

Dissipative Non-Self-Adjoint Sturm-Liouville Operators and Completeness of Their Eigenfunctions

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Dedicated to Professor ZhiJiang Cao on the occasion of his seventieth birthday

Abstract. In this paper, non-self-adjoint Sturm-Liouville operators in Weyl's limit-circle case are studied. We first determine all the non-self-adjoint boundary conditions yielding dissipative operators for each allowed Sturm-Liouville differential expression. Then, using the characteristic determinant, the completeness of the system of eigenfunctions and associated functions for these dissipative operators is proved.

Key words: Sturm-Liouville differential operators, dissipative operators, eigenfunctions, completeness, characteristic determinant.

§1. Introduction

Non-self-adjoint spectral problems have more and more applications. For example, interesting non-classical wavelets can be obtained from eigenfunctions and associated functions for non-self-adjoint spectral problems. Thus, such problems receive more and more attention, especially the discreteness of the spectrum and the completeness of eigenfunctions.

The non-self-adjointness of spectral problems can be caused by one or more of the following factors: the non-linear dependence of the problems on the spectral parameter, the non-symmetricness of the differential expressions used, and the non-self-adjointness of the boundary conditions (BC's) involved. Next, we recall some results in these three categories.

Non-self-adjoint spectral problems associated with differential operators having only a discrete spectrum and depending polynomially on the spectral parameter have been considered by Gohberg and Krein [6] and by Keldysh [11]. They studied the spectrum and principal functions of such problems and showed the completeness of the principal functions in the corresponding Hilbert function spaces.

Non-self-adjoint differential operators whose spectrum may have a continuous part have been investigated by Glazman [5], Sims [19], Marchenko [14] and Race [17]. They

obtained some important results concerning the spectrum and principal functions of such operators in $L^2(a, b)$ generated by the differential expression

$$(1.1) \quad l(y) := -y'' + qy \quad \text{on } [0, +\infty)$$

together with J -self-adjoint BC's, where q is a complex-valued function. Some results of Glazman and Sims have been extended to the even high order case by Race [18], Kamimura [10] and Wang [21], [22].

Regular non-self-adjoint differential operators generated by symmetric differential expressions together with non-self-adjoint BC's have been investigated by Naimark in [15]. The singular case has been considered by Guseinov and Tuncay [7]. They studied the characteristic determinant associated with the Sturm-Liouville differential expression $l(y)$ in Weyl's limit-circle (LC) case and with a real-valued potential q together with separated BC's

$$(1.2) \quad \begin{cases} \cos \alpha y(0) + \sin \alpha y'(0) = 0, \\ [y, u](\infty) + (h_1 + ih_2)[y, v](\infty) = 0, \end{cases}$$

where h_1 and h_2 are real numbers, u and v are certain maximal domain functions, and $[y, u]$ is the Lagrange bracket of y and u . The completeness of the system of eigenfunctions and associated functions is proved for $h_2 > 0$ by Allahverdiev and Canoglu [2] using self-adjoint dilations of dissipative operators and the characteristic function.

In this paper, we generalize the results of [2]. More precisely, we study non-self-adjoint operators generated by the Sturm-Liouville differential expression

$$(1.3) \quad l(y) := [-(py)'] + qy/w \quad \text{on } (a, b)$$

in Weyl's LC case together with non-self-adjoint BC's, where $-\infty \leq a < b \leq +\infty$, $1/p$, q and w are real-valued functions on (a, b) which are integrable on each finite segment $[c, d] \subset (a, b)$, $p, w > 0$ almost everywhere on (a, b) , and the BC's can be either separated or coupled. We first determine all the non-self-adjoint BC's yielding dissipative operators, see Theorem 2.12. Then, the completeness of the system of eigenfunctions and associated functions of these dissipative operators is proved, see Theorem 2.30.

The methods of this paper can be applied to dissipative operators generated by quasi-differential expressions of higher order. The only difference is that when the order is ≥ 5 , the so obtained description of the dissipative BC's will be in terms of the semi-definiteness of a certain matrix constructed from the coefficient matrix of the BC's. See (or We omit the details.)

After having finished this paper, we noticed the recent paper [1]. Comparing to [1], our description of the dissipative BC's is more explicit, and our proof of the completeness is more direct.

This paper is organized as follows. In Section 2, we introduce our notation and state the main results of this paper. The determination of all the dissipative BC's is achieved in Section 3. In Section 4, we review the characteristic function and the characteristic determinant. The completeness of eigenfunctions and associated functions is studied in Section 5.

§2. Notation and Main Results

For any $m, n \in \mathbb{N}$, we use $M_{m,n}(\mathbb{C})$ to denote the vector space of m by n complex matrices, and $M_{m,n}^*(\mathbb{C})$ its open subset consisting of the elements with the maximal rank $\min\{m, n\}$. When a capital Greek or Latin letter other than Y stands for a matrix, the entries of the matrix will be denoted by the corresponding lower case letter with two indices. Let $GL(2, \mathbb{C})$ be the set of invertible complex matrices in dimension 2.

Throughout this paper, we suppose that the differential expression $l(y)$ defined by (1.3) is in Weyl's LC case, i.e., all the solutions of the differential equation

$$(2.1) \quad -(py')' + qy = \lambda wy \quad \text{on } (a, b)$$

are in the weighted Hilbert space $L_w^2((a, b), \mathbb{C})$ with weight w . Let

$$(2.2) \quad \mathcal{D}_{\max} = \left\{ f \in L_w^2((a, b), \mathbb{C}); f, pf' \in AC_{\text{loc}}((a, b), \mathbb{C}), l(y) \in L_w^2((a, b), \mathbb{C}) \right\},$$

the (maximal) domain of $l(y)$. Here, $AC_{\text{loc}}((a, b), \mathbb{C})$ stands for the set of complex-valued functions on (a, b) that are absolutely continuous on each finite segment $[c, d] \subset (a, b)$. Then, for any $y, z \in \mathcal{D}_{\max}$, their Lagrange bracket

$$(2.3) \quad [y, z](x) = y(x) \overline{(pz')(x)} - (py')(x) \overline{z(x)}$$

has finite limits at both a and b , and hence $[y, z]$ is continuous on $[a, b]$.

Two real-valued functions f and g in \mathcal{D}_{\max} are said to form a **boundary condition basis** if

$$(2.4) \quad [f, g](a) = [f, g](b) = 1.$$

Fix a point $c \in (a, b)$, then a particular choice of BC basis consists of $\theta(\cdot, 0)$ and $\tau(\cdot, 0)$, where for each $\lambda \in \mathbb{C}$, $\theta(\cdot, \lambda)$ and $\tau(\cdot, \lambda)$ are the solutions of (2.1) satisfying the initial conditions

$$(2.5) \quad \theta(c, \lambda) = 1, \quad (p\theta')(c, \lambda) = 0, \quad \tau(c, \lambda) = 0, \quad (p\tau')(c, \lambda) = 1.$$

This is because for any $\lambda \in \mathbb{R}$, $[\theta(\cdot, \lambda), \tau(\cdot, \lambda)] \equiv 1$ on $[a, b]$. If (2.1) is regular at a , i.e., $1/p, q, w \in L((a, a'), \mathbb{R})$ for some $a' \in (a, b)$, then a can also be chosen as c . There is a similar statement about b .

In this paper, we study the differential operators L generated by the differential expression (1.3) and **boundary conditions** of the form

$$(2.6) \quad \begin{cases} a_{11}[y, f](a) + a_{12}[y, g](a) + b_{11}[y, f](b) + b_{12}[y, g](b) = 0, \\ a_{21}[y, f](a) + a_{22}[y, g](a) + b_{21}[y, f](b) + b_{22}[y, g](b) = 0, \end{cases}$$

where the coefficient matrix

$$(2.7) \quad \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \end{pmatrix}$$

of (2.6) belongs to $M_{2,4}^*(\mathbb{C})$, i.e., has rank 2. In other words, given the BC (2.6), the operator L is the restriction of $l(y)$ to the subspace $\mathcal{D}(L)$ of \mathcal{D}_{\max} consisting of all the functions in \mathcal{D}_{\max} satisfying (2.6):

$$(2.8) \quad Ly = l(y) = [-(py')' + qy]/w, \quad y \in \mathcal{D}(L).$$

The BC (2.6) is **degenerated** if either the left or the right half of its coefficient matrix equals zero.

DEFINITION 2.9. [6, p.175] A (linear) operator T , acting in a complex Hilbert space \mathcal{H} and having domain $\mathcal{D}(T)$, is said to be **dissipative** if $\text{Im}(Ty, y) \geq 0$ for all $y \in \mathcal{D}(T)$.

REMARK 2.10. If T is self-adjoint, i.e., $T = T^*$, then for any $y \in \mathcal{D}(T) = \mathcal{D}(T^*)$,

$$(2.11) \quad (Ty, y) = (y, T^*y) = (y, Ty) = \overline{(Ty, y)},$$

and hence $\text{Im}(Ty, y) = 0$. Therefore, self-adjoint operators are dissipative.

The first main result of this paper is the following explicit characterization of all dissipative BC's, i.e., all BC's making the operator L dissipative.

THEOREM 2.12. *The Sturm-Liouville operator L generated by the differential expression (1.3) and a boundary condition (2.6) is dissipative if and only if up to a $\text{GL}(2, \mathbb{C})$ factor on the left, the coefficient matrix of (2.6) has one of the following four forms:*

$$(2.13) \quad \begin{pmatrix} 1 & a_{12} & 0 & b_{12} \\ 0 & a_{22} & -1 & b_{22} \end{pmatrix},$$

where a_{12} , a_{22} , b_{12} and b_{22} satisfy

$$(2.14) \quad \text{Im}(a_{12} + b_{22}) \geq 0, \quad 4\text{Im} a_{12} \text{Im} b_{22} \geq |a_{22} - \overline{b_{12}}|^2;$$

$$(2.15) \quad \begin{pmatrix} 1 & a_{12} & b_{11} & 0 \\ 0 & a_{22} & b_{21} & -1 \end{pmatrix},$$

where a_{12} , a_{22} , b_{11} and b_{21} satisfy

$$(2.16) \quad \text{Im}(a_{12} - b_{21}) \geq 0, \quad -4\text{Im} a_{12} \text{Im} b_{21} \geq |a_{22} + \overline{b_{11}}|^2;$$

$$(2.17) \quad \begin{pmatrix} a_{11} & 1 & 0 & b_{12} \\ a_{21} & 0 & -1 & b_{22} \end{pmatrix},$$

where a_{11} , a_{21} , b_{12} and b_{22} satisfy

$$(2.18) \quad \text{Im}(a_{11} - b_{22}) \leq 0, \quad -4\text{Im} a_{11} \text{Im} b_{22} \geq |a_{21} + \overline{b_{12}}|^2;$$

$$(2.19) \quad \begin{pmatrix} a_{11} & 1 & b_{11} & 0 \\ a_{21} & 0 & b_{21} & -1 \end{pmatrix},$$

where a_{11} , a_{21} , b_{11} and b_{21} satisfy

$$(2.20) \quad \operatorname{Im}(a_{11} + b_{21}) \leq 0, \quad 4\operatorname{Im} a_{11} \operatorname{Im} b_{21} \geq |a_{21} - \overline{b_{11}}|^2.$$

PROOF. See Section 3. ■

It is interesting to examine the dissipativeness of separated BC's. Here we only do this for a typical class of them, while the others can be discussed similarly.

COROLLARY 2.21. *When the coefficient matrix of (2.6) can be written into the form*

$$(2.22) \quad \begin{pmatrix} 1 & a_{12} & 0 & 0 \\ 0 & 0 & -1 & b_{22} \end{pmatrix},$$

the operator L is dissipative if and only if $\operatorname{Im} a_{12} \geq 0$ and $\operatorname{Im} b_{22} \geq 0$.

PROOF. By Theorem 2.12, in this case L is dissipative if and only if

$$(2.23) \quad \operatorname{Im}(a_{12} + b_{22}) \geq 0, \quad \operatorname{Im} a_{12} \operatorname{Im} b_{22} \geq 0,$$

which is equivalent to $\operatorname{Im} a_{12} \geq 0$ and $\operatorname{Im} b_{22} \geq 0$. ■

REMARK 2.24. Direct calculations yield that when the coefficient matrix of (2.6) can be written into the form in (2.13), (2.6) is self-adjoint if and only if

$$(2.25) \quad \operatorname{Im} a_{12} = \operatorname{Im} b_{22} = a_{22} - \overline{b_{12}} = 0.$$

There are similar statements about the forms in (2.15), (2.17) and (2.19). So, a *generic dissipative BC is non-self-adjoint*.

REMARK 2.26. Assume that the differential expression (1.3) is regular, i.e., $1/p$, q and w are actually integrable on the whole interval (a, b) . Then, by Naimark's Patching Lemma (see Chapter V Section 17.3 Lemma 2 in [15]), there are real-valued functions $f, g \in \mathcal{D}_{\max}$ such that

$$(2.27) \quad f(a) = f(b) = 0, \quad (pf')(a) = (pf')(b) = 1,$$

$$(2.28) \quad g(a) = g(b) = -1, \quad (pg')(a) = (pg')(b) = 0,$$

and hence they form a BC basis. Then, (2.6) takes the more commonly seen form

$$(2.29) \quad \begin{cases} a_{11}y(a) + a_{12}(py')(a) + b_{11}y(b) + b_{12}(py')(b) = 0, \\ a_{21}y(a) + a_{22}(py')(a) + b_{21}y(b) + b_{22}(py')(b) = 0. \end{cases}$$

So, in the regular case, even when the BC's are given in the regular BC form (2.29), the conclusions of Theorem 2.12, Corollary 2.21 and Remark 2.24 are still true. There are similar statements for the cases where only one of the two end points is regular.

The geometry of the space of self-adjoint BC's has been investigated in [12], is the base for studying the dependence of the spectrum on BC, and reveals many new properties of the spectrum (see also [8], [3], [9] and [16]). Thus, it is natural to undertake the same task for the space of dissipative BC's. We plan to pursue this in further publications.

Recall that a complex number λ_0 is called an eigenvalue of an operator T if there exists a non-zero element $y_0 \in \mathcal{D}(T)$ such that $Ty_0 = \lambda_0 y_0$; in this case, y_0 is called an eigenfunction of T for λ_0 . The eigenfunctions for λ_0 span a subspace of $\mathcal{D}(T)$, called the eigenspace for λ_0 ; and the geometric multiplicity of λ_0 is the dimension of its eigenspace.

A non-zero element $y \in \mathcal{D}(T)$ is called a root function of T for a complex number λ_0 if $(T - \lambda_0 I)^n y = 0$ for some $n \in \mathbb{N}$. In this case, λ_0 must be an eigenvalue. The root functions for λ_0 span a linear subspace of $\mathcal{D}(T)$, called the root lineal for λ_0 ; and the **algebraic multiplicity** of λ_0 is the dimension of its root lineal. *The algebraic multiplicity of any eigenvalue of L is finite* (see Chapter 1 Section 2 in [15]). If an element $y \in \mathcal{D}(T)$ is not an eigenfunction for λ_0 , then it is a root function for λ_0 if and only if there is a $k \in \mathbb{N}$ such that y_0 is an eigenfunction for λ_0 provided we set $y_k = y$ and $y_{j-1} = Ty_j - \lambda_0 y_j$ for $j = k, \dots, 1$. A root function is called an **associated function** if it is not an eigenfunction.

In general, the system of eigenfunctions and associated functions of L is not complete in $L_w^2((a, b), \mathbb{C})$. For example, if (2.6) is degenerated, then L does not have any eigenvalue, and hence the system is empty.

The other main result of this paper is the following theorem claiming the completeness when L is dissipative.

THEOREM 2.30. *If the Sturm-Liouville operator L generated by the differential expression (1.3) and a boundary condition (2.6) is dissipative, then its system of eigenfunctions and associated functions is complete in the Hilbert space $L_w^2((a, b), \mathbb{C})$.*

PROOF. See Section 5. ■

Combining Theorem 2.30 and Corollary 2.21, we immediately obtain the following consequence.

COROLLARY 2.31. *If the coefficient matrix of (2.6) can be written into the form in (2.22) with $\text{Im } a_{12} \geq 0$ and $\text{Im } b_{22} \geq 0$, then the system of eigenfunctions and associated functions of L is complete in $L_w^2((a, b), \mathbb{C})$.*

To end this section, we make the following comments.

REMARK 2.32. An operator T is said to have a definite imaginary if either $\text{Im}(Ty, y) \geq 0$ for all $y \in \mathcal{D}(T)$ or $\text{Im}(Ty, y) \leq 0$ for all $y \in \mathcal{D}(T)$, i.e., if either T or $-T$ is dissipative. Actually, the conclusion of Theorem 2.30 is true for all Sturm-Liouville operators L , generated by (1.3) and BC's (2.6), with definite imaginaries. Moreover, from the proof of Theorem 2.12 in Section 3 we can see that if L has a definite imaginary, then (2.6) can be rewritten so that its coefficient matrix has one of the forms given in (2.13), (2.15), (2.17)

and (2.19); when the coefficient matrix has the form specified by (2.13), L has a definite imaginary if and only if the second inequality in (2.14) is satisfied; when the coefficient matrix has the form given in (2.15), L has a definite imaginary if and only if the second inequality in (2.16) is fulfilled; etc.

§3. Dissipative Operators

In this section, we prove Theorem 2.12 and present a couple of other results about dissipative operators.

As mentioned in Second 2, a BC is just a system of two linearly independent homogeneous algebraic equations on $[y, f](a)$, $[y, g](a)$, $[y, f](b)$ and $[y, g](b)$ with a coefficient matrix in $M_{2,4}^*(\mathbb{C})$; and hence equivalent algebraic systems give the same BC. Two algebraic systems are equivalent if and only if their coefficient matrices differ by a $GL(2, \mathbb{C})$ factor on the left (corresponding to row operations on coefficient matrices, of course); in this case, we say that the two coefficient matrices are **equivalent**.

Write the coefficient matrix of (2.6) as $(C | D)$, where $C, D \in M_{2,2}(\mathbb{C})$. Then, it is equivalent to

$$(3.1) \quad \begin{pmatrix} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{pmatrix}$$

for some $b_{11}, b_{12}, b_{21}, b_{22} \in \mathbb{C}$ if and only if $\det C \neq 0$, to a matrix of the form given by (2.13) if and only if

$$(3.2) \quad \det \begin{pmatrix} c_{11} & d_{11} \\ c_{21} & d_{21} \end{pmatrix} \neq 0,$$

..., to

$$(3.3) \quad \begin{pmatrix} a_{11} & a_{12} & -1 & 0 \\ a_{21} & a_{22} & 0 & -1 \end{pmatrix}$$

for some $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{C}$ if and only if $\det D \neq 0$. Therefore, the coefficient matrix of each BC is equivalent to a matrix in one of the forms given by (3.1), (2.13), (2.15), (2.17), (2.19) and (3.3); and the degenerated BC's are the only BC's with coefficient matrices not equivalent to matrices in any of the forms given by (2.13), (2.15), (2.17) and (2.19).

The proof of Theorem 2.12 will need the following result about the Lagrange bracket.

LEMMA 3.4. *If h and k are real-valued functions in \mathcal{D}_{\max} , then for any two elements y and z of \mathcal{D}_{\max} ,*

$$(3.5) \quad [y, z] [h, k] = [y, h] \overline{[z, k]} - \overline{[z, h]} [y, k] \quad \text{on } [a, b].$$

PROOF. Direct calculations verify the equality on (a, b) , while the equality at the end points are obtained by taking limits. ■

Now, we are ready to prove Theorem 2.12.

PROOF of Theorem 2.12. Let $y \in \mathcal{D}(L)$. By Green's formula,

$$(3.6) \quad 2\mathrm{Im}(Ly, y) = (Ly, y) - (y, Ly) = [y, y](b) - [y, y](a).$$

Then, applying (3.5) and (2.4),

$$(3.7) \quad \begin{aligned} 2\mathrm{Im}(Ly, y) &= [y, f](b) \overline{[y, g](b)} - \overline{[y, f](b)} [y, g](b) \\ &\quad - [y, f](a) \overline{[y, g](a)} + \overline{[y, f](a)} [y, g](a). \end{aligned}$$

If the coefficient matrix of (2.6) is equivalent to the matrix in (2.13), then

$$(3.8) \quad [y, f](a) = -a_{12}[y, g](a) - b_{12}[y, g](b), \quad [y, f](b) = a_{22}[y, g](a) + b_{22}[y, g](b),$$

and hence

$$(3.9) \quad 2\mathrm{Im}(Ly, y) = \begin{pmatrix} \overline{[y, g](a)} & \overline{[y, g](b)} \end{pmatrix} \begin{pmatrix} r & c \\ \bar{c} & s \end{pmatrix} \begin{pmatrix} [y, g](a) \\ [y, g](b) \end{pmatrix},$$

where

$$(3.10) \quad r = 2\mathrm{Im} a_{12}, \quad c = i(\overline{a_{22}} - b_{12}), \quad s = 2\mathrm{Im} b_{22}.$$

Note that the 2 by 2 matrix in (3.9) is Hermitian. The eigenvalues of the Hermitian matrix are

$$(3.11) \quad \frac{r + s \pm \sqrt{(r - s)^2 + 4|c|^2}}{2},$$

and they are both non-negative if and only if

$$(3.12) \quad r + s \geq 0, \quad rs \geq |c|^2,$$

i.e., if and only if (2.14) is true. Since in this case $[y, g](a)$ and $[y, g](b)$ can be arbitrary by Naimark's Patching Lemma, $\mathrm{Im}(Ly, y) \geq 0$ for all $y \in \mathcal{D}(L)$ if and only if (2.14) is satisfied.

Similarly, L is not dissipative when (2.6) is degenerated, and we prove the dissipativeness conditions for the situations where the coefficient matrix of (2.6) is equivalent to one of the matrices in (2.15), (2.17) and (2.19). Since the coefficient matrix of each non-degenerated BC is equivalent to a matrix in one of the forms given by (2.13), (2.15), (2.17) and (2.19), the proof is finished. ■

REMARK 3.13. Actually, when the coefficient matrix of (2.6) is equivalent to $(I | B)$ for some $B \in M_{2,2}(\mathbb{C})$, the operator L is dissipative if and only if

$$(3.14) \quad \mathrm{Im}(b_{11} \overline{b_{21}} + b_{12} \overline{b_{22}}) \leq 0, \quad 4\mathrm{Im}(b_{11} \overline{b_{21}}) \mathrm{Im}(b_{12} \overline{b_{22}}) \geq |1 - b_{11} \overline{b_{22}} + \overline{b_{12}} b_{21}|^2;$$

and when the coefficient matrix of (2.6) is equivalent to $(A \mid -I)$ for some $A \in M_{2,2}(\mathbb{C})$, the operator L is dissipative if and only if

$$(3.15) \quad \operatorname{Im}(a_{11} \overline{a_{21}} + a_{12} \overline{a_{22}}) \geq 0, \quad 4\operatorname{Im}(a_{11} \overline{a_{21}}) \operatorname{Im}(a_{12} \overline{a_{22}}) \geq |a_{11} \overline{a_{22}} - \overline{a_{12}} a_{21} - 1|^2.$$

Now, we discuss general dissipative operators.

THEOREM 3.16. *Let T be an invertible operator. Then, $-T$ is dissipative if and only if the inverse operator T^{-1} of T is dissipative.*

PROOF: Assume that $-T$ is dissipative. Then, for all $y \in \mathcal{D}(T)$,

$$(3.17) \quad \operatorname{Im}(y, Ty) = -\operatorname{Im}(Ty, y) = \operatorname{Im}(-Ty, y) \geq 0.$$

Hence, for any $z \in \mathcal{D}(T^{-1})$,

$$(3.18) \quad \operatorname{Im}(T^{-1}z, z) = \operatorname{Im}(T^{-1}z, T(T^{-1}z)) \geq 0,$$

since $T^{-1}z \in \mathcal{D}(T)$. So, T^{-1} is dissipative.

The sufficiency can be deduced from the already proven necessity applied to T^{-1} , since $(-T^{-1})^{-1} = -T$. ■

For a densely defined operator T , we can always introduce two operators

$$(3.19) \quad T_{\operatorname{Im}} := (T - T^*)/(2i), \quad T_{\operatorname{Re}} := (T + T^*)/2.$$

Their domains are $\mathcal{D}(T) \cap \mathcal{D}(T^*)$. So, if $\mathcal{D}(T^*) \supseteq \mathcal{D}(T)$, then $T = T_{\operatorname{Re}} + iT_{\operatorname{Im}}$, and the dissipativeness of T is equivalent to the non-negativeness of its imaginary component T_{Im} ; if $\mathcal{D}(T^*) \subseteq \mathcal{D}(T)$, then $T^* = T_{\operatorname{Re}} - iT_{\operatorname{Im}}$; and if T is a bounded operator on \mathcal{H} , then T_{Re} and T_{Im} are self-adjoint. Note that when $\mathcal{D}(T) \cap \mathcal{D}(T^*)$ is dense, T_{Re} and T_{Im} are not self-adjoint in general, since the domains of their adjoints may be different from $\mathcal{D}(T) \cap \mathcal{D}(T^*)$.

Since T_{Im} is discussed here, next we prove a related result for later use.

LEMMA 3.20. *Assume that a densely defined operator T is invertible and has a dense range. If \mathcal{E} and \mathcal{F} are linear complements of*

$$(3.21) \quad \{y \in \mathcal{D}(T) \cap \mathcal{D}(T^*); Ty = T^*y\}$$

in $\mathcal{D}(T)$ and $\mathcal{D}(T^)$, respectively, then the range of $(T^{-1})_{\operatorname{Im}}$ is contained in $\mathcal{E} \oplus \mathcal{F}$.*

PROOF. Since the range $\mathcal{R}(T)$ of T is dense, T^* is also invertible. From $(T^*)^{-1} = (T^{-1})^*$ we obtain that

$$(3.22) \quad (T^{-1})_{\operatorname{Im}} = (T^{-1} - (T^*)^{-1})/(2i).$$

Let \mathcal{C} denote the subspace given by (3.21). Then, $\mathcal{D}(T) = \mathcal{C} \oplus \mathcal{E}$ and $\mathcal{D}(T^*) = \mathcal{C} \oplus \mathcal{F}$. Hence, $\mathcal{D}(T^{-1}) = \mathcal{R}(T) = T(\mathcal{C}) \oplus T(\mathcal{E})$ and $\mathcal{D}((T^*)^{-1}) = \mathcal{R}(T^*) = T(\mathcal{C}) \oplus T^*(\mathcal{F})$. So,

$\mathcal{D}(T^{-1}) \cap \mathcal{D}((T^*)^{-1}) = T(\mathcal{C}) \oplus (T(\mathcal{E}) \cap T^*(\mathcal{F}))$. From (3.22) we then see that for any $y \in T(\mathcal{C})$, $(T^{-1})_{\text{Im}}(y) = 0$; and for all $y \in T(\mathcal{E}) \cap T^*(\mathcal{F})$, $(T^{-1})_{\text{Im}}(y) \in \mathcal{E} \oplus \mathcal{F}$. Therefore, our claim is true. ■

§4. Characteristic Function and Characteristic Determinant

In this section, to prepare for the proof of Theorem 2.30, we review Green's function and use it to study the inverse of L . We also recall some basics about the characteristic determinant of nuclear operators.

The algebraic system (2.6) using a general BC basis can always be rewritten into a similar system using the particular BC basis consisting of $u := \theta(\cdot, 0)$ and $v := \tau(\cdot, 0)$. So, from now on we will give BC's only in terms of u and v , i.e., we will only use the form

$$(4.1) \quad \begin{cases} U_1(y) := a_{11}[y, u](a) + a_{12}[y, v](a) + b_{11}[y, u](b) + b_{12}[y, v](b) = 0, \\ U_2(y) := a_{21}[y, u](a) + a_{22}[y, v](a) + b_{21}[y, u](b) + b_{22}[y, v](b) = 0 \end{cases}$$

of the BC's.

For each $\lambda \in \mathbb{C}$, the two functions $\theta(\cdot, \lambda)$ and $\tau(\cdot, \lambda)$ form a fundamental system of solutions of (2.1), and hence determine the eigenvalues of L . Moreover, using u and v we can rewrite (2.1) as a regular first order system and hence obtain properties of θ and τ . More precisely, we have the following results.

LEMMA 4.2. *For all $x \in [a, b]$,*

$$(4.3) \quad \phi_{11}(x, \lambda) := [\theta(\cdot, \lambda), u](x), \quad \phi_{12}(x, \lambda) := [\tau(\cdot, \lambda), u](x),$$

$$(4.4) \quad \phi_{21}(x, \lambda) := [\theta(\cdot, \lambda), v](x), \quad \phi_{22}(x, \lambda) := [\tau(\cdot, \lambda), v](x)$$

are entire functions of λ with growth order ≤ 1 and minimal type: for any $i, j = 1, 2$ and $\varepsilon > 0$, there exists a finite constant $C_{i,j,\varepsilon}$ such that

$$(4.5) \quad |\phi_{i,j}(x, \lambda)| \leq C_{i,j,\varepsilon} e^{\varepsilon|\lambda|} \quad \forall \lambda \in \mathbb{C}.$$

Denote by $(A_{2 \times 2} | B_{2 \times 2})$ the coefficient matrix of (4.1), and set $\Phi = (\phi_{ij})_{2 \times 2}$. Then, a complex number is an eigenvalue of L if and only if it is a zero of the entire function

$$(4.6) \quad \Delta(\lambda) := \begin{vmatrix} U_1(\theta(\cdot, \lambda)) & U_1(\tau(\cdot, \lambda)) \\ U_2(\theta(\cdot, \lambda)) & U_2(\tau(\cdot, \lambda)) \end{vmatrix} = \det(A\Phi(a, \lambda) + B\Phi(b, \lambda)).$$

PROOF. For a simple proof of the first claim, see Lemma 1.1 in [4]; while the second claim can be verified using the uniqueness of solution of linear initial value problems on $[y, u]$ and $[y, v]$. ■

The important function $\Delta(\lambda)$ is called the **characteristic function** of L . Note that when the algebraic system (4.1) is replaced by an equivalent one, $\Delta(\lambda)$ gets a non-zero constant factor. When $\Delta(\lambda) \not\equiv 0$: the **analytic multiplicity** of an eigenvalue λ_0 is the order of λ_0 as a zero of $\Delta(\lambda)$; it is known that *the algebraic multiplicity of any eigenvalue of L is equal to the analytic multiplicity of the eigenvalue* (see Chapter 1 Section 2 in [15]). About $\Delta(\lambda)$, we have the following direct consequence of the first claim of Lemma 4.2.

COROLLARY 4.7. *The entire function $\Delta(\lambda)$ is also of growth order ≤ 1 and minimal type: for any $\varepsilon > 0$, there exists a finite constant C_ε such that*

$$(4.8) \quad |\Delta(\lambda)| \leq C_\varepsilon e^{\varepsilon|\lambda|} \quad \forall \lambda \in \mathbb{C},$$

and hence

$$(4.9) \quad \limsup_{\lambda \rightarrow \infty} \frac{\ln |\Delta(\lambda)|}{|\lambda|} \leq 0.$$

From (4.8) we can deduce the following properties of the zeros of $\Delta(\lambda)$.

LEMMA 4.10. *Assume that $\Delta(\lambda) \not\equiv 0$. If we denote by $\{\lambda_j\}$ a sequence of all zeros of $\Delta(\lambda)$, counting analytic multiplicity, then*

(1) *there exists a finite limit*

$$(4.11) \quad \lim_{r \rightarrow +\infty} \sum_{|\lambda_j| \leq r} \frac{1}{\lambda_j};$$

(2) *the number $n(r)$ of zeros λ_j lying in the disk $|\lambda| < r$ has the limit*

$$(4.12) \quad \lim_{r \rightarrow +\infty} \frac{n(r)}{r} = 0;$$

(3) *when $\Delta(0) \neq 0$, one has that*

$$(4.13) \quad \Delta(\lambda) = \Delta(0) \lim_{r \rightarrow +\infty} \prod_{|\lambda_j| \leq r} \left(1 - \frac{\lambda}{\lambda_j}\right) \quad \forall \lambda \in \mathbb{C}.$$

PROOF. See ? in [13]. ■

It is possible that $\Delta \equiv 0$, i.e., every complex number is an eigenvalue of L . However, this does not happen when L is dissipative, i.e., we have the following result.

LEMMA 4.14. *If L is dissipative, then the eigenvalues of L form a discrete subset of \mathbb{C} .*

PROOF. First, consider the case where the coefficient matrix of (4.1) is given by (2.13). Then, from the second inequality in (2.14) we obtain that

$$(4.15) \quad \begin{aligned} 4\operatorname{Im} a_{12} \operatorname{Im} b_{22} + 2\operatorname{Re} a_{22} \operatorname{Re} b_{12} - 2\operatorname{Im} a_{22} \operatorname{Im} b_{12} \\ \geq (\operatorname{Re} a_{22})^2 + (\operatorname{Re} b_{12})^2 + (\operatorname{Im} a_{22})^2 + (\operatorname{Im} b_{12})^2 \\ \geq -2\operatorname{Re} a_{22} \operatorname{Re} b_{12} + 2\operatorname{Im} a_{22} \operatorname{Im} b_{12}, \end{aligned}$$

and hence

$$(4.16) \quad \operatorname{Re} a_{22} \operatorname{Re} b_{12} - \operatorname{Im} a_{22} \operatorname{Im} b_{12} + \operatorname{Im} a_{12} \operatorname{Im} b_{22} \geq 0.$$

Let r be a square root of the above left hand side, and set

$$(4.17) \quad (\psi_{ij}(\lambda))_{2 \times 2} = \Phi_{ij}(b, \lambda) \Phi_{ij}(a, \lambda)^{-1}.$$

Then, direct calculations yield that

$$(4.18) \quad \begin{aligned} \Delta(\lambda) &= a_{22} + b_{12} + a_{12}\psi_{11}(\lambda) - \psi_{12}(\lambda) \\ &\quad + (a_{22}b_{12} - a_{12}b_{22})\psi_{21}(\lambda) + b_{22}\psi_{22}(\lambda) \\ &= a_{22} + b_{12} - 2r + \Delta_1(\lambda) + i\Delta_2(\lambda), \end{aligned}$$

where

$$(4.19) \quad \begin{aligned} \Delta_1(\lambda) &= 2r + (\operatorname{Re} a_{12})\psi_{11}(\lambda) - \psi_{12}(\lambda) \\ &\quad + (r^2 - \operatorname{Re} a_{12} \operatorname{Re} b_{22})\psi_{21}(\lambda) + (\operatorname{Re} b_{22})\psi_{22}(\lambda), \end{aligned}$$

$$(4.20) \quad \Delta_2(\lambda) = (\operatorname{Im} a_{12})\psi_{11}(\lambda) + c\psi_{21}(\lambda) + (\operatorname{Im} b_{22})\psi_{22}(\lambda),$$

with

$$(4.21) \quad c = \operatorname{Re} a_{22} \operatorname{Im} b_{12} + \operatorname{Im} a_{22} \operatorname{Re} b_{12} - \operatorname{Re} a_{12} \operatorname{Im} b_{22} - \operatorname{Im} a_{12} \operatorname{Re} b_{22} \in \mathbb{R}.$$

Since $\Delta_1(\lambda)$ is the characteristic function of the self-adjoint BC with coefficient matrix

$$(4.22) \quad \begin{pmatrix} 1 & \operatorname{Re} a_{12} & 0 & r \\ 0 & r & -1 & \operatorname{Re} b_{22} \end{pmatrix},$$

it is not constant on \mathbb{R} . Thus, from (4.18) and the reality of $\Delta_2(\lambda)$ on \mathbb{R} we deduce that $\Delta(\lambda)$ is also not constant on \mathbb{R} . Therefore, $\Delta(\lambda) \not\equiv 0$.

Similarly, we prove the claim for the other cases. ■

From now on, we assume that L is dissipative. By the above lemma, replacing q by $q + sw$ for some constant $s \in \mathbb{R}$ if necessary, we may suppose that zero is not an eigenvalue

of L (i.e., $\ker L = \{0\}$). Thus, the inverse operator L^{-1} of L exists. To find an explicit formula for L^{-1} , we use Green's function (see [15]). Let

$$(4.23) \quad G(x, \xi) = \frac{1}{\Delta(0)} \begin{vmatrix} u(x) & v(x) & g(x, \xi) \\ U_1(u) & U_1(v) & U_1(g(\cdot, \xi)) \\ U_2(u) & U_2(v) & U_2(g(\cdot, \xi)) \end{vmatrix},$$

where

$$(4.24) \quad g(x, \xi) = \frac{1}{2} \begin{cases} u(x)v(\xi) - u(\xi)v(x) & \text{if } a < \xi \leq x < b, \\ u(\xi)v(x) - u(x)v(\xi) & \text{if } a < x \leq \xi < b. \end{cases}$$

Then,

$$(4.25) \quad \int_a^b \int_a^b |G(x, \xi)|^2 w(x) dx w(\xi) d\xi < +\infty,$$

and hence the integral operator B defined by

$$(4.26) \quad Bf = \int_a^b G(\cdot, \xi) y(\xi) w(\xi) d\xi \quad \forall y \in L_w^2((a, b), \mathbb{C})$$

is a Hilbert-Schmidt operator. So, B is compact; but, non-self-adjoint, in general. It is easy to verify that B is the inverse of L : $B = L^{-1}$. Thus, 0 is not in the spectrum $\sigma(L)$ of L , and the root lineals of B coincide with those of L . Therefore, about the essential spectrum $\sigma_e(L)$ of L we have that

$$(4.27) \quad \sigma_e(L) = \{1/\lambda; \lambda \in \sigma_e(B) \setminus \{0\}\} = \emptyset,$$

and the completeness in $L_w^2((a, b), \mathbb{C})$ of the system of eigenfunctions and associated functions of L is equivalent to the completeness of the system of B .

For a compact operator K (acting on the whole space \mathcal{H}), we denote by $\nu(K) \in \mathbb{N} \cup \{+\infty\}$ the sum of the algebraic multiplicities of all non-zero eigenvalues of K , and by $\{\mu_j(K)\}_{j=1}^{\nu(K)}$ a sequence of all non-zero eigenvalues of K counting algebraic multiplicity and with non-increasing moduli. Recall that if K is a nuclear operator, then $\sum_{j=1}^{\nu(K)} |\mu_j(K)| < +\infty$; and if K is a Hilbert-Schmidt operator, then $\sum_{j=1}^{\nu(K)} |\mu_j(K)|^2 < +\infty$.

When K is a nuclear operator, the product

$$(4.28) \quad \det(I - \mu K) = \prod_{j=1}^{\nu(K)} (1 - \mu \mu_j(K))$$

converges uniformly on every compact subset of \mathbb{C} and hence determines an entire function of the variable μ , called the **characteristic determinant** of K and denoted by $D_K(\mu)$.

For a Hilbert-Schmidt operator K , the product

$$(4.29) \quad \prod_{j=1}^{\nu(K)} (1 - \mu \mu_j(K)) e^{\mu \mu_j(K)}$$

also converges uniformly on every compact subset of \mathbb{C} and hence defines an entire function of μ , called the **regularized characteristic determinant** of K and denoted by $\tilde{D}_K(\mu)$.

Let S and T be bounded operators such that $S - T$ is a nuclear operator. If $1/\mu \notin \sigma(T)$, i.e., the operator $I - \mu T$ has a bounded inverse defined on the whole space \mathcal{H} , then

$$(4.30) \quad (I - \mu S)(I - \mu T)^{-1} = I - \mu(S - T)(I - \mu T)^{-1}$$

with $\mu(S - T)(I - \mu T)^{-1}$ being also a nuclear operator. Consequently, the determinant

$$(4.31) \quad D_{S/T}(\mu) = \det[(I - \mu S)(I - \mu T)^{-1}]$$

makes sense and is called the **determinant of perturbation** of T by $K = S - T$.

LEMMA 4.32. [6, p.172] *Let S and T be Hilbert-Schmidt operators such that their difference $S - T$ is a nuclear operator. If $1/\mu \notin \sigma(T)$, then*

$$(4.33) \quad D_{S/T}(\mu) = \frac{\tilde{D}_S(\mu)}{\tilde{D}_T(\mu)} e^{\mu \operatorname{tr}(T-S)},$$

where, for a nuclear operator K , $\operatorname{tr} K$ denotes its trace.

LEMMA 4.34. [6, p.177] *If S and T are bounded dissipative operators such that $S - T$ is a nuclear operator, then for any $\theta_0 \in (0, \pi/2)$, the limit*

$$(4.35) \quad \lim_{\rho \rightarrow +\infty} \frac{\ln |D_{S/T}(\rho e^{i\theta})|}{\rho} = 0$$

converges uniformly in θ on the interval $(\pi/2 - \theta_0, \pi/2 + \theta_0)$.

§5. Completeness of Eigenfunctions

In this section, we give a proof of Theorem 2.30. One of the main ideas of the proof is the following result.

LEMMA 5.1. [6, p.227] *Let T be a compact dissipative operator on \mathcal{H} such that $\operatorname{tr} T_{\operatorname{Im}} < +\infty$. Then, the system of root functions of T is complete in \mathcal{H} if and only if*

$$(5.2) \quad \sum_{j=1}^{\nu(T)} \operatorname{Im} \mu_j(T) = \operatorname{tr} T_{\operatorname{Im}}.$$

We now return to the integral operator B defined by (4.26), the inverse of L . Set $B = B_1 + iB_2$ with $B_1 = B_{\text{Re}}$ and $B_2 = B_{\text{Im}}$. By the discussions of Section 4, B and B_1 are Hilbert-Schmidt operators, and B_1 is self-adjoint. It seems to us that the following result has not appeared in the literature.

LEMMA 5.3. *The operator B_1 is the inverse of the Sturm-Liouville operator generated by the differential expression (1.3) and a unique boundary condition.*

Of course, the BC used for the inverse of B_1 is self-adjoint. Note that this result is true in general, i.e., we only assume that B is the inverse of L .

PROOF. It can be verified by straightforward, even though lengthy, calculations. ■

Now, we are ready to prove Theorem 2.30.

PROOF of Theorem 2.30. Note that $-B$ is dissipative by Theorem 3.16. Moreover, since $Ly = L^*y$ for all minimal domain functions y and $\mathcal{D}(L)$ and $\mathcal{D}(L^*)$ are only complex 2-dimensional extensions of the minimal domain, B_2 is a finite rank operator by Lemma 3.20. Thus, from Lemma 4.32 we obtain that for any μ such that $1/\mu \notin \sigma(-B)$,

$$(5.4) \quad D_{-B_1/(-B)}(\mu) = \frac{\tilde{D}_{-B_1}(\mu)}{\tilde{D}_{-B}(\mu)} e^{\mu \text{tr}(B_1 - B)} = \frac{\tilde{D}_{-B_1}(\mu)}{\tilde{D}_{-B}(\mu)} e^{-i\mu \text{tr} B_2}.$$

Set $m = \nu(B)$, and let $\{1/\lambda_j\}_{j=1}^m$ be the eigenvalues of B . Then, $\{\lambda_j\}_{j=1}^m$ are the eigenvalues of L . Denote by Δ_{-L} the characteristic function of $-L$, then we have that

$$(5.5) \quad \tilde{D}_{-B}(\mu) = \prod_{j=1}^m \left(1 + \frac{\mu}{\lambda_j}\right) e^{-\mu/\lambda_j} = \frac{\Delta_{-L}(\mu)}{\Delta_{-L}(0)} \exp\left(-\mu \sum_{j=1}^m \frac{1}{\lambda_j}\right),$$

since the algebraic multiplicity of each $-\lambda_j$ equals its analytic multiplicity. Let $\{1/r_j\}_{j=1}^{+\infty} \subset \mathbb{R}$ be the eigenvalues of the self-adjoint operator B_1 . Then,

$$(5.6) \quad \tilde{D}_{-B_1}(\mu) = \prod_{j=1}^{+\infty} \left(1 + \frac{\mu}{r_j}\right) e^{-\mu/r_j},$$

and hence

$$(5.7) \quad D_{-B_1/(-B)}(\mu) = \frac{\prod_{j=1}^{+\infty} (1 + \mu/r_j) e^{-\mu/r_j}}{\prod_{j=1}^m (1 + \mu/\lambda_j) e^{-\mu/\lambda_j}} e^{-i\mu \text{tr} B_2}.$$

Note that $\text{Im} \lambda_j \geq 0$ for each j since L is dissipative. So, for any $t > 0$, we have that $-it \notin \sigma(L)$ and hence $1/(it) \notin \sigma(-B)$. Putting $\mu = it$ with $t > 0$ in (5.7), noting that the r_j 's are all real and passing to the modulus, we get that

$$(5.8) \quad \frac{1}{t} \ln |D_{-B_1/(-B)}(it)| = \frac{1}{t} \ln \left| \prod_{j=1}^{+\infty} \left(1 + \frac{it}{r_j}\right) \right| - \frac{1}{t} \ln \left| \prod_{j=1}^m \left(1 + \frac{it}{\lambda_j}\right) \right| \\ + \text{tr} B_2 - \sum_{j=1}^m \text{Im} \frac{1}{\lambda_j}.$$

In virtue of Lemmas 4.34, 5.3 and 4.7, one has that

$$(5.9) \quad \lim_{t \rightarrow +\infty} \frac{1}{t} \ln |D_{-B_1/(-B)}(it)| = 0$$

and

$$(5.10) \quad \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \left| \prod_{j=1}^{+\infty} \left(1 + \frac{it}{r_j}\right) \right| \leq 0, \quad \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \left| \prod_{j=1}^m \left(1 + \frac{it}{\lambda_j}\right) \right| \leq 0.$$

Since $\text{Im } \lambda_j \geq 0$ for each j , we have the following estimates: for any $t > 0$ and each j ,

$$(5.11) \quad \left|1 + \frac{it}{\lambda_j}\right|^2 \geq \left(1 + \frac{t \text{Im } \lambda_j}{|\lambda_j|^2}\right)^2 \geq 1, \quad \left|1 + \frac{it}{r_j}\right|^2 = 1 + \frac{t^2}{|r_j|^2} \geq 1,$$

which imply that

$$(5.12) \quad \ln \left| \prod_{j=1}^{+\infty} \left(1 + \frac{it}{r_j}\right) \right| \geq 0, \quad \ln \left| \prod_{j=1}^m \left(1 + \frac{it}{\lambda_j}\right) \right| \geq 0.$$

From (5.10) and (5.12) one then deduces that

$$(5.13) \quad \lim_{t \rightarrow +\infty} \frac{1}{t} \ln \left| \prod_{j=1}^{+\infty} \left(1 + \frac{it}{r_j}\right) \right| = \lim_{t \rightarrow +\infty} \frac{1}{t} \ln \left| \prod_{j=1}^m \left(1 + \frac{it}{\lambda_j}\right) \right| = 0.$$

Now, taking the limit $t \rightarrow +\infty$ in (5.8) and making use of (5.9) and (5.13), we get that

$$(5.14) \quad \sum_{j=1}^m \text{Im} \frac{1}{\lambda_j} = \text{tr} B_2.$$

Therefore, by Lemma 5.1, the system of eigenfunctions and associated functions of $-B$ is complete in $L_w^2((a, b), \mathbb{C})$, and hence the same is true for L . ■

As a direct consequence of Theorem 2.30, we have the following fact.

COROLLARY 5.15. *If L is dissipative, then it has infinitely many eigenvalues.*

PROOF. Since each root lineal of L is finite dimensional, the completeness in $L_w^2((a, b), \mathbb{C})$ of the system of eigenfunctions and associated functions of L implies that L has infinitely many eigenvalues. ■

Finally, we mention that the above proof of Theorem 2.30 actually yields the following general result.

THEOREM 5.16. *If T is a compact dissipative operator on \mathcal{H} such that $\operatorname{tr} T_{\operatorname{Im}} < +\infty$,*

$$(5.17) \quad \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \left| \prod_{j=1}^{\nu(T)} (1 - it\mu_j(T)) \right| \leq 0, \quad \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \left| \prod_{j=1}^{\nu(T_{\operatorname{Re}})} (1 - it\mu_j(T_{\operatorname{Re}})) \right| \leq 0,$$

then

$$(5.18) \quad \sum_{j=1}^{\nu(T)} \operatorname{Im} \mu_j(T) = \operatorname{tr} T_{\operatorname{Im}},$$

and hence the system of root functions of T is complete in \mathcal{H} .

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