Main Tools/Classical Results
Properties of Digraph Spectra
Some digraph spectra
Tournaments and their Spectral Properties
Totally nonnegative (ordered) digraphs
Cospectral Digraphs
Energy of Digraphs
Laplacian Eigenvalues of Digraphs

Spectra of Digraphs

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LA’09, Northern Illinois University
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Dedicated to Biswa Datta on his recent birthday #68
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Digraph $D$: set $V$ of vertices, and set $E$ of edges which are ordered pairs of not necessarily distinct vertices. So loops are allowed.

An edge $(u, v)$ ($u \rightarrow v$) contributes 1 to the outdegree of $u$ and 1 to the indegree of $v$. A loop contributes 1 to both the indegree and outdegree of $u$.

So we have an outdegree vector $R = (r_1, r_2, \ldots, r_n)$ and indegree vector $S = (s_1, s_2, \ldots, s_n)$ where

$$r_1 + r_2 + \cdots + r_n = s_1 + s_2 + \cdots + s_n.$$
**Digraph** $D$: set $V$ of **vertices**, and set $E$ of **edges** which are ordered pairs of not necessarily distinct vertices. So loops are allowed.

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$$r_1 + r_2 + \cdots + r_n = s_1 + s_2 + \cdots + s_n.$$
Order the vertices in some way: $v_1, v_2, \ldots, v_n$. The adjacency matrix is the $(0, 1)$-matrix $A = [a_{ij}]$ of order $n$ where

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge from } v_i \text{ to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

A different ordering results in the (similar) matrix $PAP^T$ for some permutation matrix $P$.

In particular, the digraph is strongly connected iff the matrix $A$ is irreducible, i.e. no $P$ such that $PAP^T = \begin{bmatrix} A_1 & O_{r,n-r} \\ \ast & A_2 \end{bmatrix}$. 

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Example of a digraph $D$
Adjacency Matrix $A$ of $D$

$$
\begin{bmatrix}
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1 & 0 & 0 & 1 & 1 \\
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Digraphs of order $n \leftrightarrow (0,1)$-matrices of order $n$

$D_n \leftrightarrow A_n$

Outdegree vector of $D$ is the row sum vector $(r_1, r_2, \ldots, r_n)$ of $A$.
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Definitions

Characteristic polynomial \( \chi_D(x) \) of \( D \), minimum polynomial of \( D \), spectrum (eigenvalues) \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of \( D \), ... are those of its adjacency matrix \( A \).

Unlike for symmetric matrices, the eigenvalues of \( D \) need not be real numbers. Convention is:

\[
|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|.
\]

The spectral radius of \( D \) is \( \rho(D) = |\lambda_1| \).

Example \( \bar{K}_n \) (all possible edges) \( \longleftrightarrow \) \( J_n \) (all 1s matrix):
eigenvalues are \( n, 0, \ldots, 0 \).
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**Example** $K_n$ (all possible edges) $\leftrightarrow J_n$ (all 1s matrix): eigenvalues are $n, 0, \ldots, 0$. 
A digraph is **regular** provided the indegree and outdegree of each vertex is some constant \( d \). For such a \( D \), \( \rho(D) = d \).

**Theorem** (Hoffman/McAndrew 1965) There is a polynomial such that \( p(A) = J_n \) iff \( D \) is a strongly connected, regular digraph. For such a \( D \), the polynomial \( p(x) \) of smallest degree is unique and is

\[
H_D(x) = \frac{nq(x)}{q(d)}
\]

where \((x - d)q(x)\) is the minimum polynomial of \( D \). The integer \( d \) is the largest real solution of the equation \( H_D(x) = n \).
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where $(x - d)q(x)$ is the minimum polynomial of $D$. The integer $d$ is the largest real solution of the equation $H_D(x) = n$. 
• \( \rho(D) \) is an eigenvalue,

• If \( D \) is strongly connected, there is a (unique) positive eigenvector corresponding to \( \rho(D) \).

• \( \min\{r_1, r_2, \ldots, r_n\} \leq \rho(D) \leq \max\{r_1, r_2, \ldots, r_n\} \). Equality on the right iff equality on the left iff \( D \) is regular.

• If \( k \) is the GCD of cycle lengths of \( D \), then the spectrum of \( D \) is invariant under a rotation of the complex plane about the origin by the angle \( 2\pi/k \).
Some Definitions

- $\mathcal{D}_n(e) =$: all digraphs with $n$ vertices and $e$ edges ($e \leq n^2$).
- $\mathcal{D}(e) =$: all digraphs with $e$ edges, number of vertices not specified.
- $\mathcal{D}_n(e \uparrow) \subseteq \mathcal{D}_n(e)$ such that the vertices can be ordered $v_1, v_2, \ldots, v_n$ so that if $(v_p, v_q)$ is an edge, then $(v_i, v_j)$ is an edge for all $i$ and $j$ with $1 \leq i \leq p, 1 \leq j \leq q$.
- $\mathcal{D}_n(e \downarrow) \subseteq \mathcal{D}_n(e)$ such that the vertices can be ordered $w_1, w_2, \ldots, w_n$ so that if $(w_p, w_q)$ is an edge, then $(w_i, w_j)$ is an edge for all $i$ and $j$ with $1 \leq i \leq p$ and $q \leq j \leq n$.
- $\mathcal{D}(e \uparrow)$ and $\mathcal{D}(e \downarrow)$ defined similarly.
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- $\mathcal{D}(e \uparrow)$ and $\mathcal{D}(e \downarrow)$ defined similarly.
Example of $D_6(25 \uparrow)$ and $D_6(25 \downarrow)$

In terms of the adjacency matrix:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]
Theorem: The maximum (respectively, minimum) spectral radius among graphs in $\mathcal{D}(e)$ occurs among the graphs in $\mathcal{D}(e \uparrow)$ (respectively, $\mathcal{D}(e \downarrow)$).

Thus

$$\max\{\rho(D) : D \in \mathcal{D}(e)\} = \max\{\rho(D) : D \in \mathcal{D}(e \uparrow)\}$$

and

$$\min\{\rho(D) : D \in \mathcal{D}(e)\} = \min\{\rho(D) : D \in \mathcal{D}(e \downarrow)\}.$$ 

Similar conclusions hold with $\mathcal{D}_n(e \uparrow)$ in place of $\mathcal{D}(e \uparrow)$ and $\mathcal{D}_n(e \downarrow)$ in place of $\mathcal{D}(e \downarrow)$.

Proof uses Perron-Frobenius theory.
Theorem: The maximum (respectively, minimum) spectral radius among graphs in $\mathcal{D}(e)$ occurs among the graphs in $\mathcal{D}(e \uparrow)$ (respectively, $\mathcal{D}(e \downarrow)$).

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Proof uses Perron-Frobenius theory.
Schwarz’s Theorem 1964

**Theorem:** The maximum (respectively, minimum) spectral radius among graphs in \( \mathcal{D}(e) \) occurs among the graphs in \( \mathcal{D}(e \uparrow) \) (respectively, \( \mathcal{D}(e \downarrow) \)).

Thus

\[
\max \{ \rho(D) : D \in \mathcal{D}(e) \} = \max \{ \rho(D) : D \in \mathcal{D}(e \uparrow) \}
\]

and

\[
\min \{ \rho(D) : D \in \mathcal{D}(e) \} = \min \{ \rho(D) : D \in \mathcal{D}(e \downarrow) \}.
\]

Similar conclusions hold with \( \mathcal{D}_n(e \uparrow) \) in place of \( \mathcal{D}(e \uparrow) \) and \( \mathcal{D}_n(e \downarrow) \) in place of \( \mathcal{D}(e \downarrow) \).

Proof uses Perron-Frobenius theory.
Geršgorin’s Theorem

**Theorem:** Let $R^0 = (r_1^0, r_2^0, \ldots, r_n^0)$ and $S^0 = (s_1^0, s_2^0, \ldots, s_n^0)$ be the outdegree and indegree vectors, respectively, of the digraph $D^0$ obtained from $D$ by removing any loops. Then the spectrum of $D$ is contained in the region of the complex plane defined by the union $\Gamma(D)$ of the disks

$$\{z : |z - a_{ii}| \leq r_i^0\} \quad (i = 1, 2, \ldots, n).$$

Here $A = [a_{ij}]$ is the adjacency matrix of $D$. Since $A$ is a (0,1)-matrix, the disks have centers at $(0, 0)$ or $(1, 0)$ (thus only two are needed). If $A$ has no loops, then this is no better than what one gets from the Perron-Frobenius theory.
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Generalizations of Geršgorin’s Theorem

**Theorem:** (Same notation) The spectrum of $D$ is contained in the region of the complex plane determined by the union of the lemniscates defined by the cycles $\gamma$ of $D$:

$$B_\gamma(D) = \{z : \prod_{i \in \gamma} |z - a_{ii}| \leq \prod_{i \in \gamma} r_i^0\}.$$

In general,

$$\min_\gamma \left\{ \left( \prod_{i \in \gamma} r_i \right)^{1/|\gamma|} \right\} \leq \rho(D) \leq \max_\gamma \left\{ \left( \prod_{i \in \gamma} r_i \right)^{1/|\gamma|} \right\}.$$
**Theorem:** (Same notation) The spectrum of $D$ is contained in the region of the complex plane determined by the union of the lemniscates defined by the cycles $\gamma$ of $D$:

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In general,

$$\min \left\{ \left( \prod_{i \in \gamma} r_i \right)^{1/|\gamma|} \right\} \leq \rho(D) \leq \max \left\{ \left( \prod_{i \in \gamma} r_i \right)^{1/|\gamma|} \right\}.$$
max \( \rho(D) \) with \( m^2 \) or \( m^2 + 1 \) edges

(B+Hoffman, 1985)

- \( \max\{\rho(D) : D \in \mathcal{D}(m^2)\} = m \), with equality iff \( D = K_m \).

- \( \max\{\rho(D) : D \in \mathcal{D}(m^2 + 1)\} \), with equality iff apart from isolated vertices, (1) \( D \) is a complete digraph of order \( m \) with one additional edge; or, (2) \( m = 1 \) and the two edges of \( D \) join two distinct vertices in opposite directions; or, (3) \( m = 2 \) and apart from isolated vertices, \( D \) is obtained from the complete digraph of order 3 by removing a complete digraph of order 2.
max $\rho(D)$ with $m^2$ or $m^2 + 1$ edges

(B+Hoffman, 1985)

• $\max\{\rho(D) : D \in D(m^2)\} = m$, with equality iff $D = \overrightarrow{K}_m$.

• $\max\{\rho(D) : D \in D(m^2 + 1)\}$, with equality iff apart from isolated vertices, (1) $D$ is a complete digraph of order $m$ with one additional edge; or, (2) $m = 1$ and the two edges of $D$ join two distinct vertices in opposite directions; or, (3) $m = 2$ and apart from isolated vertices, $D$ is obtained from the complete digraph of order 3 by removing a complete digraph of order 2.
Friedland (1985)

- For $1 \leq l \leq 2m$,

$$\max\{\rho(D) : D \in \mathcal{D}(m^2 + l)\} \leq \frac{m + \sqrt{m^2 + 2l}}{2}.$$  

Equality if $l = 2m$ and, apart from isolated vertices, $D$ is obtained from $K_{m+1}$ by removing a loop at one vertex.

- $\max\{\rho(D) : D \in \mathcal{D}(m^2 + 2m - 3)\} \leq \frac{m-1 + \sqrt{m^2 + 6m - 7}}{2}$. For $m \geq 3$, equality holds if and only if $D$ is obtained from $K_{m+1}$ by removing a complete digraph $\mathrel{\rightarrow\leftarrow} K_2$ of order 2 (a zero matrix of order 2 in lower right).
Bounds for $\max \rho(D), 1$

Friedland (1985)

- For $1 \leq l \leq 2m$,

$$\max \{\rho(D) : D \in \mathcal{D}(m^2 + l)\} \leq \frac{m + \sqrt{m^2 + 2l}}{2}.$$  

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- $\max \{\rho(D) : D \in \mathcal{D}(m^2 + 2m - 3)\} \leq \frac{m-1+\sqrt{m^2+6m-7}}{2}$. For $m \geq 3$, equality holds if and only if $D$ is obtained from $\leftrightarrow K_{m+1}$ by removing a complete digraph $\leftrightarrow K_2$ of order 2 (a zero matrix of order 2 in lower right).
Friedland (1985)

- For \( l \geq 2 \), there exists a constant \( C_l \) such that if \( m \geq C_l \), a digraph \( D^* \in \mathcal{D}(m^2 + l) \) satisfying

\[
\rho(D^*) = \max\{\rho(D) : D \in \mathcal{D}(m^2 + l)\}
\]

is obtained from \( K_m \) by including a new vertex \( u \) and edges in both directions joining \( u \) and \( \lfloor l/2 \rfloor \) vertices of \( K_m \), and, if \( l \) is odd, a edge in either direction joining \( u \) and an additional vertex of \( K_m \).
min $\rho(D)$ with $e$ edges

- If $e \leq \binom{n}{2}$, then $\tilde{\rho}(n, e) = 0$: there is a digraph $D \in \mathcal{D}_n(e)$ such that every edge is of the form $(i, j)$ with $i > j$ (the adjacency matrix has 0s on and above the main diagonal).

- If 
  \[
  \binom{n}{2} < e \leq \binom{n+1}{2},
  \]
  then $\tilde{\rho}(n, e) = 1$: there is a digraph whose adjacency matrix has 0s above the main diagonal and at least one 1 on the main diagonal.

So assume that $e > \binom{n+1}{2}$. 

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\min \rho(D) \text{ with } e \text{ edges}

- If \(e \leq \binom{n}{2}\), then \(\tilde{\rho}(n, e) = 0\): there is a digraph \(D \in \mathcal{D}_n(e)\) such that every edge is of the form \((i, j)\) with \(i > j\) (the adjacency matrix has 0s on and above the main diagonal).

- If

\[
\binom{n}{2} < e \leq \binom{n + 1}{2},
\]

then \(\tilde{\rho}(n, e) = 1\): there is a digraph whose adjacency matrix has 0s above the main diagonal and at least one 1 on the main diagonal).

So assume that \(e > \binom{n+1}{2}\).
\[
\min \rho(D) \text{ with } e \text{ edges}
\]

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B+Solheid (1986)

Let \( n \geq 2 \) and \( 0 \leq \tau \leq \lfloor n/2 \rfloor \lceil n/2 \rceil \). Then

\[
\tilde{\rho}(n, n^2 - \tau) = n + \sqrt{n^2 - 4\tau}.
\]

If \( D \in \mathcal{D}_n(n^2 - \tau) \), then \( \rho(D) = \tilde{\rho}(n, n^2 - \tau) \) iff there are \( p \) and \( q \) with \( p + q = n \), such that \( D \) is obtained by taking the complete digraph \( \leftrightarrow K_n \), partitioning its vertices into sets \( U \) and \( W \) of cardinalities \( p \) and \( q \), respectively, and then removing any \( \tau \) edges from the vertices in \( U \) to the vertices in \( W \).
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Let \( n \geq 2 \) and \( 0 \leq \tau \leq \lfloor n/2 \rfloor \lceil n/2 \rceil \). Then

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If $n = 7$ and $\tau = 6$, an example with equality is the digraph with adjacency matrix

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
$$
In the remaining cases, \( \tilde{\rho}(n, n^2 - \tau) \) can be sandwiched between two consecutive integers: Let \( 1 \leq k \leq n \), and let \( n = qk + l \) where \( q \) is a positive integer and \( 0 \leq l < k \). Define

\[
\tau_{n,k} = \frac{q(q-1)}{2}k^2 + qkl.
\]

• Let \( 0 \leq \tau < \binom{n}{2} \). Let \( 1 \leq k \leq n - 1 \) be such that

\[
\tau_{n,k+1} \leq \tau < \tau_{n,k}.
\]

Then

\[
k < \tilde{\rho}(n, n^2 - \tau) \leq k + 1.
\]
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$$k < \tilde{\rho}(n, n^2 - \tau) \leq k + 1.$$
Adding edges to $\Delta_n$

$\Delta_n$ is the **transitive tournament** on $n$ vertices: there is an edge from $i$ to $j$ iff $n \geq i > j \geq 1$. For example, $\Delta_5$ has adjacency matrix

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 \\
\end{bmatrix}
$$

$\Delta_n$ has all eigenvalues equal to 0 and so spectral radius 0.

**Question:** (B+Hwang, 1996) How should one add $d$ new edges to maximize (minimize) the spectral radius?
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**Question**: (B+Hwang, 1996) How should one add $d$ new edges to maximize (minimize) the spectral radius?
If $d = 1$, the maximum spectral radius occurs only by putting the new edge from 1 to $n$:

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}
$$

In fact, if $d \leq n - 2$, in order to achieve the maximum spectral radius one must add the edge from 1 to $n$. 
Example

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1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
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$$

In fact, if $d \leq n - 2$, in order to achieve the maximum spectral radius one must add the edge from 1 to $n$. 
Theorem: Let $d$ be a positive integer. Then for $n$ sufficiently large, a digraph with maximal spectral radius obtained by adding $d$ new edges to $\Delta_n$ has the property that the new edges (1s) have an upper staircase pattern. For example, with $n = 7$ and $d = 8$, one possibility is:

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
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Theorem:

- For $1 \leq d \leq n(n - 1)/2$, the minimum spectral radius is attained by a matrix with a staircase pattern.

An example of such a matrix with a staircase pattern, with $n = 7$ and $d = 8$ is:

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\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
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1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]
Minimum Spectral Radius continued

Theorem continued:

• If $1 \leq d < n$, then the minimum spectral radius equals

$$\begin{cases} 
2 & \text{if } 1 \leq d \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
\frac{3+\sqrt{5}}{2} & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq d \leq \left\lfloor \frac{2n}{3} \right\rfloor, \\
3 & \text{if } \left\lfloor \frac{2n}{3} \right\rfloor + 1 \leq d \leq n - 1.
\end{cases}$$

• If $n \geq 3$ and $d = n$, the minimum spectral radius equals

$$\begin{cases} 
3 & \text{if } n \equiv 3 \mod 3, \\
2 + \sqrt{2} & \text{if } n \equiv 1 \text{ or } 2 \mod 3, \ n \neq 2, 5, \\
\frac{5+\sqrt{5}}{2} & \text{if } n = 5.
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  \]
There is an impressive series of three papers by L. Kolotilina (2005-06-06) that extends and generalizes many classical bounds for the spectral radius of a nonnegative matrix. When specialized to digraphs they give very interesting conclusions. For instance:

**Theorem** Let $D$ be a digraph of order $n$ with a positive outdegree vector $R = (r_1, r_2, \ldots, r_n)$. Then for each $\alpha$ with $0 \leq \alpha \leq 1$,

$$\min \left\{ r_i^\alpha r_j^{1-\alpha} : (v_i, v_j) \in E \right\} \leq \rho(D) \leq \max \left\{ r_i^\alpha r_j^{1-\alpha} : (v_i, v_j) \in E \right\}.$$

Conditions are given for equality to hold on each side.
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The **Manhattan street digraph** (view on a torus) $M_2(4, 4)$ is:

$$M_2(n_1, n_2)$$ is defined/drawn in a similar way. There is a more general $k$-dimensional version.

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![Manhattan Street Digraph](image)

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Theorem (Comellas, Dalfó, Fiol, 2008): The eigenvalues of $M_2(n_1, n_2)$ are

$$0, \pm \sqrt{2 \cos \left(\frac{4\pi k}{n_1}\right) + 2 \cos \left(\frac{4\pi l}{n_2}\right)} \quad (0 \leq k \leq \frac{n_1}{2} - 1, 0 \leq l \leq \frac{n_2}{2} - 1)$$

In addition, the geometric multiplicity of each nonzero eigenvalue equals its algebraic multiplicity, while the geometric multiplicity of the eigenvalue 0 is at least $(n_1 n_2)/2$, and equals $(n_1 n_2)/2$ if $n_i \not\equiv 0 \mod 4$ for $i = 1$ and 2.

There are some results for the $k$-dimensional Manhattan street digraph.
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There are some results for the $k$-dimensional Manhattan street digraph.
Wrapped butterfly digraphs have been studied for their application in network theory. Let $d$ and $n$ be positive integers. The \textit{wrapped butterfly digraph} $B_d(n)$ has vertices

$$\{(l; x) = (l; x_0, x_1, \ldots, x_{n-1}) : 0 \leq l \leq n - 1, 0 \leq x_i \leq d - 1\}.$$  

(The $l$ in a vertex is called its \textbf{level}.)

Edges are:

$$\{(l; x_0, x_1, \ldots, x_{n-1}) \rightarrow (l + 1; x_0, x_1, \ldots, x_{l-1}, \alpha, x_{l+1}, \ldots, x_{n-1})$$

for every integer $\alpha$ with $0 \leq \alpha \leq d - 1$. Here addition in the first component is modulo $n$ and in the other components modulo $d$ (thus the use of the word \textit{wrapped}).
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In general, $B_d(n)$ is a strongly connected digraph of order $nd^n$ and has diameter $2n - 1$; it is also regular of degree $d$. 
In general, $B_d(n)$ is a strongly connected digraph of order $nd^n$ and has diameter $2n - 1$; it is also regular of degree $d$. 
Theorem (Comellas, Fiol, Gimbert, Mitjana, 2008): The spectrum of the wrapped butterfly digraph $B_d(n)$ is

\[ 0 \quad [n(d^n - 1)], \quad d \quad [1], \quad d\omega^1 \quad [1], \quad d\omega^2 \quad [1], \ldots, \quad d\omega^{n-1} \quad [1] \]

where $\omega = e^{2\pi i/n}$ and the quantities in the brackets are the algebraic multiplicities.
A **tournament** $T$ is a digraph in which between each pair of distinct vertices there is exactly one edge (no loops).

The outdegree sequence of a tournament is usually called its **score sequence**. The adjacency matrix of a tournament is a **tournament matrix**.

Example: $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$ is a tournament matrix.
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is a tournament matrix.
Let $T$ denote a tournament with score sequence

$$R = (r_1, r_2, \ldots, r_n) \text{ where } r_1 \leq r_2 \leq \cdots \leq r_n.$$ 

Then

$$A + A^t = J_n - I_n \quad (J_n \text{ the all 1s matrix}).$$

This equation leads to special spectral properties of tournaments. For instance, the rank of a tournament matrix $A$ of order $n$ is at least $n - 1$ and so if 0 is an eigenvalue of $A$, then it is a simple eigenvalue.
Tournament Spectra

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This equation leads to special spectral properties of tournaments. For instance, the rank of a tournament matrix $A$ of order $n$ is at least $n - 1$ and so if 0 is an eigenvalue of $A$, then it is a simple eigenvalue.
Theorem Brauer and Gentry (Bull. AMS: 1968, LAA: 1972) The real part of each eigenvalue of a tournament $T$ of order $n$ is at least $-1/2$. Moreover,

$$\min\{(r_1 r_2 r_3)^{1/3}, (r_1 r_3)^{1/2}\} \leq \rho(T) \leq \frac{n - 1}{2}.$$

Equality occurs on the right if and only if $T$ is a regular tournament (and so $n$ must be odd).

Also, for each eigenvalue $\lambda$ of $T$,

$$|\text{Im} \lambda| \leq \frac{1}{2} \cot(\pi/2n).$$
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NIU LA’09, DeKalb, August 12–14, 2009
**Conjecture:** RAB and Li (Disc. Math: 1983) For \( n \) even, the maximum spectral radius \( \bar{\rho}_n \) of a tournament of order \( n \) equals the spectral radius of the nearly regular tournament with adjacency matrix

\[
\begin{bmatrix}
\frac{L_{n/2}}{L_{n/2}} & L^t_{n/2} + I_{n/2} \\
L^t_{n/2} & \frac{L_{n/2}}{L_{n/2}}
\end{bmatrix}, \text{ where } L_{n/2} = \\
\begin{bmatrix}
0 & 1 & 1 & 1 & \cdots & 1 \\
0 & 0 & 1 & 1 & \cdots & 1 \\
0 & 0 & 0 & 1 & \cdots & 1 \\
0 & 0 & 0 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix},
\]

is the adjacency matrix of the transitive tournament of order \( n/2 \).

These tournaments have been called Brualdi-Li tournaments.
Progress on the Conjecture for $\rho_n$, $n$ even

**Theorem:** Kirkland (LAMA: 1991, LAA: 1997, LAA:2003): If $T$ is a nearly regular tournament of order $n = 2m$, then

$$\rho(T) \geq \frac{m - 1}{2} - \sqrt{m^2 - 1}.$$

For every regular tournament of order $m$ with adjacency matrix $S$, the nearly regular tournament $T$ of order $n$ with adjacency matrix

$$\begin{bmatrix}
S & S^t \\
S^t + I_m & S
\end{bmatrix}$$

has this spectral radius.
Progress on the Conjecture for $\bar{\rho}_n$, $n$ even

(Kirkland continued)

If $n$ is even, then

$$
\bar{\rho}_n = \frac{n-1}{2} - \frac{\gamma_n}{n} + O\left(\frac{1}{n^2}\right), \text{ where}
$$

$$
0.377453 \ldots \approx \frac{2 \cdot 3^{2/3} - 3^{4/3} + 13}{34} \leq \gamma_n \leq \frac{e^2 - 1}{2(e^2 + 1)} \approx 0.380797 \ldots.
$$

Moreover, for $n$ sufficiently large a tournament of order $n$ with maximum spectral radius must be nearly regular.
**Conjecture:** RAB and Li (Disc. Math: 1983) Let $\tilde{\rho}_n$ denote the minimum spectral radius of a strongly connected tournament of order $n$. Then $\tilde{\rho}_n$ equals the spectral radius of the tournament $\tilde{T}_n$ with adjacency matrix

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 1 & 0 & \cdots & 0 \\
1 & 1 & 0 & 0 & 1 & \cdots & 0 \\
1 & 1 & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & 1 & 1 & \cdots & 0
\end{bmatrix}.
$$
Theorem: Kirkland (LAA: 1996) Let $T$ be a strongly connected tournament of order $n$. Then

$$\rho(T) \geq \rho(\tilde{T}_n),$$

with equality if and only if $T$ is isomorphic to $\tilde{T}_n$.

Remark: de Caen, Gregory, Kirkland, Pullman, and Maybee (LAA: 1997)

$$\rho(\tilde{T}_n) \rightarrow 2.4844353 \ldots.$$
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Remark: de Caen, Gregory, Kirkland, Pullman, and Maybee (LAA: 1997)

$$\rho(\tilde{T}_n) \rightarrow 2.4844353 \ldots$$
Consider the digraph $D$ with adjacency matrix

\[
A = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
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0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}.
\]

All square submatrices have a nonnegative determinant; thus $A$, respectively, $D$, is \textit{totally nonnegative}.

The eigenvalues are: 0, 0, 0, 0.5858, 2, 3.4142.
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1 & 1 & 1 & 1 & 1 & 0 \\
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Example

Consider the digraph $D$ with adjacency matrix

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\end{bmatrix}.$$

All square submatrices have a nonnegative determinant; thus $A$, respectively, $D$, is **totally nonnegative**.

The eigenvalues are: 0, 0, 0, 0.5858, 2, 3.4142.
Total nonnegativity of a digraph depends on the order in which the vertices are listed.

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
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is totally nonnegative, but interchanging rows 1 and 2 and interchanging columns 1 and 2 gives the matrix

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0 & 0 & 0 & 0
\end{bmatrix}
\]

which is not totally nonnegative.
A Second Example

Total nonnegativity of a digraph depends on the order in which the vertices are listed.

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
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A Third Example

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

has positive eigenvalues 1, 1, 1, but is not totally nonnegative. Total nonnegativity implies nonnegativity of all eigenvalues, but not conversely.
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**Theorem** (McKay et al, 2004) If all the eigenvalues of a digraph $D$ are positive, then the digraph has no cycles of length $> 1$.

Such a digraph has an adjacency matrix with all 1s on the main diagonal and all 0s above the main diagonal; thus all eigenvalues equal 1. This digraph need not be totally nonnegative: recall the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

**Corollary** An *irreducible* $(0, 1)$-matrix of order $n \geq 2$ with all eigenvalues nonnegative is singular (0 must be an eigenvalue).

**Question** raised: Investigate digraphs all of whose eigenvalues are real and nonnegative.

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**Theorem** Let $D$ be a digraph of order $n$ with $r$ positive eigenvalues and $n - r$ zero eigenvalues. Assume that $D$ has exactly $r$ loops. Then $D$ has no cycles of length greater than 1.

**Proof Outline:** $A$ the adjacency matrix of $D$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_n$. Then

$$1 = \frac{\text{trace}(A)}{r} = \frac{\lambda_1 + \cdots + \lambda_r}{r} \geq \left( \prod_{i=1}^{r} \lambda_i \right)^{1/r} \geq 1.$$ 

This implies that $\lambda_1 = \cdots = \lambda_r = 1$. Now use the P-F theory to conclude that $A$ has $r$ irreducible components equal to $I_1 = [1]$ and $n - r$ irreducible components equal to $O_1 = [0]$. 
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Remark

It is not hard to show that if \( A \) has \( n - 1 \) positive and one zero eigenvalue, then the trace of \( A \) equals \( n - 1 \) or \( n \). The preceding theorem takes care of the case of trace equal to \( n - 1 \). If trace of \( A \) equals \( n \), \( A \) need not be triangularizable. For example,

\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ (trace } = 2, \text{ eigenvalues } 0, 2), \text{ and}
\]

\[
A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ (trace } = 3, \text{ and eigenvalues } 0, 0.3820, 2.6180). 
\]
Theorem (RAB and Kirkland, 2009): An $m$ by $n$ (0,1)-matrix is totally nonnegative iff it has no submatrix equal to one of

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

More on totally nonnegative digraphs ((0,1)-matrices) in a forthcoming paper by RAB and Kirkland. E.g. An irreducible totally nonnegative (0,1)-matrix of order $n$ has 0 as an eigenvalue of multiplicity at least $\lceil n/2 \rceil$. 
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Digraphs $D_1$ and $D_2$ are **cospectral** provided they are not isomorphic but have the same spectrum (may be complex).

**Example** (Krishnamurthy and Parthasarathy, 1974, 1975): Let $D_i$ be the digraph obtained from a directed cycle $(v_1, v_2, \ldots, v_{2k}, v_1)$ of length $2k$ by adding two new vertices, $u$ and $w$, and inserting edges $(v_1, u), (u, v_1), (w, v_i), (v_i, w)$. Then for $2 \leq i \leq k + 1$, the resulting strongly connected digraphs of order $n = 2k + 2$ are cospectral, indeed have characteristic polynomial equal to

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\lambda^{2k+2} - 2\lambda^{2k} + \lambda^{2k-2} - \lambda^2.
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Note: It is easy to find non-strongly connected examples.
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Three cospectral, strongly connected digraphs of this type.
Gutman (1978) defined the energy of a graph to be the sum of the absolute values of its eigenvalues.

Nikiforov (2007) defined the energy of a matrix $A$ to be the sum of the singular values of $A$ (the positive square roots of the eigenvalues of the $p$-s-$d$ symmetric matrix $AA^T$):

$$E(A) = \sigma_1 + \sigma_2 + \cdots,$$

and gave a general upper bound for $E(A)$ which for digraphs with $q$ edges becomes:

$$E(D) \leq \frac{q}{n} + \sqrt{(n - 1) \left(q - \frac{q^2}{n^2}\right)}$$

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e(D) = \sum_{i=1}^{n} |\text{Re}(\lambda_i)|.
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Coulson’s integral formula for energy of graphs is extended to low energy of digraphs and the McClelland identity is extended:

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Chung (2005) introduced **Laplacians** of (strongly connected) digraphs $D$. Outdegrees: $r_1, r_2, \ldots, r_n$; Indegrees: $s_1, s_2, \ldots, s_n$. Let $P = [p_{ij}]$ be the matrix of order $n$ defined by

$$p_{ij} = \begin{cases} \frac{1}{r_i} & \text{if } (v_i, v_j) \text{ is an edge}, \\ 0 & \text{otherwise}. \end{cases}$$

$P$ is the (irreducible) transition matrix of a random walk on $D$ (all row sums are 1 and $\rho(P) = 1$), and $P$ has a unique, normalized left eigenvector $\phi$ for 1: $\phi P = \phi, \sum_{i=1}^{n} \phi_i = 1$.

$$L(D) = I_n - \frac{\phi^{1/2} P \Phi^{-1/2} + \Phi^{-1/2} P^T \phi^{1/2}}{2},$$

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If $D$ is a symmetric digraph (a graph), then $\phi = \frac{1}{d}(r_1, r_2, \ldots, r_n)$ where $d = \sum_{i=1}^{n} r_i$. $\mathcal{L}(D)$ is the symmetric matrix $I_n - XAX$ where $A$ is the adjacency matrix of $D$ and

$$X = \text{diag} \left( \frac{1}{\sqrt{r_1}}, \frac{1}{\sqrt{r_2}}, \ldots, \frac{1}{\sqrt{r_n}} \right).$$

Thus, the Laplacian of a symmetric digraph is the so-called normalized Laplacian. $\mathcal{L}(D)$ is a singular, positive semidefinite symmetric matrix with eigenvalues $\lambda_0 = 0 \leq \lambda_1 \leq \cdots \leq \lambda_n$, called the Laplacian eigenvalues or Laplacian spectrum of $D$. 
Theorem (Chung 2006): The diameter of $D$ is at most

$$\left\lfloor \frac{2 \min \left\{ \log \left( \frac{1}{\phi_i} \right) : 1 \leq i \leq n \right\}}{\log \frac{2}{2-\lambda_1}} \right\rfloor + 1,$$

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