The Self-adjoint Extensions of Singular Differential Operators with a Real Regularity Point

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Abstract. This paper deals with the self-adjoint domains of a singular symmetric differential expression \( l(y) \) with a middle deficiency index, under the condition that \( \Pi(L_0) \cap \mathbb{R} \neq \emptyset \), where \( \Pi(L_0) \) is the regularity domain of the corresponding minimal operator \( L_0 \). A complete analytic description of the self-adjoint domains of \( l(y) \) is obtained by giving a new decomposition of the maximal operator domain \( D_M \) using the \( L^2 \)-solutions of the equation \( l(y) = \lambda_0 y \) with \( \lambda_0 \in \Pi(L_0) \cap \mathbb{R} \). The description is independent of the properties of \( l(y) \) at infinity, the singular point of \( l(y) \).

Key words: symmetric differential operator; self-adjoint extensions; regularity domain; deficiency indices.

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Introduction

The description of self-adjoint domains of symmetric differential expression \( l(y) \) is a fundamental problem in the theory of differential operators. It has a profound background in physics and applied mathematics. In 1960’s, Everitt and his collaborators generalized the Titchmarsh-Weyl methods, and gave some results on self-adjoint domains in some special cases (e.g., the limit circle case and the limit point case [5]). Afterwards, using the theory of linear operators, Cao and Sun [1, 2, 3, 9] obtained a complete and direct characterization of all self-adjoint extensions of symmetric differential operators. In particular, Sun gave a new decomposition of the maximal operator domain of \( l(y) \) of order \( n \). Using the decomposition, a complete description of the self-adjoint extensions of the minimal operators \( L_0 \) generated by \( l(y) \) was given; and it is the first such description in the case of middle deficiency indices. In 1990, Evans and Ibrahim [4] generalized the result to more general differential expressions \( M \), and gave a characterization of all the regularly solvable operators and their adjoints generated by \( M \). By Cao and Sun’s methods, Shang [8] got the \( J \)-self-adjoint extensions of a \( J \)-symmetric \( l(y) \), where \( J \) denotes complex conjugation. Fu [6] extended Sun’s result to the characterization of self-adjoint extensions of differential operators in direct sum spaces. In this century, Wang and Sun [10] gave a complex symplecto-geometric description of \( J \)-symmetric differential operators. However,
we notice that the determination of boundary conditions depends on the properties of the solutions of \( l(y) = \lambda y \) at infinity, the singular point of \( l(y) \). So, it is difficult to realize the self-adjoint extensions.

In this paper, under the condition that \( \Pi(L_0) \cap \mathbb{R} \neq \emptyset \), we use the solutions of \( l(y) = \lambda_0 y \) with \( \lambda_0 \in \Pi(L_0) \cap \mathbb{R} \) and Cao’s [2] and Sun’s [9] methods to give a complete analytic description of self-adjoint domains of \( l(y) \) with middle deficiency indices. This result improves the conclusion of [9], and the determination of boundary conditions is independent of the properties of \( l(y) \) at infinity.

1 Preliminaries

Definition 1.1 Let \( T \) be a linear operator in a Hilbert space \( H \). The set

\[
\Pi(T) = \{ z \in \mathbb{C} : \text{there exists a constant } k(z) > 0 \text{ such that } \| (z - T)f \| \geq k(z) \| f \| \text{ for all } f \in D(T) \}
\]

is called the regularity domain of \( T \).

Lemma 1.1 [7] Let \( T \) be a linear operator in a Hilbert space \( H \), then \( \Pi(T) \) is open.

Definition 1.2 Let \( \lambda \) be a complex number. The subspace \( R(\lambda - T)^{\perp} \) is called the defect space of \( T \) and \( \lambda \). The cardinal number \( n_\lambda = \text{dim} R(\lambda - T)^{\perp} \) is called the deficiency index of \( T \) and \( \lambda \).

Lemma 1.2 [7] If \( T \) is a symmetric operator, then \( \mathbb{C} \setminus \mathbb{R} \subseteq \Pi(T) \), and the deficiency index of \( T \) is constant on each connected subset of \( \Pi(T) \).

Let \( n_+ \) and \( n_- \) denote the deficiency indices of the symmetric operator \( T \) associated with the upper and lower half-planes, respectively. The pair \((n_+, n_-)\) are called the deficiency indices of \( T \). Denote \( \text{def} T = (n_+, n_-) \).

Lemma 1.3 Let \( T \) be a symmetric operator in a complex Hilbert space. If \( \Pi(T) \cap \mathbb{R} \neq \emptyset \), then \( T \) has self-adjoint extensions.

Proof Since \( \Pi(T) \cap \mathbb{R} \neq \emptyset \), by Lemma 1.1, we have \( \Pi(T) \) is connected. By Lemma 1.2, we get \( n_+ = n_- \); and hence \( T \) has equal deficiency indices. Therefore, \( T \) has self-adjoint extensions.

From Definition 1.2 and Lemma 1.2, we have

Lemma 1.4 If \( \lambda_0 \in \Pi(T) \cap \mathbb{R} \), and \( n_{\lambda_0} \) is the deficiency index of \( T \) and \( \lambda_0 \), then \( \text{def} T = (n_{\lambda_0}, n_{\lambda_0}) \).

Throughout this paper, we assume that

\[
l(y) = \sum_{j=0}^{n} p_{n-j}(t)y^{(j)}, \quad t \in [0, \infty),
\]

is a singular symmetric differential expression of order \( n \) with equal deficiency indices \((m, m)\), where \( p_0(t), p_1(t), \ldots, p_n(t) \) are complex functions satisfying suitable differentiable conditions on \([0, \infty)\), and \([\frac{n+1}{2}] \leq m \leq n \) with \([\frac{n+1}{2}]\) denoting the integer part of
Let $L_M$ and $L_0$ denote the maximal operator and the minimal operator defined by the differential expression $l(y)$ restricted to the sets $D_M$ and $D_0$, respectively. For any matrix $A$, we denote its transpose by $A^T$ and its complex conjugate transpose by $A^*$.

Let $[\cdot, \cdot]$ denote the Langrange bilinear form associated with $l(y)$: for all $y, z \in D_M$,

$$[y, z](t) = \sum_{i=1}^{n} \sum_{j+k=i-1}^{n} (-1)^{j} y^{(k)}(p_{n-m}z)^{(j)} = \sum_{j,k=1}^{n} q_{jk}(t)y^{(k-1)}z^{(j-1)}.$$ 

Set $Q(t) = (q_{jk}(t))_{n \times n}$, and call $Q(t)$ the matrix of Langrange bilinear form associated with $l(y)$. By short calculations, we obtain $\det Q(t) = (p_0(x))^n \neq 0$, i.e., $Q(t)$ is non-singular.

Suppose $\Pi(L_0) \cap \mathbb{R} \neq \emptyset$. Let $\lambda_0 \in \Pi(L_0) \cap \mathbb{R}$. Because the deficiency indices of $l(y)$ are $(m, m)$, by Lemma 1.4, we obtain that $l(y) = \lambda_0 y$ has exactly $m$ linearly independent square integrable solutions on $[0, \infty)$.

Let $z_1, \cdots, z_n$ be functions in $D_M$ satisfying the following conditions:

$$z_i^{(k-1)}(0) = \delta_{ik}, \quad z_i^{(k-1)}(1) = 0 \quad \text{for} \quad k, i = 1, \cdots, n,$$

$$z_i(t) = 0 \quad \text{for} \quad i = 1, \cdots, n, \quad t \geq 1,$$  

(1.1)

where $\delta_{ik}$ is the Kronecker delta (for the existence of these functions, see Sect. 17 Lemma 2 in [7]).

**Theorem 1.1** Let $\lambda_0 \in \Pi(L_0) \cap \mathbb{R}$. Then, the equation $l(y) = \lambda_0 y$ has $2m - n$ linearly independent square integrable solutions $\theta_1, \cdots, \theta_{2m-n}$ on $[0, \infty)$ satisfying

$$\text{rank } E = 2m - n,$$

$$D_M = D_0 + \text{span}\{z_1, \cdots, z_n\} + \text{span}\{\theta_1, \cdots, \theta_{2m-n}\},$$  

(1.2)

where

$$E = ([\theta_{ij}]_{0 \leq i, j \leq 2m-n}).$$  

(1.3)

**Proof** From the proof of Theorem 1 of [9], we have

$$D_M = D_0 + \text{span}\{z_1, \cdots, z_n\} + \text{span}\{\varphi_1, \cdots, \varphi_{2m-n}\},$$  

(1.4)

where $\varphi_1, \cdots, \varphi_{2m-n} \in L^2[0, \infty)$ are the solutions of $l(y) = \lambda y$ or $l(y) = \overline{\lambda} y$ with $\text{Im}(\lambda) \neq 0$ such that \text{rank}([\varphi_i, \varphi_j]_{1 \leq i, j \leq 2m-n}) = 2m - n$. Because $l(y) = \lambda_0 y$ has $m$ linearly independent square integrable solutions $\theta_1, \cdots, \theta_m$ on $[0, \infty)$, by (1.4), we have

$$\theta_i = y_{0i} + \sum_{s=1}^{n} d_{is}z_s + \sum_{j=1}^{2m-n} c_{ij}\varphi_j,$$  

(1.5)

where each $y_{0i} \in D_0$. Since $z_s(t) = 0$ for $t \geq 1$ and $[y_{0i}, \theta_i]_{(\infty)} = 0$, we obtain

$$([\theta_{kl}]_{1 \leq k, l \leq m}) = ([\sum_{j=1}^{2m-n} c_{kj}\varphi_j, \sum_{j=1}^{2m-n} c_{lj}\varphi_j]_{(\infty)})_{1 \leq k, l \leq m} = C([\varphi_i, \varphi_j]_{(\infty)})_{1 \leq i, j \leq 2m-n} C^*,$$

where $C = (c_{ij})_{m \times (2m-n)}$. Hence,

$$\text{rank}([\theta_{kl}]_{1 \leq k, l \leq m}) \leq \text{rank}([\varphi_i, \varphi_j]_{(\infty)})_{1 \leq i, j \leq 2m-n} = 2m - n.$$
Since \( l(\theta_i) = \lambda_0 \theta_i \) and \( \lambda_0 \in \mathbb{R} \), from Green’s formula, we have
\[
[\theta_k, \theta_l](\infty) = [\theta_k, \theta_l](0), \quad k, l = 1, \ldots, m.
\] (1.6)
So,
\[
([\theta_k, \theta_l](\infty))_{m \times m}^T = ([\theta_k, \theta_l](0))_{m \times m}^T = (R(\theta_0)(0)Q(0)(R(\theta_0)(0))^T)^{m \times m} = \Theta^*(0)Q(0)\Theta(0),
\]
where \( R(\theta_0)(0) = (\theta_k(0), \theta'_k(0), \ldots, \theta^{(n-1)}_k(0)) \), and \( \Theta(t)_{n \times m} \) denotes the Wronskian matrix of \( \{\theta_i(t); i = 1, \ldots, m\} \). Since \( \text{rank}(Q(0)) = n \) and \( \text{rank}(\Theta(0)) = \text{rank}(\Theta^*(0)) = m \), it follows that \( \text{rank}([\theta_k, \theta_l](\infty))_{m \times m} \geq 2m - n \). Here we have used the fact that \( \text{rank}(MN) \geq \text{rank}M + \text{rank}N - n \) for any matrices \( M = M_{u \times n} \) and \( N = N_{n \times v} \). Hence,
\[
\text{rank}([\theta_k, \theta_l](\infty))_{m \times m} = \text{rank}([\theta_k, \theta_l](0))_{m \times m} = 2m - n.
\] (1.7)
Then, by \( C([\varphi_i, \varphi_j](\infty))_{1 \leq i, j \leq 2m-n}C^T = ([\theta_k, \theta_l](\infty))_{m \times m}^T = \Theta^*(0)Q(0)\Theta(0), \) we know \( \text{rank}C \geq 2m - n \). Since \( \text{rank}C_{m \times (2m-n)} \leq 2m - n \), we have \( \text{rank}C = 2m - n \). Without loss of generality, we can assume that the first \( 2m - n \) rows of \( C \) are linearly independent. Denote \( C_1 = (c_{ij})_{1 \leq i, j \leq 2m-n} \), then \( \text{rank}C_1 = 2m - n \). Using (1.5), we have
\[
([\theta_i, \theta_j](\infty))_{1 \leq i, j \leq 2m-n} = C_1([\varphi_i, \varphi_j](\infty))_{1 \leq i, j \leq 2m-n}C_1^*.
\]
So, \( \text{rank}([\theta_i, \theta_j](\infty))_{1 \leq i, j \leq 2m-n} = 2m - n \). Therefore, there exist \( 2m - n \) linearly independent solutions, say \( \theta_1, \ldots, \theta_{2m-n} \), of \( l(y) = \lambda_0 y \) such that \( \text{rank}E = 2m - n \). It remains to show that each \( y \in D_M \) can be uniquely written as the following form:
\[
y = y_0 + \sum_{s=1}^{n} d_s z_s + \sum_{j=1}^{2m-n} \tau_j \theta_j,
\]
where \( y_0 \in D_0 \). From the equation
\[
\theta_i = y_{0i} + \sum_{s=1}^{n} d_s z_s + \sum_{j=1}^{2m-n} c_{ij} \varphi_j, \quad i = 1, \ldots, 2m - n,
\] (1.8)
and \( \text{rank}C_1 = 2m - n \), we can solve for each \( \varphi_j \) and obtain the unique representation
\[
\varphi_j = \tilde{y}_{0j} + \sum_{i=1}^{n} \tilde{c}_{ji} z_i + \sum_{s=1}^{2m-n} \tilde{b}_{js} \theta_s,
\]
where \( \tilde{y}_{0j} \in D_0 \). Using the method similar to Theorem 1 of [9], we can prove (1.2). The proof is completed. \( \blacksquare \)

**Lemma 1.5** [7] Let the deficiency indices of \( L_0 \) be \((m, m)\). Then, a linear subspace \( D \) in \( D_M \) is the domain of a self-adjoint extension of \( L_0 \) if and only if there exist functions \( v_1, \ldots, v_m \) in \( D_M \) which satisfy
(i) \( v_1, \ldots, v_m \) are linearly independent modulo \( D_0 \);
(ii) \([v_i, v_j]_\infty^0 = 0 \) (i, j = 1, ⋯, m),
and
(iii) \( D = \{ y \in D_M | [y, v_j]_\infty^0 = 0, \ j = 1, \cdots, m \}. \)
2 The Main Result

Let
\[ B_1 = (\langle \varphi_i, \varphi_j \rangle(\infty))_{1 \leq i,j \leq 2m-n}, \]  
where \( \varphi_1, \ldots, \varphi_{2m-n} \) are defined in the proof of Theorem 1.1.

Lemma 2.1 [9] Let \( l(y) \) be a singular symmetric ordinary differential expression of order \( n \) with equal deficiency indices \( (m,m) \), where \( \left\lceil \frac{n+1}{2} \right\rceil \leq m \leq n \). Then, a linear subspace \( D \) in \( D_M \) is the domain of a self-adjoint extension of \( L_0 \) if and only if there exist an \( m \times n \) matrix \( M_1 \) and an \( m \times (2m - n) \) matrix \( N_1 \) satisfying

(1') \( \text{rank}(M_1 \oplus N_1) = m \),

(2') \( M_1 Q^{-1}(0) M_1^* + N_1 B_1 N_1^* = 0 \),

and such that

(3') \( D = \{ y \in D_M : M_1 \begin{pmatrix} y(0) \\ \vdots \\ y^{(n-1)}(0) \end{pmatrix} - N_1 \begin{pmatrix} [y, \varphi_1](\infty) \\ \vdots \\ [y, \varphi_{2m-n}](\infty) \end{pmatrix} = 0 \} \).

In the following, we will use (1.8), i.e., the relation between the \( \theta_i \)'s and the \( \varphi_i \)'s, and Lemma 2.1 to prove the main result of this paper.

Let
\[ B = ([\theta_i, \theta_j](0))_{1 \leq i,j \leq 2m-n}. \]  

Theorem 2.1 Let \( l(y) \) be a singular symmetric ordinary differential expression of order \( n \) with equal deficiency indices \( (m,m) \), where \( \left\lceil \frac{n+1}{2} \right\rceil \leq m \leq n \), \( \Pi(L_0) \cap \mathbb{R} \neq \emptyset \), and \( \{\theta_1, \ldots, \theta_{2m-n}\} \) satisfy Theorem 1.1. Then, a linear subspace \( D \) in \( D_M \) is the domain of a self-adjoint extension of \( L_0 \) if and only if there exist complex matrices \( M_{m \times n} \) and \( N_{m \times (2m - n)} \) satisfying

(1) \( \text{rank}(M \oplus N) = m \),

(2) \( M Q^{-1}(0) M^* + N B N^* = 0 \),

and such that

(3) \( D = \{ y \in D_M : M \begin{pmatrix} y(0) \\ \vdots \\ y^{(n-1)}(0) \end{pmatrix} - N \begin{pmatrix} [y, \theta_1](\infty) \\ \vdots \\ [y, \theta_{2m-n}](\infty) \end{pmatrix} = 0 \} \).

Proof. Necessity. Let \( D \) be the domain of a self-adjoint extension of \( L_0 \). By Lemma 1.5, there exist \( v_1, \ldots, v_m \in D_M \) satisfying the conditions (i), (ii) and (iii). From Theorem 1.1, each \( v_i \) can be uniquely written as
\[ v_i = \tilde{y}_{0i} + \sum_{j=1}^{n} e_{ij} z_j + \sum_{j=1}^{2m-n} \tau_{ij} \theta_j, \]  

where \( \tilde{y}_{0i} \in D_0 \). Substitute (1.8) into (2.3), we have
\[ v_i = (\tilde{y}_{0i} + \sum_{k=1}^{2m-n} \tau_{ik} y_{0k}) + (\sum_{j=1}^{n} e_{ij} z_j + \sum_{k=1}^{2m-n} \sum_{j=1}^{n} \tau_{ik} d_{kj} z_j) + \sum_{k=1}^{2m-n} \sum_{j=1}^{2m-n} \tau_{ik} c_{kj} \varphi_j. \]
By Lemma 2.1, there exist matrices
\[ M_1 = V^*(0)Q(0) \quad \text{and} \quad N_1 = (\tau_{ij})_{m \times (2m-n)}(\bar{c}_{ij})_{(2m-n) \times (2m-n)} \]
satisfying the conditions \((1'),(2')\) and \((3')\), where \(V(t)_{n \times m}\) denotes the Wronskian matrix of \(\{v_i(t); \ i = 1, \cdots, m\}\).

By \((1.6)\), we have
\[ B = ([\theta_i, \theta_j](0))_{1 \leq i, j \leq 2m-n}^{2m-n} \]
\[ = ([\sum_{j=1}^{2m-n} c_{kj}\varphi_j, \sum_{j=1}^{2m-n} c_{lj}\varphi_j](\infty))_{1 \leq k, l \leq 2m-n}^{2m-n} \]
\[ = \bar{C}_1(\varphi_1, \varphi_2)(\infty)_{1 \leq i, j \leq 2m-n}^{2m-n}C_1^T = \bar{C}_1B_1C_1^T, \]
where \(C_1 = (c_{ij})_{(2m-n) \times (2m-n)}\). By the proof of Theorem 1.1, we have \(\text{rank}C_1 = 2m - n\).

Set
\[ M = M_1 \quad \text{and} \quad N = N_1\bar{C}_1^{-1}. \quad (2.4) \]

Then,
\[ \text{rank}(M \oplus N) = \text{rank}(M_1 \oplus N_1\bar{C}_1^{-1}) = \text{rank}(M_1 \oplus N_1) = m, \]
\[ MQ^{-1}(0)M^* + NBN^* = M_1Q^{-1}(0)M_1^* + N_1\bar{C}_1^{-1}\bar{C}_1B_1C_1^T(\bar{C}_1^{-1})^*N_1^* = M_1Q^{-1}(0)M_1^* + N_1B_1N_1^* = 0, \]
and for \(y \in D_M\),
\[ N \begin{pmatrix} [y, \theta_1](\infty) \\ \vdots \\ [y, \theta_{2m-n}](\infty) \end{pmatrix} = N \begin{pmatrix} [y, \sum_{j=1}^{2m-n} c_{1j}\varphi_j](\infty) \\ \vdots \\ [y, \sum_{j=1}^{2m-n} c_{2m-n,j}\varphi_j](\infty) \end{pmatrix} = N\bar{C}_1 \begin{pmatrix} [y, \varphi_1](\infty) \\ \vdots \\ [y, \varphi_{2m-n}](\infty) \end{pmatrix}. \]

So,
\[ M \begin{pmatrix} y(0) \\ \vdots \\ y^{(n-1)}(0) \end{pmatrix} - N \begin{pmatrix} [y, \theta_1](\infty) \\ \vdots \\ [y, \theta_{2m-n}](\infty) \end{pmatrix} = M_1 \begin{pmatrix} y(0) \\ \vdots \\ y^{(n-1)}(0) \end{pmatrix} - N_1 \begin{pmatrix} [y, \varphi_1](\infty) \\ \vdots \\ [y, \varphi_{2m-n}](\infty) \end{pmatrix}. \]

Therefore,
\[ D = \{y \in D_M : M \begin{pmatrix} y(0) \\ \vdots \\ y^{(n-1)}(0) \end{pmatrix} - N \begin{pmatrix} [y, \theta_1](\infty) \\ \vdots \\ [y, \theta_{2m-n}](\infty) \end{pmatrix} = 0\}. \]

**Sufficiency.** This follows by reversing the above arguments and then using Lemma 2.1. ■

**Corollary 2.1** Let \(M_1\) and \(N_1\) be the coefficient matrices for a self-adjoint domain with respect to \(\varphi_1, \cdots, \varphi_{2m-n}\), and let \(M\) and \(N\) be the coefficient matrices for the domain with respect to \(\theta_1, \cdots, \theta_{2m-n}\). Then, \(M = M_1\) and \(N = N_1\bar{C}_1^{-1}\).
As mentioned in the introduction, the new characterization of the self-adjoint domains, given in Theorem 2.1, does not use any property of $l(y)$ at $t = +\infty$, since $Q(0)$ and $B$ are defined at $t = 0$.

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**References**


