenvalue problem

$$Fy = \lambda G y; \qquad y \in D^- \cap R(I) \setminus \{0\} \tag{7.32}$$

have the same eigenvalues with the same geometric multiplicity. Therefore we call $\mathbf{B}$ the differential operator generated by (7.32). In the case $k = 0$ $\mathbf{B}$ is selfadjoint and coincides with the operator $A$ defined in Section 6 of [11].

Let $W = D^- \cap R(I)$ and first suppose that there is a nontrivial $y \in W$ with $Fy = \lambda G y$. Then $\|y\|_I > 0$ by the definiteness assumption **II** and hence $\pi(y) \notin N$.

From $Fy = \lambda Gy$ we have $(F+iG)y = (\lambda+i)Gy$ and therefore $y = R_{-i}^{-}\big((\lambda+i)y\big) = (\lambda+i)R_{-i}^{-}y$. Then we get $\pi(y) = (\lambda+i)\pi(R_{-i}^{-}y) = (\lambda+i)\Gamma_{-i}^{-}\pi(y)$. Thus $\pi(y) \in D_B$ and $(\mathbf{B} + i\,\mathrm{id}_{\mathbb{H}})\pi(y) = (\lambda+i)\pi(y)$ which implies $\mathbf{B}\pi(y) = \lambda\pi(y)$.

Suppose on the other hand, that for $\pi(y) \in \pi(W)$, $\|y\|_I \neq 0$, $\mathbf{B}\pi(y) = \lambda\pi(y)$. Then $(\mathbf{B} + i\,\mathrm{id}_{\mathbb{H}})\pi(y) = (\lambda+i)\pi(y)$ and therefore

$$\pi(y) = (\lambda+i)\Gamma_{-i}^{-}(\pi(y)) = \pi\big((\lambda+i)R_{-i}^{-}y\big).$$

Hence $\pi(y - (\lambda+i)R_{-i}^{-}y)$=0 so that $y - (\lambda+i)R_{-i}^{-}y = g \in N$. Thus

$$(F+iG)y - (\lambda+i)(F+iG)R_{-i}^{-}y = (F+iG)g$$

which simplifies to

$$(F-\lambda G)(y-g) = (i-\lambda)Gg$$

using $(F+iG)R_{-i}^{-}y = Gy$. Since $g \in N$, we have $Gg = 0$ a.e.; thus $F(y-g) = \lambda G(y-g)$ a.e., and $\lambda$ is an eigenvalue of (7.32). Finally since $\pi$ is injective on $E_\lambda(I)$, the geometric multiplicity of the eigenvalues coincide and this completes the proof of our assertion.

## 8. The relation to a Nevanlinna matrix

In [17] H. D. Niessen has shown by a very simple consideration how problems with unequal deficiency indices can be reduced to problems with equal deficiency indices. We will use here the same idea in order to prove that our Titchmarsh-Weyl matrix $M(\lambda)$ can be extended in a unique way to a Nevanlinna matrix $\hat{M}(\lambda)$. To achieve this we consider on $I = [a,b)$ the system

$$\hat{J}\hat{y}' - \hat{B}(x)\hat{y} = \lambda\hat{A}(x)\hat{y} \tag{8.1}$$

where

$$\hat{J} := \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}; \quad \hat{B}(x) := \begin{pmatrix} B(x) & 0 \\ 0 & -B(x) \end{pmatrix}; \quad \hat{A}(x) := \begin{pmatrix} A(x) & 0 \\ 0 & A(x) \end{pmatrix}.$$

Then (8.1) is a Hamiltonian system of even order $2(2t+k)$ and the matrix $\frac{1}{i}\hat{J}$ has $2t+k$ positive and $2t+k$ negative eigenvalues and for (8.1) the assumptions **I, II** and **III** of section 2 are fulfilled. To verify the assumption **III** we only have to observe that for the corresponding eigenspaces $\hat{E}_\lambda(I) = E_\lambda(I) \times E_{-\lambda}(I)$, we have the relation

$$\dim \hat{E}_i(I) = \dim E_i(I) + \dim E_{-i}(I) = \dim \hat{E}_{-i}(I) = 2t+k$$

which confirms that **III** is fulfilled. Concerning further relations between the system (2.1) and (8.1) we refer to section 7 in [17].

For the system (8.1) we choose the matrix

$$\tilde{\hat{A}} := \left[\begin{array}{ccc|ccc} A_1^* & 0 & 0 & -A_2^* & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}E_k & 0 & 0 & i\frac{1}{\sqrt{2}}E_k & 0 \\ A_2^* & 0 & 0 & A_1^* & 0 & 0 \\ \hline 0 & 0 & A_1^* & 0 & 0 & A_2^* \\ 0 & \frac{1}{\sqrt{2}}E_k & 0 & 0 & -i\frac{1}{\sqrt{2}}E_k & 0 \\ 0 & 0 & A_2^* & 0 & 0 & -A_1^* \end{array}\right].$$

For this matrix we get the relation

$$(\tilde{\hat{A}})^*\hat{J}\tilde{\hat{A}} = \begin{pmatrix} 0 & -E_{2t+k} \\ E_{2t+k} & 0 \end{pmatrix} \tag{8.2}$$

and this relation is fundamental for the equation (4.4), page 334 of [11], from which Theorem 4.7 and 4.16 as well as Corollaries (4.18) and (4.19) follow. Therefore let $\hat{W}(x,\lambda)$ be the fundamental matrix of (8.1) with the initial condition $\hat{W}(a,\lambda) = \tilde{\hat{A}}$. We introduce the matrices

$$\hat{\Theta}(x,\lambda) := \hat{W}(x,\lambda)\begin{pmatrix} E_{2t+k} \\ 0 \end{pmatrix}; \qquad \hat{\Phi}(x,\lambda) := \hat{W}(x,\lambda)\begin{pmatrix} 0 \\ E_{2t+k} \end{pmatrix}$$

and then we obtain

**Theorem 8.3.** *There exists a uniquely determined* $(2t+k)\times(2t+k)$ *matrix* $\hat{M}(\lambda)$ *defined on* $\mathbb{C}\setminus\mathbb{R}$ *such that*

$$\hat{\Theta}(\cdot,\lambda) + \hat{\Phi}(\cdot,\lambda)\cdot\hat{M}(\lambda) \in \left(\hat{R}(I)\right)^{2t+k}.$$

$\hat{M}(\lambda)$ *is a Nevanlinna matrix.*

Now we show, that this matrix $\hat{M}(\lambda)$ is uniquely determined by the $M(\lambda)$ matrix of Theorem 5.3. Let $\lambda \in \mathbb{C}^+$ and split the matrices $\hat{M}(\lambda)$ and $M(\lambda)$

$$\hat{M}(\lambda) = \begin{bmatrix} M_{11}(\lambda) & M_{12}(\lambda) & M_{13}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) & M_{23}(\lambda) \\ M_{31}(\lambda) & M_{32}(\lambda) & M_{33}(\lambda) \end{bmatrix}$$

and

$$M(\lambda) = \begin{bmatrix} m_1(\lambda) \\ m_2(\lambda) \end{bmatrix}$$

where $M_{11}(\lambda)$ and $M_{33}(\lambda)$ are $t\times t$ matrices and $M_{22}(\lambda)$ a $(k\times k)$ matrix. The number of rows and columns of the order matrices is then evident. The matrix $m_1(\lambda)$ is a $(k\times t)$-matrix and $m_2(\lambda)$ a $(t\times t)$-matrix. From the definition of $M(\lambda)$ the $t$ columns of the matrix

$$\begin{bmatrix} A_1^* & 0 & -A_2^* \\ 0 & E_k & 0 \\ A_2^* & 0 & A_1^* \end{bmatrix}\begin{bmatrix} E_t \\ M(\lambda) \end{bmatrix} \tag{8.4}$$

are the initial values for a basis of $E_\lambda(I)$ and the $2t+k$ columns of

$$\left(E_{2t+k}, 0\right)\cdot\left(\hat{\Theta}(a,\lambda) + \hat{\Phi}(a,\lambda)\hat{M}(\lambda)\right) \tag{8.5}$$

are the initial values for solutions in $E_\lambda(I)$. Hence the columns of (8.5) can be expressed as linear combinations of the columns of (8.4). Therefore with uniquely determined matrices $K_i$ $(i=1,2,3)$ we have relations

$$\begin{bmatrix} A_1^* & 0 & -A_2^* \\ 0 & E_k & 0 \\ A_2^* & 0 & A_1^* \end{bmatrix}\begin{bmatrix} E_t \\ M(\lambda) \end{bmatrix} K_1 = \begin{bmatrix} A_1^* \\ 0 \\ A_2^* \end{bmatrix} + \begin{bmatrix} -A_2^* & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}}E_k & 0 \\ A_1^* & 0 & 0 \end{bmatrix}\begin{bmatrix} M_{11}(\lambda) \\ M_{21}(\lambda) \\ M_{31}(\lambda) \end{bmatrix} \tag{8.6}$$

$$(8.7)\quad \begin{bmatrix} A_1^* & 0 & -A_2^* \\ 0 & E_k & 0 \\ A_2^* & 0 & A_1^* \end{bmatrix} \begin{bmatrix} E_t \\ M(\lambda) \end{bmatrix} K_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} E_k \\ 0 \end{bmatrix} + \begin{bmatrix} -A_2^* & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} E_k & 0 \\ A_1^* & 0 & 0 \end{bmatrix} \begin{bmatrix} M_{12}(\lambda) \\ M_{22}(\lambda) \\ M_{32}(\lambda) \end{bmatrix}$$

$$(8.8)\quad \begin{bmatrix} A_1^* & 0 & -A_2^* \\ 0 & E_k & 0 \\ A_2^* & 0 & A_1^* \end{bmatrix} \begin{bmatrix} E_t \\ M(\lambda) \end{bmatrix} K_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -A_2^* & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} E_k & 0 \\ A_1^* & 0 & 0 \end{bmatrix} \begin{bmatrix} M_{13}(\lambda) \\ M_{23}(\lambda) \\ M_{33}(\lambda) \end{bmatrix}.$$

The relation (8.6) is equivalent to the system of equations

$$\begin{aligned} \big(A_1^* - A_2^* m_2(\lambda)\big) K_1 &= A_1^* - A_2^* M_{11}(\lambda) \\ m_1(\lambda) K_1 &= \frac{i}{\sqrt{2}} M_{21}(\lambda) \\ \big(A_2^* + A_1^* m_2(\lambda)\big) K_1 &= A_2^* + A_1^* M_{11}(\lambda). \end{aligned}$$

Multiplying the first equation with $A_1$, the third with $A_2$ and then adding we get with respect to (5.1):

$$K_1 = E_t$$

and the second equation yields

$$M_{21}(\lambda) = -i\sqrt{2} m_1(\lambda).$$

Multiplying the first equation with $-A_2$, the third by $A_1$ and then adding we get again with respect to (5.1)

$$M_{11}(\lambda) = m_2(\lambda).$$

With the same method we solve the relations (8.7) and (8.8) and we obtain:

$$K_2 = 0; \qquad M_{12}(\lambda) = 0; \qquad M_{22}(\lambda) = iE_k$$

and

$$K_3 = 0; \qquad M_{23}(\lambda) = 0; \qquad M_{13}(\lambda) = 0.$$

Next consider the matrix

$$\big(0, E_{2t+k}\big)\big(\hat{\Theta}(a,\lambda) + \hat{\Phi}(\cdot,\lambda)\hat{M}(\lambda)\big).$$

The columns of this matrix form the initial values for $2t+k$ solutions in $E_{-\lambda}(I)$. By definition of $M(\lambda)$ the columns of

$$\hat{A} \cdot \begin{pmatrix} E_{t+k} \\ M(-\lambda) \end{pmatrix}$$

form the initial values for a basis in $E_{-\lambda}(I)$ and hence with uniquely determined matrices $K_4$, $K_5$, $K_6$ we have the equations

$$\begin{bmatrix} A_1^* & 0 & -A_2^* \\ 0 & E_k & 0 \\ A_2^* & 0 & A_1^* \end{bmatrix} \begin{bmatrix} E_{t+k} \\ M(-\lambda) \end{bmatrix} K_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_2^* \\ 0 & -\frac{i}{\sqrt{2}}E_k & 0 \\ 0 & 0 & -A_1^* \end{bmatrix} \begin{bmatrix} M_{11}(\lambda) \\ M_{21}(\lambda) \\ M_{31}(\lambda) \end{bmatrix} \tag{8.9}$$

$$\begin{bmatrix} A_1^*, & 0 & -A_2^* \\ 0 & E_k & 0 \\ A_2^* & 0 & A_1^* \end{bmatrix} \begin{bmatrix} E_{t+k} \\ M(-\lambda) \end{bmatrix} K_5 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}}E_k \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_2^* \\ 0 & -\frac{i}{\sqrt{2}}E_k & 0 \\ 0 & 0 & -A_1^* \end{bmatrix} \begin{bmatrix} M_{12}(\lambda) \\ M_{22}(\lambda) \\ M_{32}(\lambda) \end{bmatrix} \tag{8.10}$$

$$\begin{bmatrix} A_1^* & 0 & -A_2^* \\ 0 & E_k & 0 \\ A_2^* & 0 & A_1^* \end{bmatrix} \begin{bmatrix} E_{t+k} \\ M(-\lambda) \end{bmatrix} K_6 = \begin{bmatrix} A_1^* \\ 0 \\ A_2^* \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_2^* \\ 0 & -\frac{i}{\sqrt{2}}E_k & 0 \\ 0 & 0 & -A_1^* \end{bmatrix} \begin{bmatrix} M_{13}(\lambda) \\ M_{23}(\lambda) \\ M_{33}(\lambda) \end{bmatrix}. \tag{8.11}$$

Splitting the matrix $M(-\lambda)$ in the form $M(-\lambda) = \big(m_3(-\lambda), m_4(-\lambda)\big)$ with a $t \times t$ matrix $m_3(-\lambda)$ and a $(t \times k)$ matrix $m_4(-\lambda)$ we solve the equations in the same manner as before and get

$$K_4 = \begin{bmatrix} 0 \\ -m_1(\lambda) \end{bmatrix}; \qquad M_{31}(\lambda) = m_4(-\lambda)m_1(\lambda)$$

$$K_5 = \begin{bmatrix} 0 \\ \sqrt{2}E_k \end{bmatrix}; \qquad M_{32}(\lambda) = -\sqrt{2}m_4(-\lambda)$$

$$K_6 = \begin{bmatrix} E_t \\ 0 \end{bmatrix}; \qquad M_{33}(\lambda) = -m_3(-\lambda).$$

Thus we have finally

$$\hat{M}(\lambda) = \begin{bmatrix} m_2(\lambda) & 0 & 0 \\ -i\sqrt{2}m_1(\lambda) & iE_k & 0 \\ m_4(-\lambda)m_1(\lambda) & -\sqrt{2}m_4(-\lambda) & -m_3(-\lambda) \end{bmatrix}.$$

Using Corollary 5.13, we may further express $m_3$ and $m_4$ in terms of $m_1$ and $m_2$ yielding for $\lambda \in \mathbb{C}^+$,

$$m_3(-\lambda) = m_2(-\bar{\lambda})^*, \quad m_4(-\lambda) = im_1(-\bar{\lambda})^*.$$

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