

THE PURE IMAGINARY SPECTRUM OF TRIANGULAR INFINITE DIMENSIONAL HAMILTONIAN OPERATORS*

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Abstract In this paper, a necessary and sufficient set of conditions for the bounded invertibility of triangular infinite dimensional Hamiltonian operators is obtained. A key feature is that the diagonal elements of the triangular operators are not necessarily invertible. From this result, a characterization of the pure imaginary spectrum of triangular infinite dimensional Hamiltonian operators is deduced. Moreover, a sufficient condition for such operators to generate C_0 semigroups is given. Several examples are also presented to illustrate these results.

Key words triangular infinite dimensional Hamiltonian operators; invertibility; pure imaginary spectrum; C_0 semigroups

2000 AMS Subject Classification 47B99, 47A10, 47D60

Introduction

Many partial differential equations, especially many of those from mechanics, can be written as infinite dimensional Hamiltonian systems $\dot{u} = Hu$, where u is a function of a single variable t and takes values in $X \times X$ with X being a Hilbert space, and H is an infinite dimensional Hamiltonian operator. See, for example, [1], [2], [24] and [25]. For infinite dimensional Hamiltonian systems, many famous scholars such as Arnold^{[3],[4]}, Gel'fand^[7], Lax^[14], Magri^[16] and Olver^[18] have made important contributions to the development and final formulation of this concept. Infinite dimensional Hamiltonian systems are used in continuum mechanics including fluid mechanics, plasma mechanics and elastic media mechanics, and have origins in stability theory, motive power theory, elasticity theory, compound material mechanics, crushing mechanics, etc.

So far, there is no complete theory on the spectra of non-self-adjoint operators. In the 1950's, Glazman^[8] studied a class of non-self-adjoint operators, i.e., J-self-adjoint operators and gave some results on their spectra. Then, some interesting results were obtained for some special non-self-adjoint operators, such as transport operators^[23] and u-scalar operators^[21]. Infinite dimensional Hamiltonian operators are from the corresponding infinite dimensional Hamiltonian systems, and have deep mechanical background. The spectral theory of infinite dimensional Hamiltonian operators is the theoretical foundation of the separation of the variables method solving mechanical problems, and plays a significant role in elasticity mechanics and other related fields^{[1],[2],[24],[25]}. Therefore, the spectral theory of infinite dimensional Hamiltonian operators deserves further investigations.

Moreover, it is well known that many problems arisen in the theory of multiwire transmission lines, linear filtering and prediction and the theory of optimal control^[9] can be transformed

* This research is supported by Natural Science Foundation of China under grant 10162003.

into solving algebraic Riccati equations. However, in order to find the steady state solutions of algebraic Riccati equations, it is needed to know that the spectrum of the corresponding Hamiltonian matrices does not lie in the imaginary axis, i.e., these matrices have no pure imaginary spectrum. In this paper, we use our results together with the method of Li^[15] to study the pure imaginary spectra of infinite dimensional Hamiltonian operators.

On the other hand, the theory of semigroups of (linear) operators is a part of functional analysis. Also, the theory of semigroups of bounded operators is closely related to the solutions of abstract Cauchy problems, which are ordinary differential equations in (Banach) spaces. Usually, each “well-posed” abstract Cauchy problem gives rise to a semigroup of bounded operators. The theory of semigroups of bounded operators developed quite rapidly since the discovery of Hille-Yosida’s Theorem in 1948. By now, it has been widely applied to many fields related to analysis and became a powerful tool for solving problems in applications. Recently, results on infinite dimensional Hamiltonian operators frequently appear. See, the work of Kurina^{[10],[11],[12],[13]} and Azizov^{[5],[6]} etc. In this paper, we attempt to introduce the method of C_0 semigroups into infinite dimensional Hamiltonian systems.

Infinite dimensional Hamiltonian operators are a special class of operator matrices, and recently the research of operator matrices is very active^{[15],[17]}. In our research, we find some interesting phenomena. For example, the system of eigenfunctions of infinite dimensional Hamiltonian operators has symplectic orthogonality, and the sequence of eigenvalues of certain infinite dimensional Hamiltonian operators is symmetric about the imaginary axis and diverges to infinity. Therefore, it is not an easy thing for infinite dimensional Hamiltonian operators to generate C_0 semigroups. However, if an infinite dimensional Hamiltonian operator only has a pure imaginary spectrum, it is enough to verify that it is a dissipative operator.

The organization of this paper is as follows. We first investigate the bounded invertibility of triangular infinite dimensional Hamiltonian operators. Then, a characterization of their pure imaginary spectrum is obtained. Third, a result on C_0 semigroups generated by them is given. Finally, several examples are presented to illustrate the results of this paper.

1 Definitions and notations

Throughout this paper, we always let X be an infinite dimensional Hilbert space.

Definition 1 Suppose that $H : \mathcal{D}(H) \subseteq X \oplus X \rightarrow X \oplus X$ is a closed densely defined (linear) operator. If $(JH)^* = JH$, then H is called an infinite dimensional Hamiltonian operator, where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

with I being the identity operator on X , 0 the zero operator on X , and $(JH)^*$ the adjoint of (JH) .

For convenience, we give an equivalent definition of infinite dimensional Hamiltonian operators.

Definition 2 Suppose that $H : \mathcal{D}(H) \subseteq X \oplus X \rightarrow X \oplus X$,

$$H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix},$$

is a densely defined operator, where A is a closed densely defined operator and B and C are both self-adjoint operators, then H is called an infinite dimensional Hamiltonian operator. Moreover, we will call H a lower triangular (or an upper triangular) infinite dimensional Hamiltonian operator when B (or C) is the zero operator, and call $\dot{u} = Hu$ an infinite dimensional Hamiltonian system.

This definition of infinite dimensional Hamiltonian systems coincides with the traditional definition $\dot{u} = J \frac{\delta \mathcal{H}}{\delta u}$, where \mathcal{H} is the corresponding Hamiltonian.

Definition 3 Suppose that $A : \mathcal{D}(A) \subseteq X \rightarrow X$ is a closed operator. Then, the set composed of 0 and all pure imaginary numbers that belong to the spectrum of A is called the pure imaginary spectrum of A and denoted by $\sigma_{\text{Im}}(A)$, i.e.,

$$\sigma_{\text{Im}}(A) = \{i\omega \in \mathcal{C} : \omega \in \mathbb{R}, i\omega \in \sigma(A)\}.$$

Moreover, let $\rho_{\text{Im}}(A) = i\mathbb{R} \setminus \sigma_{\text{Im}}(A)$, where $i\mathbb{R} = \{i\omega \in \mathcal{C} : \omega \in \mathbb{R}\}$.

In this paper, we employ the subdivision of the spectrum of Stone^[20]. For an operator A , the domain, range, null space, spectrum, point spectrum, residual spectrum, continuous spectrum and resolvent set are denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\sigma(A)$, $P_\sigma(A)$, $R_\sigma(A)$, $C_\sigma(A)$ and $\rho(A)$, respectively. The identity operator, the set of complex numbers, the set of real numbers, the empty set, the real part of a complex number c , the modulus of c and the set of all bounded operators of the space X are denoted by I , \mathcal{C} , \mathbb{R} , \emptyset , $\text{Re}(c)$, $|c|$ and $\mathcal{L}(X)$, respectively. In addition, $\langle \cdot, \cdot \rangle_X$, $\langle \cdot, \cdot \rangle_{X \oplus X}$ and $\|\cdot\|_X$, $\|\cdot\|_{X \oplus X}$ are the scalar products and norms of the corresponding spaces, respectively.

2 Main results

In this section, the main results of this paper and their proofs are given. Without loss of generality, we only discuss the case of upper triangular infinite dimensional Hamiltonian operators.

Lemma 1 Suppose that H is an infinite dimensional Hamiltonian operator in a Hilbert space $X \oplus X$. If H has a densely defined inverse, then H^{-1} is also an infinite dimensional Hamiltonian operator.

Proof Note that $J^* = J^{-1} = -J$, $H^* = JHJ$, and $\mathcal{D}(H)$ is dense in $X \oplus X$ from the Definition 1. Thus,

$$(H^{-1})^* = (H^*)^{-1}$$

since $\mathcal{N}(H) = \{0\}$ and $\mathcal{R}(H)$ is dense. So, we have $(H^{-1})^* = JH^{-1}J$, that is $(JH^{-1})^* = JH^{-1}$. Hence, H^{-1} is also an infinite dimensional Hamiltonian operator. ■

Remark Note that $\mathcal{N}(A) \oplus \{0\} \subseteq \mathcal{N}(H)$. Thus, if H is invertible, then so is A . On the other hand, from $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$ we see that A^* is invertible if and only if $\mathcal{R}(A)$ is dense (in X). This implies that if A is invertible and $\mathcal{R}(A)$ is dense, then A^* is also invertible, and hence so is H . Therefore, as to invertibility of H , the only interesting case is the one where $\mathcal{R}(A)$ is not dense.

Remark For applications, we are actually interested in bounded inverses (defined on the whole space $X \oplus X$) for infinite dimensional Hamiltonian operators H . Since H is closed, its inverse H^{-1} , when exists, is also closed. So, by the Closed Graph Theorem, H has a bounded inverse if and only if it has an inverse defined on the whole space, i.e., if and only if $H : \mathcal{D}(H) \rightarrow X \oplus X$ is a bijection. By Lemma 1, bounded inverses of infinite dimensional Hamiltonian operators are also infinite dimensional Hamiltonian operators.

Motivated by the last remark, we have the following results on the existence of a bounded inverse for an infinite dimensional Hamiltonian operator.

Theorem 2 Suppose that

$$H = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix}$$

is an upper triangular infinite dimensional Hamiltonian operator in $X \oplus X$, then H has an inverse defined on $X \oplus X$ if and only if the following conditions are fulfilled:

- (i) $\mathcal{R}(A^*) = X$;
- (ii) in the block representation

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

under the decompositions

$$X = \mathcal{N}(-A^*)^\perp \oplus \mathcal{N}(-A^*) = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \rightarrow X = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp,$$

B_4 has an inverse defined on $\mathcal{R}(A)^\perp$. Moreover, in this case, $\mathcal{R}(A)$ is closed.

Proof Suppose that H has an inverse defined on $X \oplus X$. By Lemma 1, its inverse is an infinite dimensional Hamiltonian operator and can be denoted by

$$M = \begin{bmatrix} C & D \\ E & -C^* \end{bmatrix},$$

where E is not necessarily the zero operator. Note that

$$HM = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix} \begin{bmatrix} C & D \\ E & -C^* \end{bmatrix} = I_{X \oplus X}, \quad I_{\mathcal{D}(H)} = \begin{bmatrix} C & D \\ E & -C^* \end{bmatrix} \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix} = MH,$$

i.e.,

$$\begin{cases} AC + BE = I, \\ AD - BC^* = 0, \\ -A^*E = 0, \\ A^*C^* = I_X, \end{cases} \quad \begin{cases} CA = I, \\ CB - DA^* = 0, \\ EA = 0, \\ EB + C^*A^* = I, \end{cases}$$

where B , D and E are all self-adjoint operators. Hence, a necessary condition for H to have a bounded inverse is that there exists some operator C such that $A^*C^* = I_X$, and hence $\mathcal{R}(A^*) = X$.

To complete the proof of the necessity, it suffices to prove the condition (ii). Clearly, $\mathcal{R}(A^*)$ is closed, and $\mathcal{N}(A) = \{0\}$. Also, by the Closed Range Theorem [22], $\mathcal{R}(A)$ is closed since A is a closed densely defined operator and $\mathcal{R}(A^*)$ is closed. Thus, we have the following decompositions:

$$\begin{aligned} X \oplus X &= X \oplus \mathcal{N}(-A^*)^\perp \oplus \mathcal{N}(-A^*) = X \oplus \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \\ &\rightarrow X \oplus X = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus X. \end{aligned} \quad (1)$$

Under the decompositions in (1), H and M have the following block representations, respectively:

$$H = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & B_2 \\ 0 & B_3 & B_4 \\ 0 & -A_1^* & 0 \end{bmatrix}, \quad (2)$$

$$M = \begin{bmatrix} C & D \\ E & -C^* \end{bmatrix} = \begin{bmatrix} C_1 & C_2 & D \\ E_1 & E_2 & -C_1^* \\ E_3 & E_4 & -C_2^* \end{bmatrix}. \quad (3)$$

Note that $\mathcal{D}(A_1) = \mathcal{D}(A)$ is dense in X . Now we will verify that A_1 is a closed operator. Let $(\mathcal{D}(A_1) \supseteq)\{x_n\} \rightarrow x \in X$ and $A_1x_n \rightarrow y \in \mathcal{R}(A)$. Since $\mathcal{D}(A_1) = \mathcal{D}(A)$, we have $Ax_n = A_1x_n \rightarrow y \in X$. Because A is a closed operator in X , we have an $x \in \mathcal{D}(A)$ such that $Ax = y$, and so $A_1x = Ax = y$. Therefore, A_1 is a closed densely defined operator.

From the (2,1)-blocks in $MH = I_{\mathcal{D}(H)}$ and the (3,1)-blocks in $MH = I_{\mathcal{D}(H)}$, we have $E_1A_1 = E_3A_1 = 0$. Since $\mathcal{R}(A_1) = \mathcal{R}(A)$, we have $E_1 = E_3 = 0$. From the (1,1)-blocks in $HM = I_{X \oplus X}$ and those in $MH = I_{\mathcal{D}(H)}$, we have

$$A_1C_1 + B_1E_1 + B_2E_3 = A_1C_1 = I_{\mathcal{R}(A)}, \quad C_1A_1 = I_{\mathcal{D}(A_1)},$$

respectively, which shows that A_1 has an inverse defined on $\mathcal{R}(A)$ and $C_1 = A_1^{-1}$.

Since E is a self-adjoint operator, we have $E_2 = E_3^* = 0$. Therefore,

$$M = \begin{bmatrix} A_1^{-1} & C_2 & D \\ 0 & 0 & -(A_1^*)^{-1} \\ 0 & E_4 & -C_2^* \end{bmatrix}.$$

From the (2,2)-blocks in $HM = I_{X \oplus X}$ and the (3,3)-blocks in $MH = I_{\mathcal{D}(H)}$, we have

$$B_4E_4 = I_{\mathcal{R}(A)^\perp}, \quad E_4B_4 = I_{\mathcal{D}(B_4)},$$

i.e., B_4 has an inverse defined on $\mathcal{R}(A)^\perp$ and $E_4 = B_4^{-1}$. From the (2,3)-blocks in $HM = I_{X \oplus X}$, we have

$$-B_3(A_1^*)^{-1} - B_4C_2^* = 0,$$

i.e., $C_2^* = -B_4^{-1}B_3(A_1^*)^{-1}$, and hence $C_2 = -A_1^{-1}B_2B_4^{-1}$. Moreover, from the (1,2)-blocks in $MH = I_{\mathcal{D}(H)}$, we can get

$$A_1^{-1}B_1 + C_2B_3 - DA_1^* = 0,$$

i.e., $D = A_1^{-1}(B_1 - B_2B_4^{-1}B_3)(A_1^*)^{-1}$. Thus, we have completed the proof of the necessity.

Conversely, Suppose that the conditions (i) and (ii) are satisfied. By $\mathcal{R}(A^*) = X$, $\mathcal{R}(A)$ is closed, and we still have the decompositions in (1). Under these decompositions, the infinite dimensional Hamiltonian operator H is given by (2), where A_1 , A_1^* and B_4 all have a bounded inverse. For the operator M in (3), take $C_1 = A_1^{-1}$, $C_2 = -A_1^{-1}B_2B_4^{-1}$, $D = A_1^{-1}(B_1 - B_2B_4^{-1}B_3)(A_1^*)^{-1}$, $E_1 = E_2 = E_3 = 0$ and $E_4 = B_4^{-1}$. Through simple calculations we have $HM = I_{X \oplus X}$ and $MH = I_{\mathcal{D}(H)}$, whence H has an inverse defined on $X \oplus X$, and its inverse is

$$H^{-1} = \begin{bmatrix} A_1^{-1} & -A_1^{-1}B_2B_4^{-1} & A_1^{-1}(B_1 - B_2B_4^{-1}B_3)(A_1^*)^{-1} \\ 0 & 0 & -(A_1^*)^{-1} \\ 0 & B_4^{-1} & -B_4^{-1}B_3(A_1^*)^{-1} \end{bmatrix}.$$

The theorem is proved. ■

Remark In the proof of Theorem 2, we used the fact that H is an infinite dimensional Hamiltonian operator. For other operator matrices, the assertions do not necessarily hold. For a counterexample, see Example 2 of Section 3.

If for an upper triangular infinite dimensional Hamiltonian operator, $\mathcal{R}(A) = X$, then the corresponding decompositions become

$$\begin{aligned} X \oplus X &= X \oplus \mathcal{N}(-A^*)^\perp \oplus \mathcal{N}(-A^*) = X \oplus X \oplus \{0\} \\ &\rightarrow X \oplus X = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus X = X \oplus \{0\} \oplus X, \end{aligned}$$

and hence we have $B_4 : \{0\} \rightarrow \{0\}$, which can be considered to have a bounded inverse.

For the finite dimensional case, Theorem 2 can be reduced to:

Corollary 3 An upper triangular Hamiltonian matrix

$$H = \begin{bmatrix} A & B \\ 0 & -A^\top \end{bmatrix}$$

is invertible if and only if A is invertible.

Remark It is enough to note the following facts in order to prove the Corollary 3: as far as a matrix A is concerned, the invertibility of A is equivalent to that of A^\top and is equivalent to $\mathcal{R}(A) = X$. Besides, from this corollary we see that the invertibility of maps (or operators) in finite dimensional spaces has essential differences with that in infinite dimensional spaces.

Now, we give a characterization of the pure imaginary spectrum of upper triangular infinite dimensional Hamiltonian operators.

Theorem 4 Suppose that

$$H = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix}$$

is an upper triangular infinite dimensional Hamiltonian operator in $X \oplus X$, then $\lambda \in \rho_{\text{Im}}(H)$ if and only if $\mathcal{R}(\lambda I + A^*) = X$ and $B_{4,\lambda}$ has an inverse defined on $\mathcal{R}(\lambda I - A)^\perp$, where $B_{4,\lambda}$ is defined by the block representation

$$B = \begin{bmatrix} B_{1,\lambda} & B_{2,\lambda} \\ B_{3,\lambda} & B_{4,\lambda} \end{bmatrix}$$

of B under the decompositions

$$X = \mathcal{N}(\lambda I + A^*)^\perp \oplus \mathcal{N}(\lambda I + A^*) \rightarrow X = \mathcal{R}(\lambda I - A) \oplus \mathcal{R}(\lambda I - A)^\perp.$$

Proof For any $\lambda \in \rho_{\text{Im}}(H)$ we have λ is 0 or a pure imaginary. We further have $\lambda I + A^* = -(\lambda I - A)^*$, thus

$$\lambda I - H = \begin{bmatrix} \lambda I - A & -B \\ 0 & \lambda I + A^* \end{bmatrix}$$

is still an upper triangular infinite dimensional Hamiltonian operator. Therefore, according to Theorem 2, we immediately obtain the desired result. ■

From Theorem 4 and the discussions above, we immediately get the following characterization of $\sigma_{\text{Im}}(H)$.

Corollary 5 Suppose that

$$H = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix}$$

is an upper triangular infinite dimensional Hamiltonian operator in $X \oplus X$, then

$$\sigma_{\text{Im}}(H) = \{\lambda \in i\mathbb{R} : \mathcal{R}(\lambda I + A^*) \neq X\} \cup \{\lambda \in i\mathbb{R} : B \text{ has no block representation mentioned in Theorem 4 such that } B_{4,\lambda} \text{ has an inverse defined on } \mathcal{R}(\lambda I - A)^\perp\}.$$

Finally, we will give a theorem on C_0 semigroups generated by upper triangular infinite dimensional Hamiltonian operators.

Theorem 6 Suppose that

$$H = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix}$$

is an upper triangular infinite dimensional Hamiltonian operator in $X \oplus X$, if $\sigma(H) \subseteq \sigma_{\text{Im}}(H)$ and there exists an $\alpha \in \mathbb{R}$ such that

$$\text{Re}\langle Au, u \rangle_X + \text{Re}\langle Bv, u \rangle_X - \text{Re}\langle A^*v, v \rangle_X \leq \alpha \|w\|_{X \oplus X}^2, \quad w = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{D}(H),$$

then H generates a C_0 semigroup e^{tH} on $X \oplus X$.

Proof Since H is an infinite dimensional Hamiltonian operator, H is a closed densely defined operator in $X \oplus X$. Note that $\sigma(H) \subseteq \sigma_{\text{Im}}(H)$, i.e., the spectrum of H only consists of the pure imaginary spectrum, so $(0, \infty) \subseteq \rho(H)$. From the assumptions of the theorem we have

$$\begin{aligned} \text{Re}\langle (H - \alpha I)w, w \rangle_{X \oplus X} &= \text{Re}\left\langle \begin{bmatrix} A - \alpha I & B \\ 0 & -A^* - \alpha I \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle_{X \oplus X} \\ &= \text{Re}\langle Au, u \rangle_X + \text{Re}\langle Bv, u \rangle_X - \text{Re}\langle A^*v, v \rangle_X - \alpha \|w\|_{X \oplus X}^2 \\ &\leq 0, \end{aligned}$$

whence $H - \alpha I$ is a dissipative operator in $X \oplus X$. From Lumer-Phillips' Theorem [19] it follows that H generates a C_0 semigroup e^{tH} on $X \oplus X$. ■

3 Examples

In this section, we give some examples to illustrate the results of this paper.

Example 1 Let $A : l^2 \rightarrow l^2$ be the operator defined by

$$Ax = (x_1 + 2x_2, x_1, x_2, \dots, x_{n-1}, \dots), \quad x = (x_1, x_2, \dots) \in l^2.$$

By simple calculations, we have

$$A^*x = (x_1 + x_2, 2x_1 + x_3, x_4, \dots, x_{n+1}, \dots), \quad x = (x_1, x_2, \dots) \in l^2.$$

Then,

$$H = \begin{bmatrix} A & I \\ 0 & -A^* \end{bmatrix}$$

is an infinite dimensional Hamiltonian operator on $l^2 \oplus l^2$, and it has an inverse defined on $l^2 \oplus l^2$.

For a triangular infinite dimensional Hamiltonian operator, if its diagonal elements are both invertible, then it is invertible without any additional conditions. First of all, we will prove that the diagonal element A^* here is not invertible in order to show the effectiveness of our results. In fact, if $A^*x = (x_1 + x_2, 2x_1 + x_3, x_4, x_5, \dots, x_{n+1}, \dots) = 0$, then $x = (c, -c, -2c, 0, \dots, 0, \dots)$, where c is an arbitrary constant. Hence, A^* is not invertible.

In the following, we will prove $\mathcal{R}(A^*) = l^2$. For an arbitrary $y = (y_1, y_2, y_3, \dots) \in l^2$, the expression $y = A^*x$ will hold provided that $x = (0, y_1, y_2, \dots, y_{n-1}, \dots)$, whence $\mathcal{R}(A^*) = l^2$.

Since $\mathcal{N}(-A^*)^\perp = \mathcal{R}(A)$ and $\mathcal{N}(-A^*) = \mathcal{R}(A)^\perp$, under the decompositions

$$l^2 = \mathcal{N}(-A^*)^\perp \oplus \mathcal{N}(-A^*) \rightarrow l^2 = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp,$$

the identity operator I can be written as

$$\begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix},$$

where $I_1 : \mathcal{N}(-A^*)^\perp \rightarrow \mathcal{R}(A)$ and $I_2 : \mathcal{N}(-A^*) \rightarrow \mathcal{R}(A)^\perp$. It is clear that I_2 is invertible.

We know that the conditions of Theorem 2 are satisfied from the things proved above, whence the upper triangular infinite dimensional Hamiltonian operator

$$H = \begin{pmatrix} A & I \\ 0 & -A^* \end{pmatrix}$$

has an inverse defined on $l^2 \oplus l^2$. In fact, we can even obtain its inverse from the proof of Theorem 2: under the decompositions

$$\begin{aligned} l^2 \oplus l^2 &= l^2 \oplus \mathcal{N}(-A^*)^\perp \oplus \mathcal{N}(-A^*) = l^2 \oplus \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \\ &\rightarrow l^2 \oplus l^2 = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus l^2, \end{aligned}$$

we have

$$H^{-1} = \begin{bmatrix} A_1^{-1} & 0 & A_1^{-1}(A_1^*)^{-1} \\ 0 & 0 & (A_1^*)^{-1} \\ 0 & I_2 & 0 \end{bmatrix},$$

where A_1 is the operator from l^2 onto $\mathcal{R}(A)$ with $A_1x = Ax$, $x \in l^2$.

Theorem 2 does not necessarily hold for other operator matrices, see the following counterexample.

Example 2 Suppose that the operator $A : X \rightarrow X$ is defined in Example 1, i.e.,

$$Ax = (x_1 + 2x_2, x_1, x_2, \dots)$$

for any $x = (x_1, x_2, \dots) \in l^2$. Obviously, $\mathcal{R}(I) = l^2$, and I_2 (this notation has the same meaning as that in Example 1) has an inverse defined on $\mathcal{R}(A)^\perp$, but the operator matrix

$$N = \begin{bmatrix} A & I \\ 0 & I \end{bmatrix}$$

is even not invertible.

In fact, the operator matrix N is not an infinite dimensional Hamiltonian operator and has the following factor representation:

$$N = \begin{bmatrix} A & I \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}.$$

Thus, the invertibility of N is equivalent to that of

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}.$$

From the discussions of Example 1, we know that A is not invertible, whence N is not invertible.

In the following, we will give an example in which the pure imaginary spectrum of an infinite dimensional Hamiltonian operator is computed using Corollary 5.

Example 3 Let $X = L^2[0, \infty)$, and consider the closed densely defined operator $A : \mathcal{D}(A) \subseteq X \rightarrow X$ given by $Ax = x'$ for all $x \in \mathcal{D}(A)$, where

$$\mathcal{D}(A) = \{x \in X : x \text{ is absolutely continuous, } x' \in X\},$$

and the self-adjoint operator $B : \mathcal{D}(B) \subset X \rightarrow X$ defined by $Bx = x''$ with domain

$$\mathcal{D}(B) = \{x \in X : x, x' \text{ are absolutely continuous, } x(0) = 0, x', x'' \in X\},$$

then

$$H = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix}$$

is an infinite dimensional Hamiltonian operator, and $\sigma_{\text{Im}}(H) = i\mathbb{R}$.

Clearly H is an infinite dimensional Hamiltonian operator, and $A^* : A^*x = -x'$ with domain

$$\mathcal{D}(A^*) = \{x \in X : x \text{ is absolutely continuous, } x(0) = 0, x' \in X\}.$$

Also, it is not hard to calculate $C_\sigma(-A^*) = i\mathbb{R}$, and then $\mathcal{R}(\lambda I + A^*) \neq X$. By Corollary 5, we know that $\sigma_{\text{Im}}(H) = i\mathbb{R}$.

Example 4 Suppose that $X = L^2(-\infty, \infty)$ and $A : \mathcal{D}(A) \subseteq X \rightarrow X$ is the closed densely defined operator defined by $Ax = x'$ for all $x \in \mathcal{D}(A)$, where

$$\mathcal{D}(A) = \{x \in X : x \text{ is absolutely continuous, } x' \in X\},$$

then

$$H = \begin{bmatrix} A & I \\ 0 & -A^* \end{bmatrix}$$

is an infinite dimensional Hamiltonian operator and generates a C_0 semigroup on $X \oplus X$.

In fact, it is easy to see that H is an infinite dimensional Hamiltonian operator. By Corollary 5, we know that the spectrum of H equals the pure imaginary spectrum of H , and

$$\sigma(H) = \sigma_{\text{Im}}(H) = i\mathbb{R}.$$

On the other hand,

$$\begin{aligned} \text{Re}\langle Au, u \rangle_X + \text{Re}\langle v, u \rangle_X - \text{Re}\langle A^*v, v \rangle_X &= \langle u', u \rangle_X + \langle v, u \rangle_X + \langle v', v \rangle_X \\ &\leq |\langle v, u \rangle_X| \leq \frac{1}{2}(\|u\|_X^2 + \|v\|_X^2) = \frac{1}{2}\|w\|_{X \oplus X}^2, \end{aligned}$$

where

$$w = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{D}(H).$$

From Theorem 6, it then follows that H generates a C_0 semigroup on $X \oplus X$.

Acknowledgement The authors would like to thank Professor Hongyou Wu for his constructive suggestions. We are also grateful to the anonymous referees for their helpful comments.

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