SPECTRAL ANALYSIS OF A CLASS OF NON-SELF-ADJOINT DIFFERENTIAL OPERATORS

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Abstract. The spectrum of a class of non-self-adjoint singular differential operators in the L_2-space on the real line is determined: it equals the continuous spectrum of the operators and fills the positive semi-axis, and the spectral singularities of the spectrum and their orders are unbounded. An associated spectral expansion in principal functions is established, and an analog of Parseval formula is proved.

1. Introduction

The paper is devoted to the study of the spectrum and spectral expansion in principal functions of a differential operator L, generated by the differential expression $\ell[y] = -y'' + q(x)y$ in the space $L_2(-\infty, \infty)$ assuming that the coefficient is

$$q(x) = x^m \sum_{\alpha=1}^{\infty} q_\alpha e^{i\alpha x} \tag{1.1}$$

and the series

$$v = \sum_{\alpha=1}^{\infty} |q_\alpha| \tag{1.2}$$

converges ($m \geq 0$). Here, absolute value of the potential is bounded from above by $v|x|^m$ and tends to infinity as $|x| \longrightarrow \infty$. The operator L is non-self-adjoint and it will be self-adjoint only in the trivial case $q(x) \equiv 0$. It is found that the spectrum of the operator L, properly continuous, fills the semi-axis $[0, \infty)$ and there are spectral singularities on the continuous spectrum that coincide with the members of the form $\lambda_n = (\frac{n}{2})^2$, $n = 1, 2, 3, ...$ and have multiplicities equal to $mn + 1$. Note that the spectrum is studied in the paper [1] assuming that $m = 1$, $q_1 = 1$ and $q_\alpha = 0$ for $\alpha = 2, 3, ...$ The spectrum and Parseval formula are studied in the papers [2, 3] under $q_1 = 1$ and $q_\alpha = 0$ for $\alpha = 2, 3, ...$ In the papers [4, 5], the direct and inverse problems were completely solved for the periodic case ($m \equiv 0$).

2. Construction of Special Solutions for the Equation $\ell[y] = k^2 y$

The explicit form of special solutions of the equation, $\ell[y] = k^2 y$ given in the following theorem is of great importance in investigation.

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Theorem 1. *Let $q(x)$ be of the form (1.1), and assume that the series (1.2) converges. Then the equation*

$$-y'' + q(x)y = k^2 y \tag{2.1}$$

has a solution of the form

$$f(x,k) = e^{ikx}(1 + \sum_{n=1}^{\infty} \sum_{s=0}^{mn} \sum_{\alpha=n}^{\infty} P_{n,s}^{\alpha}(k) x^s e^{i\alpha x}). \tag{2.2}$$

Here, $P_{n,s}^{\alpha}(k)$ is a proper rational function having poles at the points $k = -\frac{\alpha}{2}, \alpha = 1,2,3,...$ and multiplicities no higher than $m\alpha + 1$ and the series (2.2) allows termwise differentiation with respect to x arbitrarily times $k \neq -\frac{n}{2}$, $n = 1,2,3,....$

Proof. We look for the solution of equation (2.1) in the form (2.2). Then we arrive at the following integral equation:

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{s=0}^{m(n+1)} \sum_{\alpha=n+1}^{\infty} P_{n+1,s}^{\alpha}(k) x^s e^{i(\alpha+k)x} \quad &= \quad \sum_{\mu=1}^{\infty} q_\mu \int_x^{\infty} \frac{\sin k(t-x)}{k} t^m e^{i(k+\mu)t} dt \\ &+ \sum_{n=1}^{\infty} \sum_{s=0}^{mn} \sum_{\alpha=n}^{\infty} \sum_{\mu=1}^{\infty} P_{n,s}^{\alpha}(k) q_\mu \\ &\times \int_x^{\infty} \frac{\sin k(t-x)}{k} t^{m+s} e^{i(k+\mu+\alpha)t} dt \end{aligned} \tag{2.3}$$

Let's transform the right hand-side of equality (2.3). Assuming

$$P_{0,0}^{0}(k) = 1, P_{0,-s}^{\alpha}(k) = 0, s = 1,2,...,m; \alpha = 1,2,3,...,$$

by passing to a new variable $\xi = t - x$ and then using Newton's binominal formula we get that the right hand-side of (2.3) equals

$$\sum_{n=0}^{\infty} \sum_{s=0}^{m(n+1)} \sum_{\alpha=n+1}^{\infty} \sum_{\mu=n}^{\alpha-1} \sum_{p=s}^{m(n+1)} P_{n,p-s}^{\mu}(k) q_{\alpha-\mu} C_p^{p-s} x^s e^{i(k+\alpha)x} \int_{x=0}^{\infty} \frac{\sin kt}{k} t^{p-s} e^{i(k+\alpha)t} dt.$$

Now by using the uniqueness of expansion in Fourier series to determine $P_{n+1,s}^{\alpha}(k)$ we get the following recursion formula:

$$P_{n+1,s}^{\alpha}(k) = \sum_{\mu=n}^{\alpha-1} \sum_{p=s}^{m(n+1)} P_{n,p-s}^{\mu}(k) q_{\alpha-\mu} C_p^{p-s} Q_{\alpha,p-s}(k), \tag{2.4}$$

where

$$Q_{\alpha,\beta}(k) = \frac{i^\beta \beta!}{2k}[\frac{1}{(2k+\alpha)^{\beta+1}} - \frac{1}{\alpha^{\beta+1}}], \beta = 0,1,2,...,mn. \tag{2.5}$$

Let r and ε be arbitrary positive numbers and the domain is

$$D(\varepsilon, r) = \{k : |2k+\alpha| \geq \varepsilon > 0, |k| < r\}.$$

Then it follows from (2.5) that if $k \in D(\varepsilon, r)$, then we have the estimation

$$|Q_{\alpha,\beta}(k)| \leq \frac{\beta!}{\alpha^{\beta+2}} C(\varepsilon, r), \tag{2.6}$$

where the number $C(\varepsilon, r)$ depends only on ε and r.

Let

$$P_n = \sum_{s=0}^{mn} \sum_{\alpha=n}^{\infty} |P_{n,s}^{\alpha}(k)|, k \in D(\varepsilon, r)$$

Then, taking the estimation (2.6) into account, we find from (2.4) that

$$P_{n+1} \leq P_n max_{0 \leq \ell \leq m(n+1)} A_{\ell,n}, \tag{2.7}$$

where

$$A_{\ell,n} = C(\varepsilon, r) v \sum_{s=0}^{\ell} C_e^{\ell-s} \frac{(\ell - s)!}{(n+1)^{\ell-s+2}}.$$

The following inequality $A_{\ell,n} \leq A_{\ell+1,n}$, $\ell = 1, 2, 3, ..., m(n+1)$ is verified immediately. Therefore,

$$\begin{aligned} max_{0 \leq \ell \leq m(n+1)} A_{\ell,n} &\leq A_{m(n+1),n} = C(\varepsilon, r) v \sum_{s=0}^{m(n+1)} C_{m(n+1)}^{m(n+1)-s} \frac{[m(n+1) - s]!}{(n+1)^{m(n+1)-s+2}} \\ &\leq C_1(\varepsilon, r) \frac{1}{(n+1)^2}, \qquad (2.8) \end{aligned}$$

where $C_1(\varepsilon, r)$ depends only on ε and r.

We find from (2.8) that series (2.2) converges uniformly in any bounded range of variation of x and $k \in D(\varepsilon, r)$. Indeed,

$$\left| \sum_{s=0}^{mn} \sum_{\alpha=n}^{\infty} P_{n,s}^{\alpha}(k) x^s e^{i(\alpha+k)} \right| \leq P_n (1 + |x|^{mn}) e^{-Imk.x}$$

and therefore the terms of the series

$$\left[1 + \sum_{n=1}^{\infty} P_n (1 + |x|)^{mn} \right] e^{-Imk.x} \tag{2.9}$$

majorize corresponding members of series (2.2) and converges uniformly at any bounded range of variation of x. The termwise differentiability of series (2.2) is easily verified. The existence of poles of the function $f(x, k)$ at the points $k = -\frac{\alpha}{2}, \alpha = 1, 2, 3, ...$ immediately follows from recursion formula (2.4). So the theorem is proved. ∎

Lemma 1. *At the point* $k = -\frac{n}{2}, n = 1, 2, 3, ...$ *the function* $f(x, k)$ *has a pole of order* $mn + 1$.

Proof. It follows from recursion formula (2.4) that

$$P_{n-1,m(n-1)}^{n-1}(k) = q_1^{n-1} P_{0,0}^{0}(k) \prod_{s=1}^{n-1} Q_{s,0}(k).$$

On the other hand $Q_{s,0}(k) = -\frac{1}{s(s+2k)}$.

Then $Q_{s,0}(-\frac{n}{2}) = \frac{1}{s(n+s)}$.

Considering $P_{0,0}^{0}(k) = 1$, we get

$$P_{n-1,m(n-1)}^{n-1}(-\frac{n}{2}) = \frac{q_1^{n-1}}{[(n-1!)]^2}.$$

Consequently,

$$\lim_{2k+n\longrightarrow 0}(2k+n)^{mn+1}P^n_{n,0}(k)=\frac{q_1(i)^{mn+2}(mn)!}{n}P^{n-1}_{n-1,m(n-1)}(-\frac{n}{2})\neq 0,$$

but

$$\lim_{2k+n\longrightarrow 0}(2k+n)^{mn+2}P^n_{n,0}(k)=0$$

Hence, it follows that the term in series (2.2) with number n has a pole of order $mn+1$ at the point $-\frac{n}{2}$, $n=1,2,3,...$ and it is easy to observe that the next terms don't change this order. ∎

3. Spectrum of Operator L

To study the spectrum of the operator L, at first calculate the kernel of the resolvent of the operator $(L-\lambda E)^{-1}$ by means of general methods. To construct the kernel of the resolvent of the operator L we prove the following lemma.

Lemma 2. *For any k with $Imk>0$ the function $f(x,k)\in L_2(0,\infty)$ and $f(x,-k)\in L_2(-\infty,0)$.*

Proof. The affirmation of the first part of the lemma is reduced to the convergence of the next iterated series:

$$\sum_{n,n'=1}^{\infty,\infty}\sum_{s,s'=0}^{mn,mn'}\sum_{\alpha=n,\alpha'=n'}^{\infty,\infty}|P^{\alpha}_{n,s}(k)||P^{\alpha'}_{n',s'}(k)|\int_0^{\infty}x^{s+s'}e^{-2Imkx}dx \tag{3.1}$$

For k with $Imk>0$ and from the inequality $P_{n+1}\leq A_{m(n+1),n}P_n$ it follows that series (3.1) converges. Applying the similar arguments we can show that $f(x,-k)\in L_2(-\infty,0)$. ∎

Lemma 3. *The functions $f(x,k)$ and $f(x,-k)$ form a fundamental system of solutions of equation (2.1) for $k\neq 0$.*

Proof. It follows from the form (2.2) and estimations (2.8) and (2.9) that $f(x,\pm k)$ allows holomorphic continuation to the upper half-plane with respect to x and

$$\lim_{Imx\longrightarrow\infty}f^{(j)}(x,\pm k)e^{\mp ikx}=(\pm ik)^j,\ \text{j=1,2,3,...,}\ \text{for}\ k\neq -\frac{n}{2}, n=1,2,3,...\ .$$

Therefore, the wronskian of the solutions $f(x,k)$ and $f(x,-k)$ is equal to $2ik$ and non-zero at $k\neq 0$. ∎

Using lemma 3, it is immediately verified that for $Imk>0$ the operator $(L-k^2E)^{-1}$ exists and is a bounded operator and its kernel is of the form:

$$R(x,t;k)=\frac{1}{2ik}\begin{cases} f(x,k)f(t,-k) & , \quad t<x\\ f(t,k)f(x,-k) & , \quad t>x.\end{cases} \tag{3.2}$$

Now let's cite a theorem on the spectrum of the operator L.

Theorem 2. *The spectrum of the operator L, continuous, fills the semi-axis* $[0,\infty)$, *and on the continuous spectrum there are spectral singularities at the points $\lambda_n=(\frac{n}{2})^2$ $(n=1,2,3,...)$ of multiplicity $mn+1$.*

Proof. It is easily established that none non-trivial linear combination of solutions (2.2) of equation (2.1) belongs to $L_2(-\infty,\infty)$. Hence, it follows that the operator L has no eigenvalues.

Let $s \neq -\frac{n}{2}$, $n = 1, 2, 3, ...$ Then

$$\lim_{\varepsilon \longrightarrow 0}[R(x,t;s+i\varepsilon) - R(x,t;s-i\varepsilon)] = \frac{i}{s}f(x,s)f(t,-s)$$

and therefore s belongs to a continuous spectrum of the operator L. It is obvious from (3.2) that on a continuous spectrum the kernel of a resolvent has poles of order $mn+1$ at the points $\lambda_n = (\frac{n}{2})^2$ $(n = 1, 2, 3, ...)$ and these points are not eigenvalues of the operator L. Consequently, they are spectral singularities (see [6], p. 306). ∎

4. Spectral Expansion in Principal Functions of the Operator L

By $\Gamma_\delta^+(\Gamma_\delta^-)$ we denote a contour formed by the segments

$$[0, \frac{1}{2} - \delta], [\frac{n}{2} + \delta, \frac{n+1}{2} - \delta] (n = 1, 2, 3, ...)$$

and semi-circles of radius δ with centers at the points $\frac{n}{2}, n = 1, 2, 3, ...$ arranged at the upper (lower) half-plane. Note that, the contour Γ_δ^+ is obtained from Γ_δ^- by turning around the angle π. In these notation it is obvious that if $z = |z|\exp(i\phi)$, $0 < \phi < \pi$, then

$$\begin{aligned} R(x,t;z) &= \frac{1}{2\pi i}\int_{|k-z|<\delta} \frac{R(x,t;k)}{k^2 - z^2}dk^2 \\ &= \frac{1}{\pi i}\int_{\Gamma_\delta^+} \frac{R(x,t;k)}{k^2 - z^2}kdk - \frac{1}{\pi i}\int_{\Gamma_\delta^-} \frac{R(x,t;k)}{k^2 - z^2}kdk \\ &+ \sum_{n=1}^{\infty} \frac{n}{(\frac{n}{2})^2 - z^2} Res_{k=\frac{n}{2}} R(x,t;k) \\ &= \frac{1}{\pi i}\int_{\Gamma_\delta^+} \frac{R(x,t;k) - R(x,t;-k)}{k^2 - z^2}kdk \\ &+ \sum_{n=1}^{\infty} \frac{n}{(\frac{n}{2})^2 - z^2} Res_{k=\frac{n}{2}} R(x,t;k) \end{aligned} \tag{4.1}$$

Using representation (3.2) it is easy to verify that

$$R(x,t;k) - R(x,t;-k) = \frac{1}{2ik}[f(x,k)f(t,-k) + f(t,k)f(x,-k)].$$

Then, it follows from (4.1) that

$$\begin{aligned} R(x,t;z) &= -\frac{1}{\pi}\int_{\Gamma_\delta^+} \frac{f(x,k)f(t,-k)}{k^2 - z^2}dk \\ &+ \frac{1}{2i}\sum_{n=1}^{\infty} \frac{1}{(\frac{n}{2})^2 - z^2} \left\{ (\frac{d}{dk})^{mn} \frac{(k - \frac{n}{2})^{mn+1}}{(mn)!} f(t,-k)f(x,k) \right\}_{k=\frac{n}{2}} \end{aligned} \tag{4.2}$$

Applying the Leibnitz law for differentiation of product to the second term at the right hand-side of equality (4.2), and then considering formula (2.2) for solving

$f(x,-k)$, we have:

$$\begin{aligned}
(\frac{d}{dk})^{mn}\frac{(k-\frac{n}{2})^{mn+1}}{(mn)!}f(t,-k)f(x,k) &= \sum_{p=0}^{mn}\frac{1}{p!(mn-p)!}[(\frac{d}{dk})^{p}f(x,k)] \\
&\times \big[(\frac{d}{dk})^{mn-p}(k-\frac{n}{2})^{mn+1}f(t,-k)\big] \\
&= \sum_{p=0}^{mn}\frac{1}{p!(mn-p)!}[(\frac{d}{dk})^{p}f(x,k)] \\
&\times \Big[(\frac{d}{dk})^{mn-p}(k-\frac{n}{2})^{mn+1}(\sum_{s=0}^{mn}P^{n}_{n,s}(-k)t^{s}e^{i(n-k)t}) \\
&+ (k-\frac{n}{2})^{mn+1}e^{-ikt}(1+\sum_{\substack{h=1\\ h\neq n}}\sum_{s=0}^{mh}\sum_{\alpha=h}^{\infty}P^{\alpha}_{h,s}(-k)t^{s}e^{i\alpha t} \\
&+ \sum_{s=0}^{mn}\sum_{\alpha=n+1}^{\infty}P^{\alpha}_{n,s}(-k)t^{s}e^{i\alpha t})\Big]
\end{aligned}$$

Considering that the second term of the last equality has no spectral singularities of the form $k=\frac{n}{2}$, $n=1,2,3,...$ equality (4.2) takes the form:

$$\begin{aligned}
R(x,t;z) &= -\frac{1}{\pi}\int_{\Gamma^{+}_{\delta}}\frac{f(x,k)f(t,-k)}{k^{2}-z^{2}}dk \\
&+ \frac{1}{2i}\sum_{n=1}^{\infty}\frac{1}{(\frac{n}{2})^{2}-z^{2}}\sum_{p=0}^{mn}\frac{1}{p!(mn-p)!} \\
&\times \left[(\frac{d}{dk})^{mn-p}(k-\frac{n}{2})^{mn+1}(\sum_{s=0}^{mn}P^{n}_{n,s}(-k)t^{s}e^{i(n-k)t})\right]_{k=\frac{n}{2}}\big[(\frac{d}{dk})^{p}f(x,k)\big]_{k=\frac{n}{2}}
\end{aligned}\tag{4.3}$$

Note that the "derivative" $\big[(\frac{d}{dk})^{p}f(x,k)\big]_{k=\frac{n}{2}}$ should be understood as the function $f^{(p)}(x,\frac{n}{2})$, arises at the right hand-side of (4.3) as a result of formal application of Leibnitz law for differentiation of product. Everywhere in sequel, where this "derivative" arises in similar situation it is understood in the meaning indicated here.

Using recursion formula (2.4) for $P^{n}_{n,s}(-k)$, having applied to the second term of equality (4.3), we find:

$$\begin{aligned}
(k-\frac{n}{2})^{mn+1}\left(\sum_{s=0}^{mn}P^{n}_{n,s}(-k)t^{s}e^{i(n-k)t}\right) &= (k-\frac{n}{2})^{mn+1}P^{n-1}_{n-1,m(n-1)}(-k) \\
&\times Q_{n,mn}(-k)e^{i(n-k)t}+(k-\frac{n}{2})^{mn+1} \\
&\times \left(\sum_{s=1}^{mn}\sum_{\ell=s}^{mn-1}P^{n-1}_{n-1,\ell-m}(-k)C^{\ell-s}_{\ell}Q_{n,\ell-s}(-k)t^{s}e^{i(n-k)t}\right).
\end{aligned}\tag{4.4}$$

At the second term at equality (4.4) each spectral property $k=\frac{n}{2}$, $n=1,2,3,...$ is a root of multiplicity $\leq mn$. Therefore

$$\lim_{k\longrightarrow\frac{n}{2}}\left[(\frac{d}{dk})^{mn-p}(k-\frac{n}{2})^{mn+1}\left(\begin{array}{c}\sum_{s=1}^{mn}\sum_{\ell=s}^{mn-1}P_{n-1,\ell-m}^{n-1}(-k)C_{\ell}^{\ell-s}\\ *Q_{n,\ell-s}(-k)t^{s}e^{i(n-k)t}\end{array}\right)\right]=0 \tag{4.5}$$

Considering (4.5) in equality (4.3), we get

$$\begin{aligned} R(x,t;z) &= -\frac{1}{\pi}\int_{\Gamma_{\delta}^{+}}\frac{f(x,k)f(t,-k)}{k^{2}-z^{2}}dk \\ &+\frac{1}{2i}\sum_{n=1}^{\infty}\frac{1}{(\frac{n}{2})^{2}-z^{2}}\sum_{p=0}^{mn}C_{mnn}^{p}\overline{f}_{n}^{(mn-p)}(t,-\frac{n}{2})f^{(p)}(x,\frac{n}{2}), \end{aligned} \tag{4.6}$$

here

$$\overline{f}_{n}(t,-k)=[(mn)!]^{-1}(k-\frac{n}{2})^{mn+1}P_{n-1,m(n-1)}^{n-1}(-k)Q_{n,mn}(-k)e^{i(n-k)t}.$$

Using formula (2.4) for $P_{n-1,m(n-1)}^{n-1}(-k)$, we find

$$\overline{f}_{n}(t,-k)=-\frac{q_{1}^{n-1}}{2(n-1)!}(\prod_{s=1}^{n-1}(s-2k)^{-1})e^{i(n-k)t}(2i)^{-mn}(\sum_{\gamma=0}^{mn}(n-2k)^{\gamma}n^{-(\gamma+1)}).$$

Consequently , for $p=0,1,2,...,mn$, $\overline{f}_{n}(t,-\frac{n}{2})\neq0$ uniformly with respect to t from any bounded range of variation.

Formula (4.6) is a spectral expansion of a kernel of resolvent of the operator L. In formula (4.6) pass to the limit as $\delta\longrightarrow0$. As a result, we find

$$\begin{aligned} R(x,t;z) &= -\frac{1}{\pi}v.p.\int_{-\infty}^{\infty}\frac{f(x,k)f(t,-k)}{k^{2}-z^{2}}dk \\ &+ \frac{1}{2i}\sum_{n=1}^{\infty}\frac{1}{(\frac{n}{2})^{2}-z^{2}}\{(\frac{d}{dk})^{mn}\overline{f}_{n}(t,-k)f(x,k)\}_{k=\frac{n}{2}} \end{aligned} \tag{4.7}$$

Here $arg\in(0,\pi)$ and the integral is understood in the sense of principal value at poles $\frac{n}{2}$, $n=+1,+2,+3,...$ along the integration path of integrand function.

Now let's cite a formula for expansion in principal functions of the operator L that is immediately derived from (4.7).

Theorem 3. *For each finite function $\Theta(x)$ from definition domain $D(L)$ it is expanded in principal functions of a continuous spectrum and spectral singularities of the operator L in the following from*

$$\Theta(x)=-\frac{1}{\pi}v.p.\int_{-\infty}^{\infty}\widehat{\Theta}(-k)f(x,k)dk+\frac{1}{2i}\sum_{n=1}^{\infty}\{(\frac{d}{dk})^{mn}\widehat{\overline{\Theta}}_{n}(t,-k)f(x,k)\}_{k=\frac{n}{2}} \tag{4.8}$$

where $\widehat{\Theta}(k)=\int_{-\infty}^{\infty}\Theta(t)f(t,k)dt,\widehat{\overline{\Theta}}_{n}(k)=\int_{-\infty}^{\infty}\Theta(t)\overline{f}_{n}(t,k)dt$.

Here the integral and series in (4.8) converge absolutely and uniformly with respect to x from any bounded range of variation.

Write equality (4.8) for $\Theta_{1}(x)=\Theta(x)$, and then multiply equality (4.8) by $\Theta_{2}(x)$ and integrate with respect to x from $-\infty$ to ∞. As a result we get the following theorem.

Theorem 4. *(Analogy of Parseval formula)For each pair of finite functions from the class* $\mathbb{C}^2(-\infty,\infty)\cap D(L)$ *we have:*

$$\int_{-\infty}^{\infty}\Theta_1(x)\Theta_2(x)dx=-\frac{1}{\pi}v.p.\int_{-\infty}^{\infty}\widehat{\Theta}_1(-k)\widehat{\Theta}_2(k)dk+\frac{1}{2i}\sum_{n=1}^{\infty}\{(\frac{d}{dk})^{mn}\widehat{\widehat{\Theta}}_{1n}(-k)\widehat{\Theta}_2(k)\}_{k=\frac{n}{2}},$$

where

$$\widehat{\Theta}_2(k)=\int_{-\infty}^{\infty}\Theta_2(x)f(x,k)dx.$$

It should be noted that up to now in the considered cases both spectral singularities themselves, and their corresponding orders are finite (see [7]) and in the paper [4] although spectral singularities are unbounded but they are of first order.

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