Stochastic-Volatility, Jump-Diffusion
Optimal Portfolio Problem
with Jump-Bankruptcy Condition:
A Control Application

Floyd B. Hanson

Universities of Illinois and Chicago

NIU LA’09
Linear and Numerical Linear Algebra:
Theory, Methods, and Applications,
12-14 August 2009, DeKalb, Illinois
Overview

1. Introduction.

2. Optimal Portfolio Problem and Underlying SVJD Model.


4. CRRA Canonical Solution to Optimal Portfolio Problem.

5. Computational Results.

6. Conclusions.
A. Histogram of S&P500 Log-Returns 1988-2008:

Figure 1: S&P500 Daily Log-Return Adjusted Closings from 1988 (post-1987) to 2008 showing long-tails of rare events. Normal kernel-smoothed graph, plus one accounting for non-central and \textit{normally} invisible but financially important, rare jumps, in \textcolor{red}{red}. 
B. Extreme Negative Tail Events for Log-Returns (’88-’08):

(a) Extreme Negative Tails.

Figure 2: Extreme Negative and Positive Log-Return Tail Events, with Thresholds POT = −0.04 and +0.048, respectively. POT means Peaks Over (or Under) Threshold. These represent the significant crashes or bonanzas during the time period. {Note: vertical scale differences.}
1. Introduction.

1.1 Early Background:

- Merton pioneered the optimal portfolio and consumption problem for geometric diffusions used HARA (hyperbolic absolute risk-aversion) utility in his lifetime portfolio (RES 1969) and general portfolio (JET 1971) papers. However, there were some errors, in particular with bankruptcy boundary conditions and vanishing consumption.

1.2 Market Jump Properties:

- *Statistical evidence* that jumps are significant in financial markets:
  - Stock and Option Prices in Ball and Torous (JFQA 1985);
  - Capital Asset Pricing Model in Jarrow and Rosenfeld (JB 1984);
  - Foreign Exchange and Stocks in Jorion (RFS 1989).

- Log-return market distributions usually *skewed negative*,
  \[ \eta_3 \equiv \frac{M_3}{(M_2)^{1.5}} < 0 \]
  compared to the skew-less normal distribution, if over sufficiently long times.

- Log-return market distributions usually *leptokurtic*,
  \[ \eta_4 \equiv \frac{M_4}{(M_2)^2} > 3 \]
  if over sufficiently long times, i.e., more peaked than normal.

- Log-return market distribution have *fatter or heavier tails* than the normal distribution’s exponentially small tails.

- *Stochastic dependence of volatility* is important.

- *Time-dependence* of rate coefficients is important, i.e., non-constant coefficients are important; and stochastic volatility.

- *Infinite state domain* is questionable for the optimal portfolio problem.
1.3 Jump-Diffusion Models:

- **Merton (JFE 1976)** in his pioneering jump-diffusion option pricing model used IID *log-normally distributed jump-amplitudes* with a compound Poisson process. Other authors have also used the normal jump-amplitude model.


- **Hanson, Westman and Zhu** (2001-2006) have a number of optimal portfolio papers using various log-return jump-amplitude distributions such as *log-discrete, normal, uniform and double-uniform distributions*.

- **Jump-diffusions give skewness and excess-leptokurtosis** to market distributions.
1.5 Stochastic Jump and Volatility Considerations:

- **Extreme jumps** in the market are relatively rare (statistical outliers) among the large number of daily fluctuations.
- Aït-Sahalia (JFE, 2004) shows difficulty in separating the jumps from the diffusion by the usual maximum likelihood methods.
- **NYSE have had circuit breakers installed** since 1988 to suppress extreme market changes, like in the 1987 crash.
- **Uniform jump-amplitudes** have the fattest of tails and finite range, consistent with circuit breakers and parsimony.
- **Bankruptcy conditions** also need to be considered for the jump-integrals of the jump-diffusion PIDE as we shall see for the optimal portfolio problem; unlike the option pricing problem.
- Andersen, Benzoni and Lund (JFE 2004) showed that both stochastic jump and volatility models are needed to explain equity returns well.
2. **Optimal Portfolio Problem and Underlying Stochastic-Volatility, Jump-Diffusion (SVJD) Return Model.**

2.1 **Stock Price Linear Stochastic Differential Equation (SDE):**

\[
dS(t) = S(t) \left( \mu_s(t) dt + \sqrt{V(t)} dG_s(t) + \nu_s(V(t), t, Q) dP_s(t; Q) \right),
\]

(1)

where

- \( S(t) = \text{stock price}, \ S(0) = S_0 > 0; \)
- \( \mu_s(t) = \text{expected rate of return} \) in absence of asset jumps;
- \( G_s(t) = \text{stock price diffusion process} \), normally distributed such that \( \mathbb{E}[dG_s(t)] = 0 \) and \( \text{Var}[dG_s(t)] = dt; \)
- \( V(t) = \text{stochastic variance} = (\text{stochastic volatility})^2 = \sigma_s^2(t); \)
- \( P_s(t; Q) = \text{Poisson jump counting process} \), Poisson distributed such that \( \mathbb{E}[dP_s(t; Q)] = \lambda_s(t) dt = \text{Var}[dP_s(t; Q)]; \)
2.1 Continued: Stock Price Dynamics:

- $\nu_s(v, t, q) = \text{Poisson jump-amplitude}$ with underlying random mark variable $q = Q$, selected for log-return so that $Q = \ln(1 + \nu_s(v, t, Q))$, such that $\nu_s(v, t, q) > -1$;

- Definition of abbreviated compound Poisson jump term:

$$S(t)\nu_s(V(t), t, Q)dP_s(t; Q) \equiv \sum_{k=P_s(t; Q)+1}^{(P_s+dP_s)(t; Q)} S(T_k^{-})\nu_s(V(T_k^{-}), T_k^{-}, Q_k);$$

(Note: $\sum_{k=P_s+1}^{P_s} A_k \equiv 0$, here if no jump, i.e., $dP_s = 0$.)

- $T_k^{-}$ is the pre-jump time and $Q_k$ is an independent and identically distributed (IID) mark realization at the $k$th jump;

- The processes $G_s(t)$ and $P_s(t) = P_s(t; Q)$ along with $Q_k$ are independent, except that $Q_k$ is conditioned on a jump-event at $T_k$. 
2.2 Double-Uniform Probability Jump-Amplitude $Q$ Mark Distribution (Zhu and Hanson, 2006):

$$
\Phi_Q(q; v, t) = p_1(v, t) \frac{q - a(v, t)}{|a|(v, t)} I\{a(v, t) \leq q < 0\} \\
+ \left( p_1(v, t) + p_2(v, t) \frac{q}{b(v, t)} \right) I\{0 \leq q < b(v, t)\} \\
+ I\{b(v, t), \leq q < \infty\}, \quad q \in [a(v, t), b(v, t)],
$$

where $a(v, t) < 0 < b(v, t), p_1(v, t) + p_2(v, t) = 1$.

- **Mark Mean:**
  \[ \mu_j(v, t) \equiv \mathbb{E}_Q[Q] = (p_1(v, t)a(v, t) + p_2(v, t)b(v, t))/2; \]

- **Mark Variance:**
  \[ \sigma_j^2(v, t) \equiv \text{Var}_Q[Q] = \\
(p_1(v, t)a^2(v, t) + p_2(v, t)b^2(v, t))/3 - \mu_j^2(v, t); \]

- **Mark motivation:** Double-uniform distribution unlinks the different behaviors in crashes and rallies. Uniform is chosen for fat tails (all tail) and difficulty (impossibility?) of estimating the jump mark distribution.
2.3 Stochastic-Volatility (Square-Root Diffusion) Model
(CIR, Econometrica 1985; Heston, RFS 1993; FPS, 2000; etc.):

dV(t) = \kappa_v(t) (\theta_v(t) - V(t)) \, dt + \sigma_v(t) \sqrt{V(t)} \, dG_v(t), \quad (2)

with

- \( V(t) \geq \min(V(t)) > 0^+ \), \( V(0) = V_0 \geq \min(V(t)) > 0^+ \);
- Log-rate \( \kappa_v(t) > 0 \); reversion-level \( \theta_v(t) > 0 \); volatility of volatility (variance) \( \sigma_v(t) > 0 \);
- \( G_v(t) = \) variance diffusion process, normally distributed such that \( \mathbb{E}[dG_v(t)] = 0 \) and \( \mathbb{V}[dG_v(t)] = dt \), with correlation \( \text{Corr}[dG_s(t), dG_v(t)] = \rho dt = \rho(t) dt \);
- Note: SDE (2) is singular for transformations as \( V(t) \to 0^+ \) due to the square root, unlike SDE (1) for \( S(t) \) where the singularity is removable through the log transformation, but Itô-Taylor chain rule or simulation applications might not be valid unless

\[ \Delta t \ll \sqrt{\varepsilon_v} \ll 1, \quad \varepsilon_v = \min(V(t)) > 0. \]
2.3 Continued: Stochastic-Volatility (Square-Root Diffusion) Model:

- **Stochastic Volatility Nonnegativity:**
  
  1. **Transformation to Perfect Square Form (see Feller’s 1951 Singular Diffusion paper for classic approach):**

     \[
     Y(t) = 2e^{\kappa_v(0, t)/2} \sqrt{V(t)},
     \]

     where

     \[
     \kappa_v(t) \equiv \int T \kappa_v(y) dy.
     \]

  2. **Desired Nonnegativity Result:**

     \[
     V(t) = e^{-\kappa_v(0, t)} \left( \frac{Y(t)}{2} \right)^2 \geq 0,
     \]

     where

     \[
     Y(t) = 2\sqrt{V_0} + \int_0^t e^{\kappa_v(0, s)/2} \left( \frac{\kappa_v \theta_v - \frac{1}{4} \sigma_v^2}{\sqrt{V}} \right) (s) ds + (\sigma_v dG_v)(s).
     \]
2.3 Continued: Stochastic-Volatility (Square-Root Diffusion) Model:

- **Stochastic Volatility Special Exact Solution:**
  - **Special Parameter Values:**
    \[ \kappa_v(t) \theta_v(t) = \frac{1}{4} \sigma_v^2(t), \quad \forall t. \]
  - **Desired Special Solution:**
    \[ V(t) = e^{-\kappa_v(0, t)} \left( \sqrt{V_0} + 0.5 \int_0^t e^{\kappa_v(0, s)} / 2 \sigma_v dG_v(s) \right)^2. \]
2.4 Wealth Portfolio with Bond, Stock and Consumption:

- **Portfolio:** Riskless asset or *bond* at price $B(t)$ and Risky asset or *stock* at price $S(t)$, with *instantaneous* portfolio change fractions $U_b(t)$ and $U_s(t)$, respectively, such that $U_b(t) = 1 - U_s(t)$.

- **Exponential Bond Price Process:**
  \[ dB(t) = r(t)B(t)dt, \quad B(0) = B_0. \]

- **SVJD Portfolio Wealth Process $W(t)$, Less Consumption $C(t)$ with Self-Financing:**
  \[
  dW(t) = W(t) \left( r(t)dt + U_s(t) \left( (\mu_s(t) - r(t))dt \\
  + \sqrt{V(t)}dG_s(t) + \nu_s(V(t), t, Q)dP_s(t; Q) \right) \right) - C(t)dt, \tag{3}
  \]
  subject to constraints $W(0) = W_0 > 0$, $W(t) > 0$,
  $v = V(t) > 0$, $0 < C(t) \leq C_0^{(\text{max})}(v, t)W(t)$ and
  $U_0^{(\text{min})}(v, t) \leq U_s(t) \leq U_0^{(\text{max})}(v, t)$, while allowing extra shortselling ($U_s(t) < 0$) and extra borrowing ($U_b(t) < 0$).
2.5 Portfolio Optimal Objective:

\[ J^*(w, v, t) = \max_{\{u, c\}} \left[ E \left[ e^{-\bar{\beta}(t, t_f)} U_w(W(t_f)) \right] \\
+ \int_t^{t_f} e^{-\bar{\beta}(t, \tau)} U_c(C(s)) d\tau \right] \]

\[ \left| W(t) = w, U_s(t) = u, C(t) = c \right]. \] (4)

where

- **Cumulative Discount:** \( \bar{\beta}(t, s) = \int_t^s \beta(\tau) d\tau \), where \( \beta(t) \) is the instantaneous discount rate.

- **Consumption and Final Wealth Utility Functions:** \( U_c(c) \) and \( U_w(w) \) are bounded, strictly increasing and concave.

- **Variable Classes:** State variables are \( w \) and \( v \), while control variables are \( u \) and \( c \).

- **Final Condition:** \( J^*(w, v, t_f) = U_w(w) \).
2.6 Absorbing Natural Boundary Condition:

Approaching bankruptcy as \( w \to 0^+ \), then by the consumption constraint as \( c \to 0^+ \) and by the objective (4),

\[
J^*(0^+, v, t) = \mathcal{U}_w(0^+) e^{-\beta(t,t_f)} + \mathcal{U}_c(0^+) \int_t^{t_f} e^{-\beta(t,s)} ds. \tag{5}
\]

- This is the simple variant of what Merton gave as a correction in his 1990 book for his 1971 optimal portfolio paper.

- However, KLASS 1986 and Sethi with Taksar 1988 pointed out that it was necessary to enforce the non-negativity of wealth and consumption.
3. **Portfolio Stochastic Dynamic Programming**

3.1 **Portfolio Stochastic Dynamic Programming PIDE:**

\[
0 = J_t^*(w, v, t) - \beta(t) J^*(w, v, t) + U_c(c^*) \\
+ (r(t) + (\mu_s(t) - r(t))u^*) \ w J_w^*(w, v, t) \\
- c^* J_w^*(w, v, t) + \frac{1}{2} v (u^*)^2 J_{ww}^*(w, v, t) \\
+ \kappa_v (\theta_v - v) J_v^*(w, v, t) + \frac{1}{2} \sigma_v^2(t) v J_{vv}^*(w, v, t) \\
+ \rho \sigma_v v u^* w J_{wv}^*(w, v, t) + \lambda(t) \left( p_1(v, t) \int_0^a(v, t) \, dq + p_2(v, t) \int_b(v, t) \, dq \right) \\
\cdot \left( J^*((1 + (e^q - 1) u^*) w, v, t) - J^*(w, v, t) \right) \ dq,
\]

(6)

where \( u^* = u^*(w, v, t) \in \left[ U_{0}^{(\text{min})}(v, t), U_{0}^{(\text{max})}(v, t) \right] \) and \( c^* = c^*(w, v, t) \in \left[ 0, C_{0}^{(\text{max})}(v, t) w \right] \) are the optimal controls if they exist, while \( J_w^*(w, v, t) \) and \( J_{ww}^*(w, v, t) \) are the continuous partial derivatives with respect to wealth \( w \) when \( 0 \leq t < t_f \). Note that \((1 + (e^q - 1) u^*(w, v, t)) w\) is a wealth argument.
3.2 **Positivity of Wealth with Jump Distribution:**

Since \((1+(e^q - 1)u^*(w, v, t))w\) is a wealth argument in (6), it must satisfy the wealth positivity condition, so

\[ K(u, q) \equiv 1 + (e^q - 1)u > 0 \]

on the support \([a(v, t), b(v, t)]\) of the jump-amplitude density \(\phi_Q(q; v, t)\).

**Lemma 3.2**  **Bounds on Optimal Stock Fraction due to Positivity of Wealth Jump Argument:**

(a) If the support of \(\phi_Q(q; t)\) is the *finite* interval \(q \in [a(v, t), b(v, t)]\) with \(a(v, t) < 0 < b(v, t)\), then \(u^*(w, v, t)\) is restricted by (6) to

\[ \frac{-1}{\nu_s(v, t, b(v, t))} < u^*(w, v, t) < \frac{-1}{\nu_s(v, t, a(v, t))}, \]  

where \(\nu_s(v, t, q) = \exp(q) - 1\).

(b) If the support of \(\phi_Q(q)\) is fully *infinite*, i.e., \((-\infty, +\infty)\), then \(u^*(w, v, t)\) is restricted by (6) to

\[ 0 < u^*(w, v, t) < 1. \]
3.2 Remarks: Non-Negativity of Wealth and Jump Distribution:

- Recall that $u$ is the stock fraction, so that short-selling and borrowing will be overly restricted in the infinite support case \[8\] where $a(v, t) \to -\infty$ and $b(v, t) \to +\infty$, t unlike the finite case \[7\] where $-\infty < a(v, t) < 0 < b(v, t) < +\infty$.
- So, unlike option pricing, finite support of the mark density makes a big difference in the optimal portfolio and consumption problem!
- Thus, it would not be practical to use either normally or double-exponentially distributed marks in the optimal portfolio and consumption problem with a bankruptcy condition.
- If $[a_{\text{min}}, b_{\text{max}}] = [\min_t (a(v, t)), \max_t (b(v, t))]$, then the overall $u^*$ range for the S&P500 data used is

$$[u_{\text{min}}, u_{\text{max}}] = [-18, +12] \subset \left( \frac{-1}{(e^{b_{\text{max}}} - 1)}, \frac{+1}{(1 - e^{a_{\text{min}}})} \right).$$
3.3 Unconstrained Optimal or Regular Control Policies:

In absence of control constraints and in presence of sufficient differentiability, the dual policy, implicit critical conditions are

- **Regular Consumption** $c^{(\text{reg})}(w, v, t)$ \{Implicitly\}:

  \[ \mathcal{U}_c'(c^{(\text{reg})}(w, v, t)) = J_w^*(w, v, t). \]  

- **Regular Portfolio Fraction** $u^{(\text{reg})}(w, v, t)$ \{Implicitly\}:

  \[
  vw^2 J^*_{ww}(w, v, t) u^{(\text{reg})}(w, v, t) = -\left(\mu_s(t) - r(t)\right) w J_w^*(w, v, t) \\
  - \rho \sigma_v(t)vw J^*_{vv}(w, v, t) \\
  - \lambda(t) w \left( \frac{p_1(v,t)}{|a(v,t)|} \int_a^0 + \frac{p_2(v,t)}{b(v,t)} \int_0^{b(v,t)} \right) \\
  \cdot (e^q - 1) J_w^* \left( t, K(u^{(\text{reg})}(w, v, t), q)w \right) dq.
  \]  

SVJD Optimal Portfolio Problem — 21 — Floyd Hanson, UIC & UofC
4. **CRRA Canonical Solution to Optimal Portfolio Problem.**

4.1 **CRRA Utilities:**

- **Constant Relative Risk-Aversion (CRRA \( \subset \) HARA) Power Utilities:**

  \[
  U_c(x) = U(x) = U_w(x) = \begin{cases} 
  x^{\gamma}/\gamma, & \gamma \neq 0 \\
  \ln(x), & \gamma = 0 
  \end{cases}, \quad x \geq 0, \gamma < 1. \tag{11}
  \]

- **Relative Risk-Aversion (RRA):**

  \[
  RRA(x) \equiv -U''(x)/(U'(x)/x) = (1 - \gamma) > 0, \quad \gamma < 1,
  \]

  i.e., negative of ratio of marginal to average change in marginal utility \((U'(x) > 0 \& U''(x) < 0)\) is a constant.

- **CRRA Canonical Separation of Variables:**

  \[
  J^*(w, v, t) = U(w)J_0(v, t), \quad J_0(v, t_f) = 1, \tag{12}
  \]

  i.e., if valid, then wealth state dependence is known and only the time-variance dependent factor \(J_0(v, t)\) need be determined.
4.2 Canonical Simplifications with CRRA Utilities:

- **Regular Consumption Control is Linear in Wealth:**

  \[ c^{(\text{reg})}(w, v, t) = w \cdot c_0^{(\text{reg})}(v, t) \equiv w/J_0^{1/(1-\gamma)}(v, t), \]  
  \[ (13) \]

  where \( c_0^{(\text{reg})}(v, t) \) is a wealth fraction, with optimal consumption

  \[ c^*(v, t) = \max \left[ \min \left[ c_0^{(\text{reg})}(v, t), C_0^{(\text{max})}(v, t) \right], 0 \right] \]

  per \( w \).

- **Regular Portfolio Fraction Control is Independent of Wealth:**

  \[ u^{(\text{reg})}(w, v, t) \equiv u_0^{(\text{reg})}(v, t) \]

  \[ = \frac{1}{(1-\gamma)v} \left( \mu_s(t) - r(t) + \rho \sigma_v(t) v(J_0, v/J_0)(v, t) \right. \]

  \[ + \lambda_s(t) I_1 \left( u_0^{(\text{reg})}(v, t), v, t \right) \]  
  \[ (14) \]

  in fixed point form, where

  \[ u^* = u_0^*(v, t) = \max \left[ \min \left[ u_0^{(\text{reg})}(v, t), U_0^{(\text{max})} \right], U_0^{(\text{max})} \right], \]

  and  \[ I_1(u, v, t) \equiv \left( \frac{p_1(v, t)}{a(v, t)} \int_a(v, t) + \frac{p_2(v, t)}{b(v, t)} \int_b(v, t) \right) (e^q - 1) K^{\gamma-1}(u, q)dq. \]
4.3 CRRA Time-Variance Dependent Component in Formal “Bernoulli” PDE ($\gamma \neq 0; \gamma < 1$):

$$0 = J_{0,t}(v, t) + (1 - \gamma) \left( g_1(v, t)J_0(v, t) + g_2(v, t)J_0^{\gamma-1}(v, t) \right)$$

$$+ g_3(v, t)J_{0,v} + \frac{1}{2} \sigma_v^2(t)vJ_{0,\sigma^2}$$

(15)

where

- **Bernoulli Coefficients** $g_1(v, t)$, $g_2(v, t)$, and $g_3(v, t)$:
  
  $g_1(v, t) = g_1(v, t; u_0^*(v, t))$, $g_2(v, t) = g_2(v, t; c_0^*(v, t), c_0^{(reg)}(v, t))$, and $g_3(v, t) = g_3(v, t; u_0^*(v, t))$, introduce implicit nonlinear dependence on $u_0^*(v, t)$, $c_0^*(v, t)$ and $c_0^{(reg)}(v, t)$, so iterations are required.

- **Formal (Implicit) Solution using Bernoulli transformation**, $J_0(v, t) = y^{1-\gamma}(v, t)^{1-\gamma}$, improving iterations:
  
  $$0 = y_t(v, t) + g_1(v, t)y(v, t) + g_4(v, t), \quad y(v, t_f) = 1,$$

  $$J_0(v, t) = \left[ e^{\overline{G}_1(v, t, t_f)} + \int_t^{t_f} g_4(v, \tau)e^{\overline{G}_1(v, t, \tau)}d\tau \right]^{1-\gamma}$$

(16)
4.3 Continued, Coefficient Function for Reference Only:

where

\[ g_1(v, t) \equiv \frac{1}{1 - \gamma} \left( -\beta(t) + \gamma (r(t) + (\mu_s(t) - r(t))u_0^*(v, t)) \right. \]

\[ - \frac{1}{2} (1 - \gamma)|v(u_0^*)^2(v, t) + \lambda_s(t) \left( I_2(u_0^*(v, t), v, t) - 1 \right) \), \]

\[ \bar{g}_1(v, t, \tau) \equiv \int_t^\tau g_1(v, s)ds. \]

\[ I_2(u, v, t) \equiv \left( \frac{p_1(v, t)}{|a|(v, t)} \int_{a(v, t)}^0 + \frac{p_2(v, t)}{b(v, t)} \int_0^{b(v, t)} \right) K^\gamma(u, q) dq, \]

\[ g_2(v, t) \equiv \frac{1}{1 - \gamma} \left( \left( \frac{c_0^*(v, t)}{c_0^{(\text{reg})(v, t)}} \right)^\gamma - \gamma \left( \frac{c_0^*(v, t)}{c_0^{(\text{reg})(v, t)}} \right) \right), \]

\[ g_3(v, t) = +\kappa_v(t)(\theta_v(t) - v) + \gamma \rho \sigma_v(t)v u_0^*(v, t), \]

\[ g_4(v, t) = g_2(v, t) + g_3(v, t)y_v(v, t) + \frac{1}{2} \sigma_v^2(t)v \left( y_{vv} - \gamma((y_v)^2/y) \right) (v, t). \]
4.4 CRRA Time-Variance Dependent Component in Formal
“Bernoulli” PDE ($\gamma = 0$; Kelly Criterion):

In this medium risk-averse case of the logarithmic CRRA utility, the formal, implicit canonical solution has two terms,

$$J^*(w, v, t) = \ln(w)J_0(v, t) + J_1(v, t), \quad (17)$$

with final boundary conditions $J_0(v, t) = 1$ and $J_1(v, t) = 0$. The regular controls satisfy,

$$c^{(reg)}(w, v, t) = wc^{(reg)}_0(v, t) \equiv w/J_0(v, t),$$
$$u^{(reg)}(w, v, t) = u^{(reg)}_0(v, t) \equiv \frac{1}{\nu}(\mu_s(t) - r(t) + \rho \sigma_v(t)(J_{0,v}/J_0)(v, t)$$
$$+ \lambda_s(t)I_1\left(u^{(reg)}_0(v, t), v, t\right)).$$
4.4 $\gamma = 0$ case continued:
The $\ln(w)$ and $w$-independent coefficients satisfies the implicit, uni-directionally-coupled PIDEs,

$$0 = J_{0,t}(v, t) - \beta(t)J_0(v, t) + g_0(v, t),$$

$$0 = J_{1,t}(v, t) - \beta(t)J_1(v, t) + \tilde{g}_2(v, t),$$

with formal solutions

$$J_0(v, t) = e^{-\beta(t; t_f)} + \int_t^{t_f} e^{-\beta(t; \tau)}g_0(v, \tau)d\tau,$$

$$J_1(v, t) = \int_t^{t_f} e^{-\beta(t; \tau)}\tilde{g}_2(v, \tau)d\tau,$$

where

$$g_0(v, t) \equiv 1 + \kappa_v(t)(\theta_v(t) - v)J_{0,v}(v, t) + \frac{1}{2}\sigma_v^2(t)vJ_{0,vv}(v, t),$$

$$\tilde{g}_2(v, t) \equiv -\ln(J_0(v, t)) - 1 + (r(t)+(\mu_s(t)-r(t))u_0^*(v, t)$$

$$-0.5v(u_0^*)^2(v, t) + \lambda_s(t)I_2^{(0)}(u_0^*(v, t), v, t)\big)J_0(v, t)$$

$$+ \kappa_v(t)(\theta_v(t) - v)J_{1,v}(v, t) + 0.5\sigma_v^2(t)vJ_{1,vv}(v, t),$$

and where $I_2^{(0)}$ has $\ln(K)$ replacing $K^\gamma$ in $I_2$. 
5. Computational Considerations and Results.

5.1 Computational Considerations:

- The primary problem is having stable computations and much smaller time-steps $\Delta t$ are needed compared to variance-steps $\Delta V$, since the computations are drift-dominated over the diffusion coefficient, in that the mesh coefficient associated with $J_{0,v}$ can be hundreds times larger than that associated with $J_{0,vv}$ for the variance-diffusion.
- Drift-upwinding is needed so the finite differences for the drift-partial derivatives follow the sign of the drift-coefficient, while central differences are sufficient for the diffusion partials.
- Iteration calculations in time, controls and volatility are sensitive to small and negative deviations, as well as the form of the iteration in terms of the formal implicitly-defined solutions.
5.2 Results for Regular $u^{(reg)}(v_p, t)$ and Optimal $u^*(v_p, t)$ Portfolio Fraction Policies, $\sigma_p = \sqrt{v_p} = 16\%$:

(a) Regular fraction policy $u^{(reg)}(v_p, t)$.

(b) Optimal fraction policy, $u^*(v_p, t)$.

Figure 3: Regular and optimal portfolio stock fraction policies, $u^{(reg)}(v_p, t)$ and $u^*(v_p, t)$ on $t \in [1999.0, 2001.0]$, while $u^*(v_p, t) \in [-18, 12]$. 
5.3 Results for Optimal Value $J^*(w, v_p, t)$ and Optimal Consumption $c^*(w, v_p, t)$, $\sigma_p = \sqrt{v_p} = 16\%$:

(a) Optimal portfolio value $J^*(w, v_p, t)$. 
(b) Optimal consumption policy $c^*(w, v_p, t)$.

Figure 4: Optimal portfolio value $J^*(w, v_p, t)$ and optimal consumption policy $c^*(w, v_p, t)$ for $(w, v_p, t) \in [0, 110] \times [1999.0, 2001.0]$, while $c^*(w, v_p, t) \in [0, 0.75 \cdot w]$ is enforced near $t = 2001$. 

SVJD Optimal Portfolio Problem — 30 — Floyd Hanson, UIC & UofC
5.4 Results for Optimal Value $J^*(w_p, v, t)$ and Optimal Consumption $c^*(w_p, v, t)$, $w_p = 55$:

(a) Optimal portfolio value $J^*(w_p, v, t)$.

(b) Optimal consumption $c^*(w_p, v, t)$.

Figure 5: Optimal portfolio value $J^*(w_p, v, t)$ and optimal consumption $c^*(w_p, v, t)$ at $w_p = 55$ for $(v, t) \in [v_{min}, 1.0] \times [1999.0, 2001.0]$, while $c^*(w_p, v, t) \in [0, 0.75 \cdot w_p]$ is enforced near $t = 2001$. 
5.5 Results for Optimal Portfolio Fraction $u^*(v, t)$:

Figure 6: Optimal portfolio fraction policy $u^*(v, t)$ for $(v, t) \in [v_{\text{min}}, 1.0] \times [1999.0, 2001.0]$, while $u^*(v, t) \in [-18, 12]$ is enforced near small variance $v = v_{\text{min}} > 0$. 
6. Conclusions

- Generalized the optimal portfolio and consumption problem for jump-diffusions to include stochastic volatility/variance.
- Confirmed significant effects on variation of instantaneous stock fraction policies due to time-dependence of interest and discount rates for SVJD optimal portfolio and consumption models.
- Showed jump-amplitude distributions with compact support are much less restricted on short-selling and borrowing compared to the infinite support case in the SVJD optimal portfolio and consumption problem.
- Noted that the CRRA reduced canonical optimal portfolio problem is strongly drift-dominated for sample market parameter values over the diffusion terms, so at least first order drift-upwinding is essential for stable Bernoulli PDE computations.