

Can Any Reduced Order Model Be Obtained Via Projection?

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NO !

Alternative title:

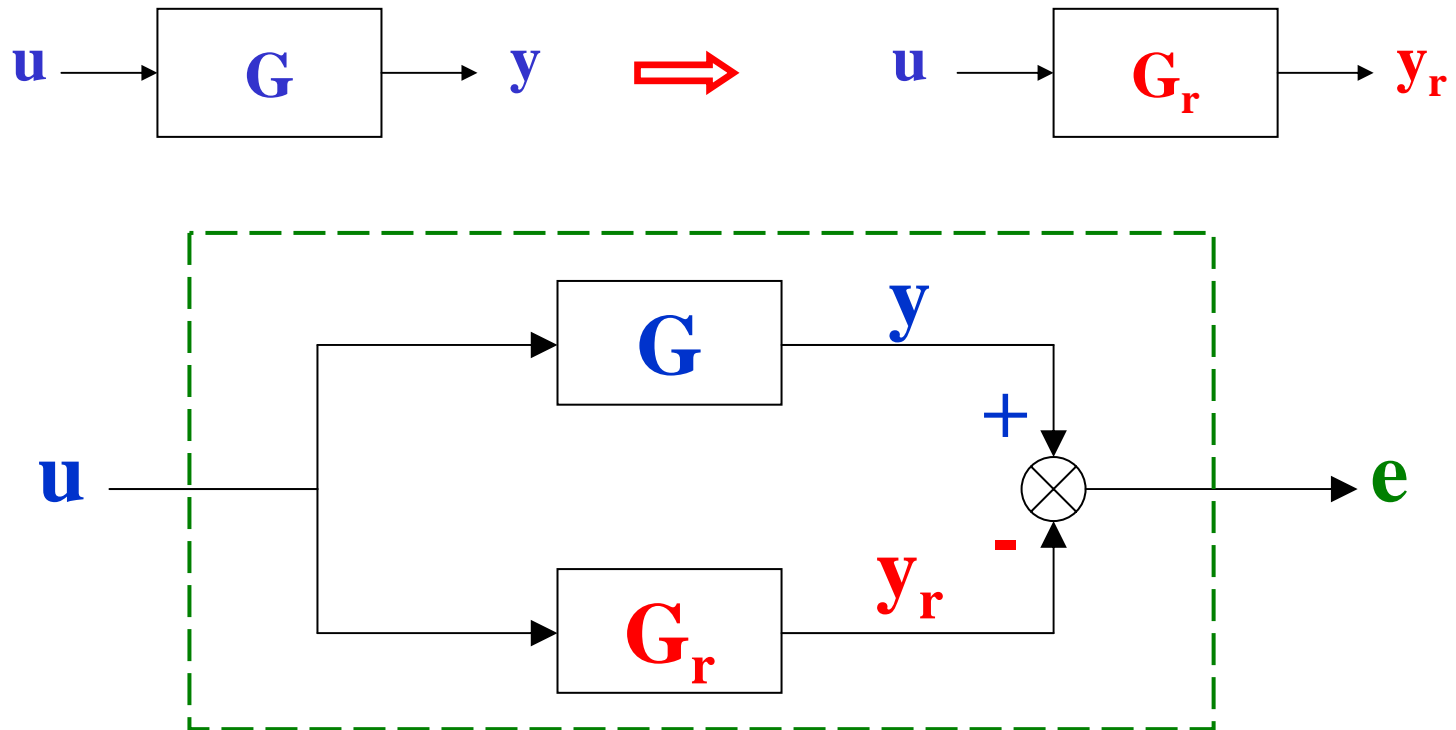
On Projection Based Model Order Reduction

LAYOUT

- Order reduction problems
- Projection based Reduced Order Models (PROM's)
- The inverse problem – from models to projection
 - General results
 - Square systems
- Projection properties of the optimal L_2 model.
- Summary

The Model Order Reduction Problem

Given a high order linear, time invariant system (model) \mathbf{G} find a reduced order model \mathbf{G}_r which is a 'good approximation' of it.



- 'good approximation' = small $\|\mathbf{e}\| = \text{small } \|\mathbf{G} - \mathbf{G}_r\|$

The Model Order Reduction Problem

Given an n -th order (McMillan degree) system $G(s)$, find an r -th order model ($r < n$) $G_r(s)$, which is a good approximation of $G(s)$.

$$\begin{aligned} \dot{x}(t) &= Ax_r(t) + Bu(t) & \Rightarrow & \quad \dot{x}_r(t) = A_r x_r(t) + B_r u(t) \\ y(t) &= Cx(t) + Du(t) & & \quad y_r(t) = C_r x(t) + D_r u(t) \end{aligned}$$

$$u \in R^m, y \in R^p, x \in R^n, x_r \in R^r$$

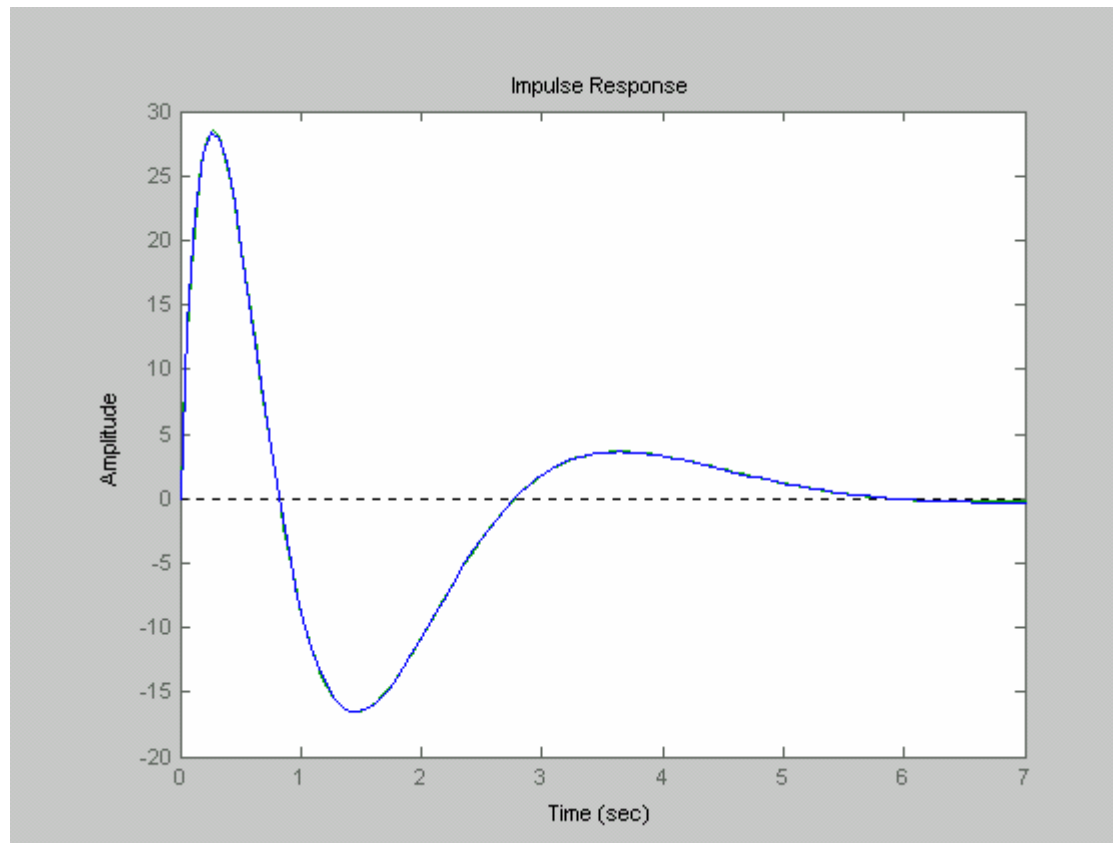
$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \Rightarrow \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array} \right]$$

- Optimization methods - L_2 , Hankel norm.
- Non-optimization methods - modal truncation, truncated balanced realization, moment matching, Routh array methods.
- Useless methods

Does it work?

$$G(s) = \frac{(2s+1)^8}{(s+1)^{10}}$$

$$G_r(s) = \frac{0.15946 (s+1586) (s^2 + 0.2705s + 0.05403)}{(s^2 + 1.674s + 1.703) (s^2 + 4.482s + 6.47)}$$



Projection based Reduced Order Models (PROM's)

Consider the $p \times m$, n -th order system

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

Many order reduction methods lead to the following r -th order model

$$G_r(s) = \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array} \right] = \left[\begin{array}{c|c} LAR & LB \\ \hline CR & D \end{array} \right]$$

Where $L \in \mathbb{R}^{r \times n}$, $R \in \mathbb{R}^{n \times r}$ and $LR = I_r$.

Define $P = RL$, then $P^2 = P$, hence P is a projection matrix

Transformation based methods

Step 1: state transformation $\mathbf{x} = \mathbf{T}\mathbf{x}'$, $\mathbf{T} = \begin{bmatrix} \mathbf{R} & \bar{\mathbf{R}} \end{bmatrix}$, $\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{L} \\ \bar{\mathbf{L}} \end{bmatrix}$

$$\begin{bmatrix} \dot{\mathbf{x}}'_1(t) \\ \dot{\mathbf{x}}'_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{L}\mathbf{A}\mathbf{R} & \mathbf{L}\mathbf{A}\bar{\mathbf{R}} \\ \bar{\mathbf{L}}\mathbf{A}\mathbf{R} & \bar{\mathbf{L}}\mathbf{A}\bar{\mathbf{R}} \end{bmatrix} \begin{bmatrix} \mathbf{x}'_1(t) \\ \mathbf{x}'_2(t) \end{bmatrix} + \begin{bmatrix} \mathbf{L}\mathbf{B} \\ \bar{\mathbf{L}}\mathbf{B} \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{C}\mathbf{R} & \mathbf{C}\bar{\mathbf{R}} \end{bmatrix} \mathbf{x}'(t) + \mathbf{D}\mathbf{u}(t)$$

If x_1 is “more important”, we can assume that $x_2 \cong 0$. Then

Step 2: truncation

$$\dot{\mathbf{x}}_r(t) = \mathbf{L}\mathbf{A}\mathbf{R}\mathbf{x}_r(t) + \mathbf{L}\mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}_r(t) = \mathbf{C}\mathbf{R}\mathbf{x}_r(t) + \mathbf{D}\mathbf{u}(t)$$

Transformation + Truncation = Projection

Other sources of projection – Moment matching methods, Krylov spaces, L_2 optimization, etc.

Examples of PROM's

Partial Fraction Expansion (Modal Truncation)

$$G(s) = \sum_{k=1}^n \frac{R_k}{s - p_k} \quad \Rightarrow \quad G_r(s) = \sum_{k=1}^r \frac{R_k}{s - p_k}$$

In state space - truncation of a *diagonal realization*.

Truncated Balanced Realization (TBR)-Moore, 1981

The state transformation (Step 1) leads to a realization where the *controllability and observability gramians are diagonal and equal*.

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Properties of PROM's

What is preserved by projection order reduction?

Practically Nothing!

Stability (or instability) is not preserved

Zeros structure at infinity (relative degree) is not preserved

Minimality (or non-minimality) is not preserved. The McMillan degree may increase (in non-minimal systems)!

Invariance results:

Assuming minimality, if there exists a realization of $G_r(s)$ which is a PROM of a certain realization of $G(s)$ then *any* realization of $G_r(s)$ is a PROM of *any* realization of $G(s)$, i.e. the problem is realization independent.

Questions

- Given $G(s)$, and $G_r(s)$, is it always possible to find a **real** projection that relates the two models ?
- Is that projection unique ?
- How can one calculate that projection ?
- Do optimal reduced order models have special PROM properties?

Q: Why is it important?

A1: It is not.

A2: Will be discussed later.

The Inverse Problem

Given (A, B, C) and (A_r, B_r, C_r) , find the projection P (equivalently L and R) that relates them.

$$LR=I_r \quad (r^2 \text{ equations})$$

$$LAR=A_r \quad (r^2 \text{ equations})$$

$$LB=B_r \quad (r \cdot m \text{ equations})$$

$$CR=C_r \quad (r \cdot p \text{ equations})$$

$(2r+m+p)r$ equations in $2nr$ unknowns (R, L).

The number of equations equals the number of unknowns when

$$r = r^* = n - (m+p)/2$$

Seemingly, the situation is as follows.

If $r < r^*$ then any $G_r(s)$ is a PROM via infinitely many projections.

If $r = r^*$ then not every $G_r(s)$ is a PROM. Those who are, can be obtained by a finite number of projections.

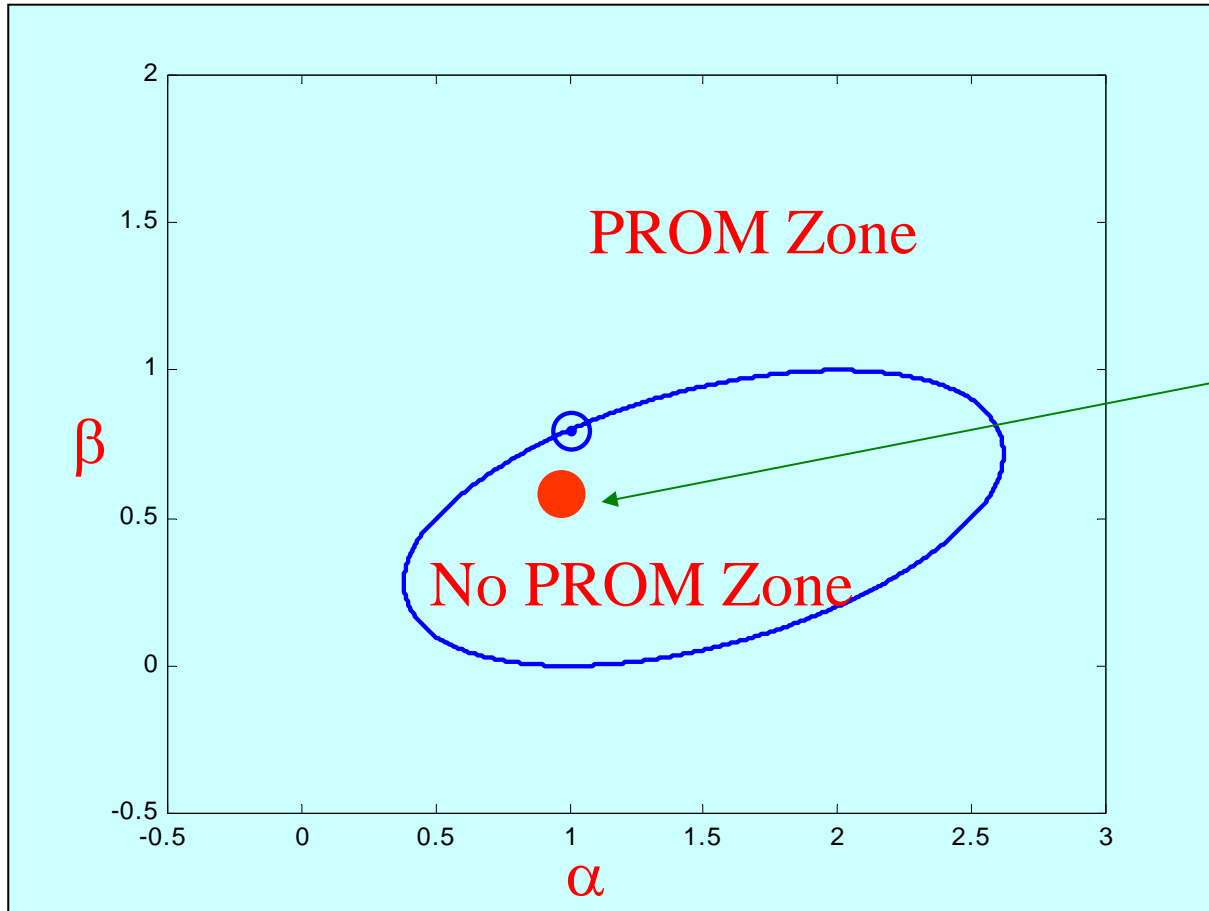
If $r > r^*$ then the class of $G_r(s)$ that are PROM has measure zero.

Q: Is it really so?

A: Only for square systems.

Case 2 ($r=r^*$): $n=2, r=1$, SISO

$$G(s) = \frac{s+1}{s^2+3s+1} \Rightarrow G_r(s) = \frac{\beta}{s+\alpha}$$



$$W(s) = \frac{s^2 + s + 1}{s^2 + 0.1s + 1}$$

$$\min \|W(s)(G - G_r)\|_2$$

Projection Calculation

Step 1: Use the *linear equations* to reduce the number of unknowns

$$LB = B_r \Rightarrow L = B_r B^+ + X B_{\perp} \quad , \quad CR = C_r \Rightarrow R = C^+ C_r + C_{\perp} Y$$

$X \in \mathbb{R}^{r \times (n-m)}$ and $Y \in \mathbb{R}^{(n-p) \times r}$ are the new unknown matrices.

Step 2: Substituting into the bilinear equations ($LR=I$ and $LAR=A_r$)

$$\tilde{X} H_i \tilde{Y} = 0 \quad i = 1, 2$$

where

$$\tilde{X} = [X \quad I_r] \quad , \quad \tilde{Y} = [Y^T \quad I_r]^T$$

$$H_1 = \begin{bmatrix} B_{\perp} C_{\perp} & B_{\perp} C^+ C_r \\ B_r B^+ C_{\perp} & B_r B^+ C^+ C_r - I_r \end{bmatrix} \quad H_2 = \begin{bmatrix} B_{\perp} A C_{\perp} & B_{\perp} A C^+ C_r \\ B_r B^+ A C_{\perp} & B_r B^+ A C^+ C_r - A_r \end{bmatrix}$$

Step 3: Replace the two equations by

$$\tilde{X} (\lambda H_1 - H_2) \tilde{Y} = 0 \quad \forall \lambda \in \mathbb{C}$$

Kronecker Canonical Form (KCF)

For every $h \times q$ pencil $\lambda E - A$ there exist square and nonsingular S and V such that

$$F(\lambda) = S(\lambda E - A)V = \begin{bmatrix} \text{blockdiag}\{F_i(\lambda)\} & 0 \\ 0 & 0 \end{bmatrix}$$

where the linear pencils $F_i(\lambda)$, $i=1, \dots, K$, which are unique, assume one of four possible structures

$$\text{type1} = \begin{bmatrix} \lambda & -1 & & \\ & \ddots & \ddots & \\ & & \lambda & -1 \end{bmatrix}, \text{type2} = \begin{bmatrix} \lambda & & & \\ -1 & \ddots & & \\ & \ddots & \lambda & \\ & & & -1 \end{bmatrix}, \text{type3} = \bar{J}\lambda - I, \text{type4} = \lambda I - \bar{F}$$

\bar{J} is nilpotent Jordan matrix and \bar{F} is in Jordan form.

Typical KCF of a non-square pencil

$$F(\lambda) = \left[\begin{array}{cccc|ccc} \lambda & -1 & 0 & 0 & & & \\ 0 & \lambda & -1 & 0 & & & \\ 0 & 0 & \lambda & -1 & & & \\ \hline & & & & \lambda & -1 & 0 \\ & & & & 0 & \lambda & -1 \\ \hline & & & & & & \lambda & -1 & 0 \\ & & & & & & 0 & \lambda & -1 \end{array} \right]$$

The sparse form of $F(\lambda)$ is the key to the subsequent derivations.

Projection Calculation (cont.)

Step 4: Transform to KCF

$$\begin{aligned}
 0 &= \tilde{X}(\lambda H_1 - H_2)\tilde{Y} \\
 &= [\tilde{X}S^{-1}] \cdot [S(\lambda H_1 - H_2)V] \cdot [V^{-1}\tilde{Y}] \\
 &= \tilde{\tilde{X}} \cdot F(\lambda) \cdot \tilde{\tilde{Y}}
 \end{aligned}$$

Step 5: Finally solving. Suppose there exists an $r \times r$ sub-pencil of zeros, then $\tilde{\tilde{X}}, \tilde{\tilde{Y}}$ are selection matrices.

$$F(\lambda) = \begin{bmatrix}
 \lambda - a_1 & 1 & 0 & 0 & 0 & 0 \\
 \boxed{0} & \lambda - a_1 & \boxed{0} & \boxed{0} & 0 & 0 \\
 0 & 0 & \lambda - a_2 & 1 & 0 & 0 \\
 \boxed{0} & 0 & 0 & \lambda - a_2 & 0 & 0 \\
 \boxed{0} & 0 & \boxed{0} & \boxed{0} & \lambda - a_3 & 1 \\
 \boxed{0} & 0 & \boxed{0} & 0 & 0 & \lambda - a_3
 \end{bmatrix}$$

$$\tilde{\tilde{X}} = \begin{bmatrix}
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
 \end{bmatrix}$$

$$\tilde{\tilde{Y}} = \begin{bmatrix}
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1 \\
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0
 \end{bmatrix}$$

$$\tilde{\tilde{X}} = \begin{bmatrix}
 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
 \end{bmatrix}$$

$$\tilde{\tilde{Y}} = \begin{bmatrix}
 1 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1 \\
 0 & 0 & 0 \\
 0 & 0 & 0
 \end{bmatrix}$$

Projection Calculation (cont.)

Step 6: Go back from $\tilde{\tilde{X}}, \tilde{\tilde{Y}}$ to L, R

$$\tilde{\tilde{X}} \rightarrow \tilde{X} \quad \tilde{X} = \tilde{\tilde{X}}S = [S_1 \quad S_2]$$

$$\tilde{X} \rightarrow X \quad \tilde{X} = c[X \quad I_r] \Rightarrow X = S_2^{-1}S_1,$$

$$X \rightarrow L \quad L = B^+B_r + S_2^{-1}S_1B_{\perp}$$

$$\tilde{\tilde{Y}} \rightarrow \dots \rightarrow R$$

An $r \times r$ zeros sub-matrix
in the KCF of $\lambda H_1 - H_2$



A projection that relates
(A, B, C) and (A_r, B_r, C_r)

Summary of the Algorithm

For given (A,B,C) and (A_r,B_r,C_r)

- If An $r \times r$ zero sub-matrix in the KCF *cannot* be found then (A_r,B_r,C_r) is *not* a PROM of (A,B,C) .
- If $r \times r$ is the *maximum* dimension possible then the number of zero sub-matrices is equal to the number of projections.
- If a *larger* sub-matrix can be found, (A,B,C) and (A_r,B_r,C_r) are related by infinitely many projections.

Square Systems ($m=p$)

- The pencil is square with dimension $m-m+r$
- The GKS consists of one *type4* block, or simply

$$F(\lambda) = \lambda I - \bar{F}$$

Alternatively, this is a generalized eigenvalue problem $\lambda H_1 - H_2$

Main results:

- If $r < n-m$ ($r < r^*$), then every (A_r, B_r, C_r) is a PROM, related to by infinitely many projections.
- If $r > n-m$, then generically every (A_r, B_r, C_r) is NOT a PROM
- If $r = n-m$ ($r = r^*$), then
 - a) The number of projections is finite.
 - b) If r is odd, and all the eigenvalues are complex, then there is no projection.

Example: A system with $n=5$, $m=2 \rightarrow r^*=3$

Case 1: $r=2$

$$F(\lambda) = \begin{bmatrix} \lambda - \sigma_1 & -\omega_1 & 0 & 0 & 0 \\ \omega_1 & \lambda - \sigma_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda - \sigma_2 & -\omega_2 & 0 \\ 0 & 0 & \omega_2 & \lambda - \sigma_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda - p_3 \end{bmatrix}$$

Case 3: $r=4$

$$F(\lambda) = \begin{bmatrix} \lambda - a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda - a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda - a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda - a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda - a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda - a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda - a_7 \end{bmatrix}$$

It is impossible to find a 4×4 block of zeros in the generic case

Example: A system with $n=5$, $m=2 \rightarrow r^*=3$

Case 1: $r=2$

$$F(\lambda) = \begin{bmatrix} \lambda - \sigma_1 & -\omega_1 & 0 & 0 & 0 \\ \omega_1 & \lambda - \sigma_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda - \sigma_2 & -\omega_2 & 0 \\ 0 & 0 & \omega_2 & \lambda - \sigma_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda - p_3 \end{bmatrix}$$

Case 3: $r=4$

$$F(\lambda) = \begin{bmatrix} \lambda & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda - a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda - a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda - a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda - a_7 \end{bmatrix}$$

4th order models must have a non-generic KCF

Case 2: $r=3$

$$F(\lambda) = \begin{bmatrix} \lambda - a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda - a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda - a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda - a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda - a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda - a_6 \end{bmatrix} \quad \frac{6!}{3!3!} = 20 \text{ solutions}$$

$$F(\lambda) = \begin{bmatrix} \lambda - a_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda - a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda - a_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda - a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda - a_3 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda - a_3 \end{bmatrix} \quad 4 \text{ solutions}$$

$$F(\lambda) = \begin{bmatrix} \lambda - \sigma_1 & \omega_1 & 0 & 0 & 0 & 0 \\ -\omega_1 & \lambda - \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda - \sigma_2 & \omega_2 & 0 & 0 \\ 0 & 0 & -\omega_2 & \lambda - \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda - \sigma_3 & \omega_3 \\ 0 & 0 & 0 & 0 & -\omega_3 & \lambda - \sigma_3 \end{bmatrix} \quad 0 \text{ solutions}$$

A Different Interpretation

Lemma (Vandendorpe and Van Dooren, 2002): The generalized eigenvalues of (H_2, H_1) are the **zeros** of $G(s) - G_r(s)$.

Corollaries:

- A reduced model cannot be obtained via projection if $r = r^*$ is odd and all the zeros of $G(s) - G_r(s)$ are complex.

Example: Is $G_r(s) = \frac{2}{s+1}$ a PROM of $G(s) = \frac{s+1}{s^2+s+1}$?

$$\frac{s+1}{s^2+s+1} - \frac{2}{s+1} = -\frac{s^2+2s+3}{(s^2+s+1)(s+1)} = -\frac{s+1 \pm j\sqrt{2}}{(s^2+s+1)(s+1)}$$

- $G_r(s)$ is a PROM of $G(s)$ with $r > r^*$ if and only if every value of s is a zero, i.e. $\det(G(s) - G_r(s)) \equiv 0$.

The Optimal L_2 Reduced Order Model (OROM)

Problem: Find $G_r(s)$ that minimizes

$$J = \|G(s) - G_r(s)\|_2$$

Method of solution

$$\frac{\partial J}{\partial A_r} = 0 \quad , \quad \frac{\partial J}{\partial B_r} = 0 \quad , \quad \frac{\partial J}{\partial C_r} = 0$$

Theorem: (Wilson, 1970; Hyland and Bernstein, 1985):

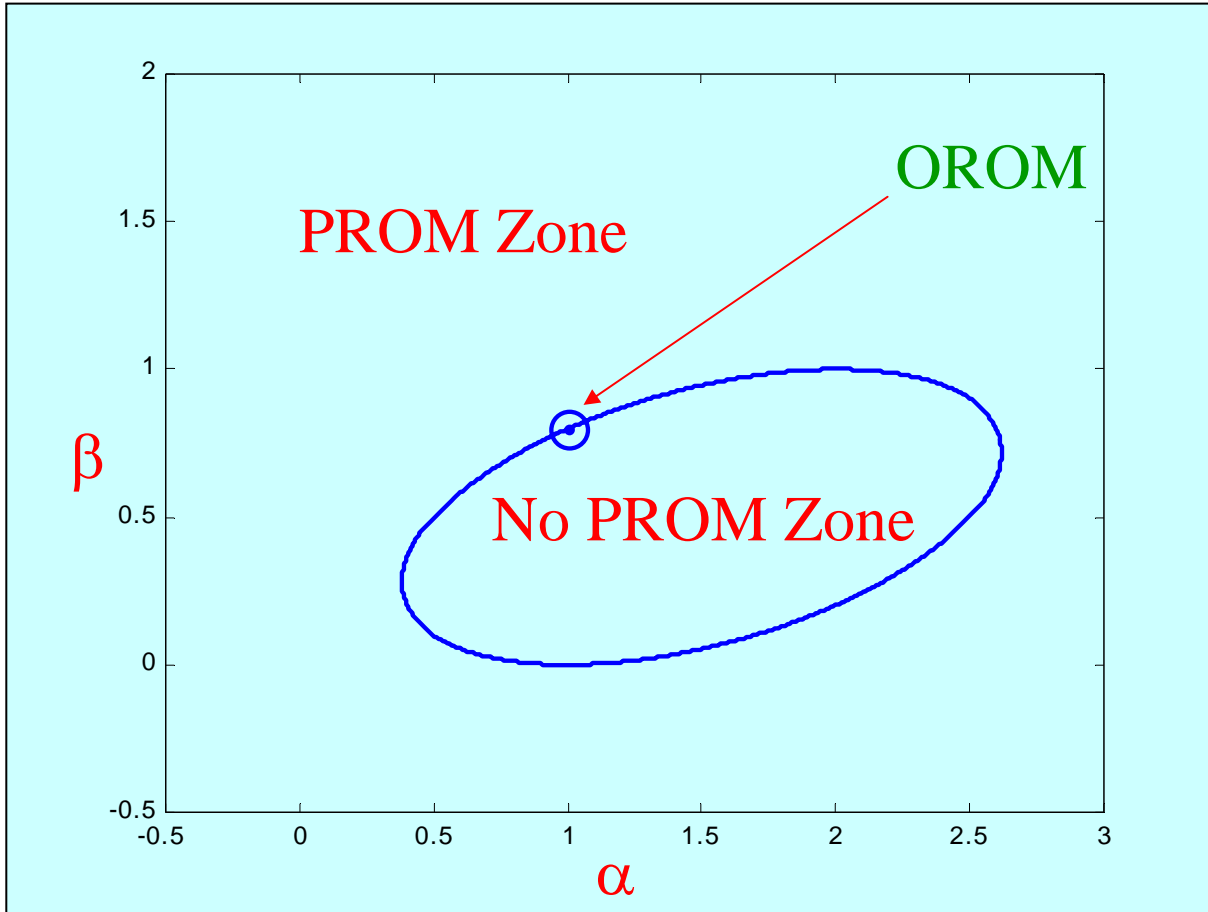
The optimal $G_r(s)$ is

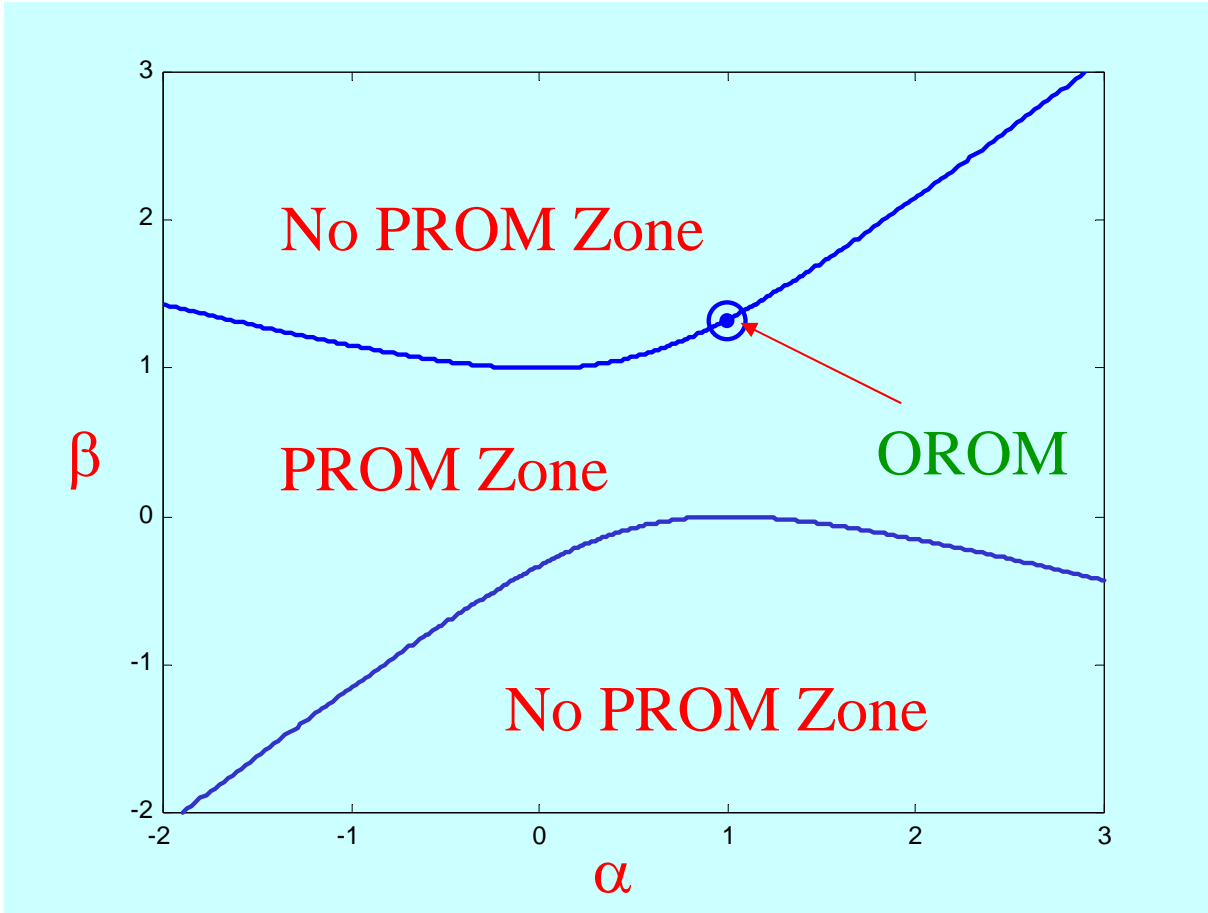
$$\dot{x}_r = L^* A R^* x_r + L^* B u$$

$$y_r(t) = C R^* x_r(t)$$

where L^* , R^* are given as ...

The L_2 optimal reduced order model is a PROM !





Main Result: Let p be a pole of $G_r^*(s)$ with multiplicity N_p . Then

- a) If the system is square, $-p$ is a zero of $G(s)-G_r^*(s)$ with multiplicity $2N_p$.
- b) If the system is non-square, $-p$ is a zero of $G(s)-G_r^*(s)$ with multiplicity N_p .
- c) Under certain conditions $G_r^*(s)$ is on the boundary of the PROM zone.

Old result (Meier and Luenberger, 1967) : In SISO systems, if p is a pole of $G_r^*(s)$ with multiplicity N_p . Then $-p$ is a zero of $G(s)-G_r^*(s)$ with multiplicity N_p+1 .

Examples:

$$G(s) = \frac{1}{(s+1)^2}, \quad G_r^*(s) = \frac{\frac{3}{8}}{s + \frac{1}{3}}, \quad E^*(s) = \frac{-\frac{3}{8}(s - \frac{1}{3})^2}{(s+1)^2(s + \frac{1}{3})}$$

$$G(s) = \frac{6.585s^2 + 46.098s + 96.439}{(s+1)(s+2)(s+3)(s+4)}, \quad G_r^*(s) = \frac{0.01018s + 6.301}{(s+1.254)^2}$$

$$E^*(s) = \frac{-0.01018(s-12.939)(s-1.254)^4}{(s+1)(s+2)(s+3)(s+4)(s+1.254)^2}$$

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{1}{s+3} \end{bmatrix}, \quad G_r^*(s) = \begin{bmatrix} \frac{0.338}{s+1.794} & \frac{0.748}{s+1.794} \end{bmatrix},$$

$$E^*(s) = \begin{bmatrix} \frac{-0.338(s-1.794)(s+1.84)}{(s+1)(s+2)(s+1.794)} & \frac{0.252(s-1.794)}{(s+3)(s+1.794)} \end{bmatrix}$$

Summary

- The existence and uniqueness properties of PROM's have been discussed.
- When viewed as a PROM, the Optimal L_2 reduced order model possesses some unique properties.