Can Any Reduced Order Model Be Obtained Via Projection?

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NO!

Alternative title:

*On Projection Based Model Order Reduction*
ORDER REDUCTION PROBLEMS

- Projection based Reduced Order Models (PROM’s)
- The inverse problem – from models to projection
  - General results
  - Square systems
- Projection properties of the optimal $L_2$ model.
- Summary
The Model Order Reduction Problem

Given a high order linear, time invariant system (model) \( G \) find a reduced order model \( G_r \) which is a 'good approximation' of it. 

\[ u \rightarrow G \rightarrow y \quad \text{\( \Rightarrow \)} \quad u \rightarrow G_r \rightarrow y_r \]

\[ u \rightarrow G \quad \text{\( \downarrow \)} \quad G \rightarrow y \quad + \quad G_r \rightarrow y_r \quad \text{\( \downarrow \)} \quad e \]

- 'good approximation' = small \( \| e \| \) = small \( \| G - G_r \| \)
The Model Order Reduction Problem

Given an $n$-th order (McMillan degree) system $G(s)$, find an $r$-th order model ($r < n$) $G_r(s)$, which is a good approximation of $G(s)$.

\[
\begin{align*}
\dot{x}(t) &= Ax_r(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\Rightarrow
\begin{align*}
\dot{x}_r(t) &= A_r x_r(t) + B_r u(t) \\
y_r(t) &= C_r x(t) + D_r u(t)
\end{align*}
\]

$u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $x \in \mathbb{R}^n$, $x_r \in \mathbb{R}^r$

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}
\]

- Optimization methods - $L_2$, Hankel norm.
- Non-optimization methods - modal truncation, truncated balanced realization, moment matching, Routh array methods.
- Useless methods
Does it work?

\[ G(s) = \frac{(2s + 1)^8}{(s + 1)^{10}} \]

\[ G_r(s) = \frac{0.15946 (s+1586) (s^2 + 0.2705s + 0.05403)}{(s^2 + 1.674s + 1.703) (s^2 + 4.482s + 6.47)} \]
Projection based Reduced Order Models (PROM’s)

Consider the $p \times m$, $n$-th order system

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Many order reduction methods lead to the following $r$-th order model

$$G_r(s) = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} = \begin{bmatrix} LAR & LB \\ CR & D \end{bmatrix}$$

Where $L \in \mathbb{R}^{r \times n}$, $R \in \mathbb{R}^{n \times r}$ and $LR=I_r$.

Define $P=RL$, then $P^2=P$, hence $P$ is a projection matrix.
Transformation based methods

**Step 1:** state transformation \( x = Tx' \), \( T = [R \quad \overline{R}] \), \( T^{-1} = \begin{bmatrix} L \\ \overline{L} \end{bmatrix} \)

\[
\begin{bmatrix}
\dot{x}'_1(t) \\
\dot{x}'_2(t)
\end{bmatrix} = \begin{bmatrix}
LAR & LAR \\
\overline{LAR} & \overline{LAR}
\end{bmatrix} \begin{bmatrix}
x'_1(t) \\
x'_2(t)
\end{bmatrix} + \begin{bmatrix}
LB \\
\overline{LB}
\end{bmatrix} u(t)
\]

\( y(t) = \begin{bmatrix}
CR & CR
\end{bmatrix} x'(t) + Du(t) \)

If \( x_1 \) is “more important”, we can assume that \( x_2 \approx 0 \). Then

**Step 2:** truncation

\( \dot{x}_r(t) = LARx_r(t) + LBu(t) \)

\( y_r(t) = CRx_r(t) + Du(t) \)

*Transformation + Truncation = Projection*

**Other sources of projection** — Moment matching methods, Krylov spaces, \( L_2 \) optimization, etc.
Examples of PROM’s

Partial Fraction Expansion (Modal Truncation)

\[ G(s) = \sum_{k=1}^{n} \frac{R_k}{s - p_k} \quad \Rightarrow \quad G_r(s) = \sum_{k=1}^{r} \frac{R_k}{s - p_k} \]

In state space - truncation of a *diagonal realization*.

Truncated Balanced Realization (TBR)-Moore, 1981

The state transformation (Step 1) leads to a realization where the *controllability and observability gramians are diagonal and equal*. 

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Properties of PROM’s

What is preserved by projection order reduction?

**Practically Nothing!**

Stability (or instability) is not preserved
Zeros structure at infinity (relative degree) is not preserved
Minimality (or non-minimality) is not preserved. The McMillan degree may increase (in non-minimal systems)!

**Invariance results:**

*Assuming minimality*, if there exists a realization of $G_r(s)$ which is a PROM of a certain realization of $G(s)$ then *any* realization of $G_r(s)$ is a PROM of *any* realization of $G(s)$, i.e. the problem is realization independent.
Questions

• Given $G(s)$, and $G_r(s)$, is it always possible to find a real projection that relates the two models?

• Is that projection unique?

• How can one calculate that projection?

• Do optimal reduced order models have special PROM properties?

**Q**: Why is it important?

**A1**: It is not.

**A2**: Will be discussed later.
The Inverse Problem

Given \((A, B, C)\) and \((A_r, B_r, C_r)\), find the projection \(P\) (equivalently \(L\) and \(R\)) that relates them.

\[
\begin{align*}
LR &= I_r \quad (r^2 \text{ equations}) \\
LAR &= A_r \quad (r^2 \text{ equations}) \\
LB &= B_r \quad (r \cdot m \text{ equations}) \\
CR &= C_r \quad (r \cdot p \text{ equations})
\end{align*}
\]

\((2r+m+p)r\) equations in \(2nr\) unknowns (\(R, L\)).

The number of equations equals the number of unknowns when

\[r = r^* = n - (m + p)/2\]
Seemingly, the situation is as follows.

If \( r < r^* \) then any \( G_r(s) \) is a PROM via infinitely many projections.

If \( r = r^* \) then not every \( G_r(s) \) is a PROM. Those who are, can be obtained by a finite number of projections.

If \( r > r^* \) then the class of \( G_r(s) \) that are PROM has measure zero.

Q: Is it really so?

A: Only for square systems.
Case 2 ($r=r^*$): $n=2$, $r=1$, SISO

\[ G(s) = \frac{s + 1}{s^2 + 3s + 1} \Rightarrow G_r(s) = \frac{\beta}{s + \alpha} \]

\[ W(s) = \frac{s^2 + s + 1}{s^2 + 0.1s + 1} \]

\[ \min \left\| W(s)(G - G_r) \right\|_2 \]
Projection Calculation

**Step 1**: Use the *linear equations* to reduce the number of unknowns

\[ LB = B_r \quad \Rightarrow \quad L = B_r B^+ + X B_\perp \quad , \quad CR = C_r \quad \Rightarrow \quad R = C^+ C_r + C_\perp Y \]

X \in \mathbb{R}^{r \times (n-m)} \text{ and } Y \in \mathbb{R}^{(n-p) \times r} \text{ are the new unknown matrices.}

**Step 2**: Substituting into the bilinear equations (LR=I and LAR=Ar)

\[ \tilde{X}H_i \tilde{Y} = 0 \quad i = 1, 2 \]

where

\[ \tilde{X} = [X \quad I_r] \quad , \quad \tilde{Y} = [Y^T \quad I_r]^T \]

\[ H_1 = \begin{bmatrix} B_\perp C_\perp & B_\perp C^+ C_r \\ B_r B^+ C_\perp & B_r B^+ C^+ C_r - I_r \end{bmatrix} \quad \text{and} \quad H_2 = \begin{bmatrix} B_\perp A C_\perp & B_\perp A C^+ C_r \\ B_r B^+ A C_\perp & B_r B^+ A C^+ C_r - A_r \end{bmatrix} \]

**Step 3**: Replace the two equations by

\[ \tilde{X}(\lambda H_1 - H_2)\tilde{Y} = 0 \quad \forall \lambda \in \mathbb{C} \]
Kronecker Canonical Form (KCF)

For every $h \times q$ pencil $\lambda E - A$ there exist square and nonsingular $S$ and $V$ such that

$$F(\lambda) = S(\lambda E - A)V = \begin{bmatrix} \text{blockdiag}\{F_i(\lambda)\} & 0 \\ 0 & 0 \end{bmatrix}$$

where the linear pencils $F_i(\lambda)$, $i=1,\ldots,K$, which are unique, assume one of four possible structures

- \textbf{type1} = \begin{bmatrix} \lambda & -1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \lambda & -1 \end{bmatrix}, \quad \textbf{type2} = \begin{bmatrix} \lambda & \cdots & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 \end{bmatrix}, \quad \textbf{type3} = \bar{J}\lambda - I, \quad \textbf{type4} = \lambda I - \bar{F}$

$\bar{J}$ is nilpotent Jordan matrix and $\bar{F}$ is in Jordan form.
Typical KCF of a non-square pencil

\[ F(\lambda) = \begin{bmatrix} 
\lambda & -1 & 0 & 0 \\
0 & \lambda & -1 & 0 \\
0 & 0 & \lambda & -1 \\
\end{bmatrix} \]

The sparse form of \( F(\lambda) \) is the key to the subsequent derivations.
Projection Calculation (cont.)

**Step 4:** Transform to KCF

\[ 0 = \tilde{X}(\lambda H_1 - H_2)\tilde{Y} \]
\[ = [\tilde{X}\tilde{S}^{-1}] \cdot [S(\lambda H_1 - H_2)V] \cdot [V^{-1}\tilde{Y}] \]
\[ = \tilde{X} \cdot F(\lambda) \cdot \tilde{Y} \]

**Step 5:** Finally solving. Suppose there exists an \( r \times r \) sub-pencil of zeros, then \( \tilde{X}, \tilde{Y} \) are selection matrices.

\[
F(\lambda) = \begin{bmatrix}
\lambda - a_1 & 1 & 0 & 0 & 0 & 0 \\
0 & \lambda - a_1 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda - a_2 & 1 & 0 & 0 \\
0 & 0 & 0 & \lambda - a_2 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda - a_3 & 1 \\
0 & 0 & 0 & 0 & 0 & \lambda - a_3
\end{bmatrix}
\]

\[
\tilde{X} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad \tilde{Y} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\tilde{X} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}, \quad \tilde{Y} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

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Step 6: Go back from $\tilde{X}, \tilde{Y}$ to $L, R$

\[ \tilde{X} \rightarrow \tilde{X} \quad \tilde{X} = \tilde{X}S = [S_1 \quad S_2] \]

\[ \tilde{X} \rightarrow X \quad \tilde{X} = c[X \quad I_r] \quad \Rightarrow \quad X = S_2^{-1}S_1, \]

\[ X \rightarrow L \quad L = B^+B_r + S_2^{-1}S_1B_\perp \]

An $r \times r$ zeros sub-matrix in the KCF of $\lambda H_1 - H_2$ \quad \iff \quad A projection that relates $(A, B, C)$ and $(A_r, B_r, C_r)$
Summary of the Algorithm

For given \((A,B,C)\) and \((A_r,B_r,C_r)\)

- If An \(r \times r\) zero sub-matrix in the KCF *cannot* be found then \((A_r,B_r,C_r)\) is *not* a PROM of \((A,B,C)\).

- If \(r \times r\) is the *maximum* dimension possible then the number of zero sub-matrices is equal to the number of projections.

- If a *larger* sub-matrix can be found, \((A,B,C)\) and \((A_r,B_r,C_r)\) are related by infinitely many projections.
Square Systems (m=p)

- The pencil is square with dimension $m-m+r$
- The GKS consists of one type4 block, or simply
  \[ F(\lambda) = \lambda I - \bar{F} \]

Alternatively, this is a generalized eigenvalue problem $\lambda H_1 - H_2$

Main results:
- If $r < n-m$ ($r < r^*$), then every $(A_r, B_r, C_r)$ is a PROM, related to by infinitely many projections.
- If $r > n-m$, then generically every $(A_r, B_r, C_r)$ is NOT a PROM
- If $r=n-m$ ($r = r^*$), then
  a) The number of projections is finite.
  b) If $r$ is odd, and all the eigenvalues are complex, then there is no projection.
**Example:** A system with $n=5$, $m=2 \rightarrow r^*=3$

Case 1: $r=2$

$$F(\lambda) = \begin{bmatrix}
\lambda - \sigma_1 & -\omega_1 & 0 & 0 & 0 \\
\omega_1 & \lambda - \sigma_1 & 0 & 0 & 0 \\
0 & 0 & \lambda - \sigma_2 & -\omega_2 & 0 \\
0 & 0 & \omega_2 & \lambda - \sigma_2 & 0 \\
0 & 0 & 0 & 0 & \lambda - p_3
\end{bmatrix}$$

Case 3: $r=4$

$$F(\lambda) = \begin{bmatrix}
\lambda - a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda - a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda - a_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda - a_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda - a_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda - a_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda - a_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda - a_7
\end{bmatrix}$$

It is impossible to find a $4 \times 4$ block of zeros in the generic case.
**Example:** A system with $n=5$, $m=2 \rightarrow r^*=3$

Case 1: $r=2$

$$F(\lambda) = \begin{bmatrix} \lambda - \sigma_1 & -\omega_1 & 0 & 0 & 0 \\ \omega_1 & \lambda - \sigma_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda - \sigma_2 & -\omega_2 & 0 \\ 0 & 0 & \omega_2 & \lambda - \sigma_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda - p_3 \end{bmatrix}$$

Case 3: $r=4$

$$F(\lambda) = \begin{bmatrix} \lambda & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda - a_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda - a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda - a_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda - a_7 \end{bmatrix}$$

4 th order models must have a non-generic KCF
Case 2: $r=3$

$$F(\lambda) = \begin{bmatrix}
\lambda - a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda - a_2 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda - a_3 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda - a_4 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda - a_5 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda - a_6 \\
\end{bmatrix} \quad \frac{6!}{3!3!} = 20 \text{ solutions}$$

$$F(\lambda) = \begin{bmatrix}
\lambda - a_1 & 1 & 0 & 0 & 0 & 0 \\
0 & \lambda - a_1 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda - a_2 & 1 & 0 & 0 \\
0 & 0 & 0 & \lambda - a_2 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda - a_3 & 1 \\
0 & 0 & 0 & 0 & 0 & \lambda - a_3 \\
\end{bmatrix} \quad 4 \text{ solutions}$$

$$F(\lambda) = \begin{bmatrix}
\lambda - \sigma_1 & \omega_1 & 0 & 0 & 0 & 0 \\
-\omega_1 & \lambda - \sigma_1 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda - \sigma_2 & \omega_2 & 0 & 0 \\
0 & 0 & -\omega_2 & \lambda - \sigma_2 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda - \sigma_3 & \omega_3 \\
0 & 0 & 0 & 0 & -\omega_3 & \lambda - \sigma_3 \\
\end{bmatrix} \quad 0 \text{ solutions}$$
A Different Interpretation

**Lemma** (Vandendorpe and Van Dooren, 2002): The generalized eigenvalues of \((H_2,H_1)\) are the *zeros* of \(G(s)-G_r(s)\).

**Corollaries:**

• A reduced model cannot be obtained via projection if \(r = r^*\) is odd and all the zeros of \(G(s)-G_r(s)\) are complex.

**Example:** Is \(G_r(s) = \frac{2}{s+1}\) a PROM of \(G(s) = \frac{s+1}{s^2+s+1}\)?

\[
\frac{s+1}{s^2+s+1} - \frac{2}{s+1} = -\frac{s^2+2s+3}{(s^2+s+1)(s+1)} = -\frac{s+1\pm j\sqrt{2}}{(s^2+s+1)(s+1)}
\]

• \(G_r(s)\) is a PROM of \(G(s)\) with \(r > r^*\) if and only if every value of \(s\) is a zero, i.e. \(\text{det}(G(s)-G_r(s))\equiv0\).
The Optimal $L_2$ Reduced Order Model (OROM)

**Problem**: Find $G_r(s)$ that minimizes

$$J = \|G(s) - G_r(s)\|_2$$

**Method of solution**

$$\frac{\partial J}{\partial A_r} = 0 \quad , \quad \frac{\partial J}{\partial B_r} = 0 \quad , \quad \frac{\partial J}{\partial C_r} = 0$$

**Theorem**: (Wilson, 1970; Hyland and Bernstein, 1985):

The optimal $G_r(s)$ is

$$\dot{x}_r = L^* A R^* x_r + L^* Bu$$

$$y_r(t) = C R^* x_r(t)$$

where $L^*$, $R^*$ are given as …

**The $L_2$ optimal reduced order model is a PROM!**
Main Result: Let \( p \) be a pole of \( G_r^*(s) \) with multiplicity \( N_p \). Then

a) If the system is square, \(-p\) is a zero of \( G(s)-G_r^*(s) \) with multiplicity \( 2N_p \).

b) If the system is non-square, \(-p\) is a zero of \( G(s)-G_r^*(s) \) with multiplicity \( N_p \).

c) Under certain conditions \( G_r^*(s) \) is on the boundary of the PROM zone.

Old result (Meier and Luenberger, 1967): In SISO systems, if \( p \) is a pole of \( G_r^*(s) \) with multiplicity \( N_p \). Then \(-p\) is a zero of \( G(s)-G_r^*(s) \) with multiplicity \( N_p + 1 \).
Examples:

\[ G(s) = \frac{1}{(s+1)^2}, \quad G_r^*(s) = \frac{3}{8} \left( \frac{s}{s+\frac{1}{3}} \right), \quad E^*(s) = \frac{-\frac{3}{8} (s - \frac{1}{3})^2}{(s+1)^2 (s+\frac{1}{3})} \]

\[ G(s) = \frac{6.585 s^2 + 46.098 s + 96.439}{(s+1)(s+2)(s+3)(s+4)}, \quad G_r^*(s) = \frac{0.01018 s + 6.301}{(s+1.254)^2} \]

\[ E^*(s) = \frac{-0.01018 (s-12.939) (s-1.254)^4}{(s+1)(s+2)(s+3)(s+4)(s+1.254)^2} \]

\[ G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{1}{s+3} \end{bmatrix}, \quad G_r^*(s) = \begin{bmatrix} \frac{0.338}{s+1.794} & \frac{0.748}{s+1.794} \end{bmatrix}, \]

\[ E^*(s) = \begin{bmatrix} -\frac{0.338 (s-1.794) (s+1.84)}{(s+1)(s+2)(s+1.794)} & \frac{0.252 (s-1.794)}{(s+3)(s+1.794)} \end{bmatrix} \]
Summary

• The existence and uniqueness properties of PROM’s have been discussed.

• When viewed as a PROM, the Optimal $L_2$ reduced order model possesses some unique properties.