M_∨- matrices : A generalization of M-matrices based on eventually nonnegative matrices

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This is joint work with Michael Tsatsomeros and Pauline van den Driessche.
Given $X \in \mathbb{R}^{n \times n}$, the spectrum of $X$ is denoted by $\sigma(X)$ and its spectral radius by $\rho(X) = \max\{|\lambda| \mid \lambda \in \sigma(X)\}$.

An $n \times n$ matrix $B = [b_{ij}]$ is nonnegative (positive), denoted by $B \geq 0$ ($B > 0$), if $b_{ij} \geq 0$ ($b_{ij} > 0$) for all $i$ and $j$. 
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$B$ is eventually nonnegative (positive), denoted by $B \stackrel{\text{v}}{\geq} 0$ ($B \stackrel{\text{v}}{>} 0$), if there exists a nonnegative integer $k_0$ such that $B^k \stackrel{\text{v}}{\geq} 0$ ($B^k \stackrel{\text{v}}{>} 0$) for all $k \geq k_0$.

We denote the smallest such nonnegative integer by $k_0 = k_0(B)$ and refer to it as the power index of $B$. 
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An $n \times n$ matrix $B = [b_{ij}]$ is nonnegative (positive), denoted by $B \geq 0 (B > 0)$, if $b_{ij} \geq 0 (b_{ij} > 0)$ for all $i$ and $j$.

$B$ is eventually nonnegative (positive), denoted by $B \geq 0^\vartriangleright (B > 0^\vartriangleright)$, if there exists a nonnegative integer $k_0$ such that $B^k \geq 0 (B^k > 0)$ for all $k \geq k_0$.

We denote the smallest such nonnegative integer by $k_0 = k_0(B)$ and refer to it as the power index of $B$.

An $n \times n$ matrix $A = [a_{ij}]$ is called

- an $M$-matrix if $A = sI - B$, where $B \geq 0$ and $s \geq \rho(B) \geq 0$
- an $M_\vartriangleright$-matrix if $A = sI - B$, where $B \geq 0^\vartriangleright$ and $s \geq \rho(B) \geq 0$
A matrix $X \in \mathbb{R}^{n \times n}$ has

- the *Perron-Frobenius property* if $\rho(X) > 0$, $\rho(X) \in \sigma(X)$ and there exists a nonnegative eigenvector corresponding to $\rho(X)$;

- the *strong Perron-Frobenius property* if, in addition to having the Perron-Frobenius property, $\rho(X)$ is a simple eigenvalue such that

$$\rho(X) > |\lambda| \quad \text{for all} \quad \lambda \in \sigma(X), \quad \lambda \neq \rho(X)$$

and the corresponding eigenvector is strictly positive.
A result of Johnson and Tarazaga (2004)

For a matrix $B \in \mathbb{R}^{n \times n}$, the following are equivalent:

(i) Both matrices $B$ and $B^T$ have the strong Perron-Frobenius property.

(ii) $B$ is eventually positive.

(iii) $B^T$ is eventually positive.
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A result of Noutsos (2006) and Elhashash/Szyld (2006)

Let $B \in \mathbb{R}^{n \times n}$ be an eventually nonnegative matrix that is not nilpotent. Then both $B$ and $B^T$ have the Perron-Frobenius property.
Some obvious properties of $M_\vee$-matrices (analogous to M-matrices)

If $A = sI - B$ is an $M_\vee$-matrix, then

(i) $s - \rho(B) \in \sigma(A)$
(ii) $\Re \lambda \geq 0$ for all $\lambda \in \sigma(A)$
(iii) $\det A \geq 0$ and $\det A = 0$ if and only if $s = \rho(B)$
(iv) if, in particular, $\rho(B) > 0$, then there exists an eigenvector $x \geq 0$ of $A$ and an eigenvector $y \geq 0$ of $A^T$ corresponding to $\lambda(A) = s - \rho(B)$
(v) if, in particular, $B \succ 0$ and $s > \rho(B)$, then in (iv) $x > 0$, $y > 0$ and in (ii) $\Re \lambda > 0$ for all $\lambda \in \sigma(A)$
An $n \times n$ matrix $B = [b_{ij}]$ is called

- **exponentially nonnegative (positive)** if $\forall t \geq 0$,
  
  $$e^{tB} = \sum_{k=0}^{\infty} \frac{t^k B^k}{k!} \geq 0 \ (e^{tB} > 0)$$

- **eventually exponentially nonnegative (positive)** if $\exists t_0 \in [0, \infty)$ such that $\forall t \geq t_0$, $e^{tB} \geq 0 \ (e^{tB} > 0)$
Exponential nonnegativity

**Definition**

An $n \times n$ matrix $B = [b_{ij}]$ is called

- *exponentially nonnegative (positive)* if $\forall t \geq 0$,
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- *eventually exponentially nonnegative (positive)* if $\exists t_0 \in [0, \infty)$ such that $\forall t \geq t_0$, $e^{tB} \geq 0 \ (e^{tB} > 0)$

For an M-matrix $A$, clearly $-A + \alpha I \geq 0$ for sufficiently large $\alpha \geq 0$, implying that

$$e^{-tA} = e^{-t\alpha} e^{-t(A-\alpha I)} \geq 0 \text{ for all } t \geq 0.$$ 

That is $-A$ is exponentially nonnegative.
An extension of the above property to $M\vee$-matrices.

**Theorem**

Let $A = sI - B \in \mathbb{R}^{n \times n}$ be an $M\vee$-matrix with $B \succ 0$ (and thus $s \geq \rho(B) > 0$). Then $-A$ is eventually exponentially positive. That is, there exists $t_0 \geq 0$ such that $e^{-tA} > 0$ for all $t \geq t_0$. 

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**Theorem**

Let $A = sl - B \in \mathbb{R}^{n \times n}$ be an $M_\vee$-matrix with $B \succ 0$ (and thus $s \geq \rho(B) > 0$). Then $-A$ is eventually exponentially positive. That is, there exists $t_0 \geq 0$ such that $e^{-tA} > 0$ for all $t \geq t_0$.

**Proof.**

Let $A = sl - B$, where $B = sl - A \succ 0$ with power index $k_0$. As $B^m > 0$ for all $m \geq k_0$, there exists sufficiently large $t_0 > 0$ so that for all $t \geq t_0$, the sum of the first $k_0 - 1$ terms of the series $e^{tB} = \sum_{m=0}^{\infty} \frac{t^m B^m}{m!}$ is dominated by the term $\frac{t^{k_0} B^{k_0}}{k_0!}$, rendering $e^{tB}$ positive for all $t \geq t_0$. It follows that $e^{-tA} = e^{-ts} e^{tB}$ is positive for all $t \geq t_0$. That is, $-A$ is eventually exponentially positive as claimed.
To obtain a similar result with eventual exponential nonnegativity, we need a definition:

The degree of 0 as a root of the minimal polynomial of $A$ is denoted by $\text{index}_0(A)$, and if $A$ is nonsingular, then $\text{index}_0(A) = 0$. 

A result of Noutsos and Tsatsomeros (2008): Let $B \in \mathbb{R}^{n \times n}$ such that $Bv \geq 0$ and $\text{index}_0(B) \leq 1$. Then $B$ is eventually exponentially nonnegative.

Theorem: Let $A = sI - B \in \mathbb{R}^{n \times n}$ be an $M_\vee$-matrix where $Bv \geq 0$ and $\text{index}_0(B) \leq 1$. Then $-A$ is an eventually exponentially nonnegative matrix.

Proof. The result follows readily from the above result and the fact that $e^{-tA} = e^{-ts}e^{tB}$. 

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To obtain a similar result with eventual exponential nonnegativity, we need a definition:

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Let $B \in \mathbb{R}^{n \times n}$ such that $B \geq 0$ and $\text{index}_0(B) \leq 1$. Then $B$ is eventually exponentially nonnegative.
To obtain a similar result with eventual exponential nonnegativity, we need a definition:

The degree of 0 as a root of the minimal polynomial of $A$ is denoted by $\text{index}_0(A)$, and if $A$ is nonsingular, then $\text{index}_0(A) = 0$.

A result of Noutsos and Tsatsomeros (2008)

Let $B \in \mathbb{R}^{n \times n}$ such that $B \geq 0$ and $\text{index}_0(B) \leq 1$. Then $B$ is eventually exponentially nonnegative.

**Theorem**

Let $A = \lambda I - B \in \mathbb{R}^{n \times n}$ be an $M_\vee$-matrix where $B \geq 0$ and $\text{index}_0(B) \leq 1$. Then $-A$ is an eventually exponentially nonnegative matrix.

**Proof.**

The result follows readily from the above result and the fact that $e^{-tA} = e^{-ts}e^{tB}$.
Corollary

Let $A = sI - B \in \mathbb{R}^{n \times n}$ be an $M_\vee$-matrix such that $B + \alpha I \geq 0$ for some $\alpha \in \mathbb{R}$ with $-\alpha \notin \sigma(B)$. Then $-A$ is an eventually exponentially nonnegative matrix.

Proof.

Since $A = (s + \alpha)I - (B + \alpha I)$ is an $M_\vee$-matrix and $B + \alpha I \geq 0$, it follows that $s + \alpha \geq \rho(B + \alpha I)$. Since $\text{index}_0(B + \alpha I) = 0 \leq 1$, the corollary follows by applying the latter theorem to $A = (s + \alpha)I - (B + \alpha I)$. 

$\square$
Consider \( A = 3I - B \), where

\[
B = \begin{bmatrix}
0 & 1 & 1 & -1 \\
1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

Since \( B \) is eventually nonnegative with \( \rho(B) = 2 \), \( A \) is an \( M_\lor \)-matrix. As \( \text{index}_0(B) = 1 \), it follows that \(-A\) is eventually exponentially nonnegative.
A second example

When \( \text{index}_0(B) > 1 \), the conclusion of the last theorem is not in general true.

\[
B = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1
\end{bmatrix}
\]

is eventually nonnegative with \( k_0(B) = 2 \) and \( \text{index}_0(B) = 2 \). Let
\[
A = sI - B \text{ with } s \geq \rho(B).
\]

As
\[
B^k = \begin{bmatrix}
2^{k-1} & 2^{k-1} & k2^{k-1} & k2^{k-1} \\
2^{k-1} & 2^{k-1} & k2^{k-1} & k2^{k-1} \\
0 & 0 & 2^{k-1} & 2^{k-1} \\
0 & 0 & 2^{k-1} & 2^{k-1}
\end{bmatrix} \quad (k = 2, 3, \ldots),
\]

it follows that the \((3,1)\) and \((4,2)\) entries of \( e^{tB} \) (and thus \( e^{-tA} \)) are negative for all \( t > 0 \). That is, \(-A\) is not eventually exponentially nonnegative.
The following well known properties of M-matrices: monotonicity, semipositivity and inverse nonnegativity are now examined in the context of $M_\vee$-matrices.
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**Theorem**

Let $A = sI - B \in \mathbb{R}^{n \times n}$, where $B \geq 0$ has power index $k_0 \geq 0$. Let the cone $K$ be defined as $K = B^{k_0} \mathbb{R}_+^n$. Consider the following conditions:

(i) $A$ is an invertible $M_\vee$-matrix

(ii) $s > \rho(B)$ (positive stability of $A$)

(iii) $A^{-1}$ exists and $A^{-1}K \subseteq \mathbb{R}_+^n$ (inverse nonnegativity)

(iv) $Ax \in K \implies x \geq 0$ (monotonicity)

Then (i)$\iff$(ii)$\implies$(iii)$\iff$(iv).

If, in addition, $B$ is not nilpotent, then all conditions (i)-(iv) are equivalent.
Remarks

(a) The above implication (iii) \( \implies \) (i) is not in general true if \( B \) is nilpotent. For example, consider
\[
B = \begin{bmatrix}
1 & -1 \\
1 & -1
\end{bmatrix} \geq 0,
\]
which has power index \( k_0 = 2 \). Thus \( K = B^2 \mathbb{R}_+^2 = \{0\} \). For any \( s < 0 \),
\( A = sI - B \) is invertible and \( A^{-1}K = \{0\} \subset \mathbb{R}_+^2 \). However, \( A \) is not an \( M_\vee \)-matrix as its eigenvalues are negative.
(b) It is well known that when an M-matrix is invertible, its inverse is nonnegative. Johnson and Tarazaga (2004) show that the inverse of a pseudo M-matrix is eventually positive. Le and McDonald (2006) show that if $B$ is an irreducible eventually nonnegative matrix with $\text{index}_0(B) \leq 1$, then there exists $t > \rho(B)$ such that for all $s \in (\rho(B), t)$, $(sl - B)^{-1} > 0$. 
(b) It is well known that when an M-matrix is invertible, its inverse is nonnegative. Johnson and Tarazaga (2004) show that the inverse of a pseudo M-matrix is eventually positive. Le and McDonald (2006) show that if $B$ is an irreducible eventually nonnegative matrix with $\text{index}_0(B) \leq 1$, then there exists $t > \rho(B)$ such that for all $s \in (\rho(B), t)$, $(sl - B)^{-1} > 0$.

The situation with the inverse of an $M_\vee$-matrix $A$ is different. Notice that condition (iii) of the last theorem is equivalent to $A^{-1}B^{k_0} \geq 0$. In general, if $A$ is an invertible $M_\vee$-matrix, $A^{-1}$ is neither nonnegative nor eventually nonnegative.
Monotonicity, semipositivity and inverse nonnegativity

For a subclass of the $M_\vee$-matrices, we have the following semipositivity result.

**Theorem**

Let $A = sI - B \in \mathbb{R}^{n \times n}$, where $B \geq 0$ has a positive eigenvector (corresponding to $\rho(B)$). Consider the following conditions:

(i) $A$ is an $M_\vee$-matrix

(ii) There exists an invertible diagonal matrix $D \geq 0$ such that the row sums of $AD$ are nonnegative

(iii) There exists $x > 0$ such that $Ax \geq 0$ (semipositivity)

Then (i)$\iff$(ii)$\iff$(iii).

If, in addition, $B$ is not nilpotent, then all conditions (i)-(iii) are equivalent.
Remarks

(a) Let $A = -\frac{1}{4} I + B$, with $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \geq 0$, and $x = [2, 1]^T$. Then (iii) above holds, but $A$ is not an $M_\vee$-matrix. Thus the implication (iii)$\implies$(i) is not in general true if $B$ is nilpotent.
Remarks

(a) Let \( A = -\frac{1}{4} I + B \), with \( B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \) \( \geq 0 \), and \( x = [2, 1]^T \). Then (iii) above holds, but \( A \) is not an \( M_\lor \) matrix. Thus the implication (iii) \( \implies \) (i) is not in general true if \( B \) is nilpotent.

(b) The existence of a positive eigenvector in the above theorem is necessary. To see this, consider \( B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \) \( \geq 0 \), which is nilpotent and has no positive eigenvector. Let

\[
A = \frac{1}{4} I - B = \begin{bmatrix} -\frac{3}{4} & -1 \\ 1 & 5/4 \end{bmatrix}.
\]

Notice that there is no \( x > 0 \) such that \( Ax \geq 0 \), but \( A \) is an \( M_\lor \) matrix. That is, (i) of the above theorem holds but not (iii).
An invertible M-matrix can be scaled to be diagonally dominant: $A De > 0$. But (ii) of the above theorem does not imply this for $M_V$-matrices because off-diagonal entries of $B \geq 0$ can be negative. For example, let

$$A = sl - B = 9.5 I - \begin{bmatrix} -0.1 & 20 & 47 \\ -0.2 & 1 & 1 \\ 0.3 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 9.6 & -20 & -47 \\ 0.2 & 8.5 & -1 \\ -0.3 & -5 & 1.5 \end{bmatrix}.$$ 

The matrix $A$ is an invertible $M_V$-matrix because $B > 0$ with $\rho(B) = 9.4834$. Letting $D = \text{diag}(w)$, where $w = [0.9799, 0.0004, 0.1996]^T$ is an eigenvector of $B$ corresponding to $\rho(B)$, gives

$$AD = \begin{bmatrix} 9.4068 & -0.0086 & -9.3819 \\ 0.1960 & 0.0036 & -0.1996 \\ -0.2940 & -0.0021 & 0.2994 \end{bmatrix}.$$ 

Note that $A De \geq 0$, however, $AD$ is not diagonally dominant.
Next we give some properties of singular $M_\vee$- matrices analogous to properties of singular M-matrices.

**Theorem**

Let $A = sI - B \in \mathbb{R}^{n \times n}$ be a singular $M_\vee$- matrix, where $B \geq 0$. Then the following hold.

(i) $A$ has rank $n - 1$.

(ii) There exists a vector $x > 0$ such that $Ax = 0$.

(iii) If for some vector $u$, $Au \geq 0$, then $u = 0$ (almost monotonicity).
The following is a comparison condition for $M_\vee$-matrices analogous to a known result for M-matrices (see e.g., Horn and Johnson).

**Theorem**

Let $A = sI - B \in \mathbb{R}^{n \times n}$ and $E = sI - F \in \mathbb{R}^{n \times n}$, where $B, F \geq 0$ are not nilpotent. Suppose that at least one of $B, B^T, F$ or $F^T$ has a positive eigenvector (corresponding to the spectral radius). If $A$ is an $M_\vee$-matrix and $A \leq E$, then $E$ is an $M_\vee$-matrix.
An important aspect of M-matrix theory is principal submatrix inheritance: every principal submatrix of an M-matrix is also an M-matrix.

As a consequence, all principal minors of an M-matrix are nonnegative (i.e., every M-matrix is a $P_0$-matrix).

These facts do not carry over to $M_\lor$-matrices as seen in the next example.
Consider
\[
B = \begin{bmatrix}
9.5 & 1 & 1.5 \\
-14.5 & 16 & 10.5 \\
10.5 & -3 & 4.5
\end{bmatrix}
\]
for which \( \rho(B) = 12 \) is a simple dominant eigenvalue having positive left and right eigenvectors. That is, \( B \) and \( B^T \) satisfy the strong Perron-Frobenius property and so \( B \triangledown \mathbf{I} > 0 \). As a consequence,
\[
A = 12.5 I - B = \begin{bmatrix}
3 & -1 & -1.5 \\
14.5 & -3.5 & -10.5 \\
-10.5 & 3 & 8
\end{bmatrix}
\]
is an invertible \( M_\triangledown \)-matrix. Clearly, \( A \) is not a \( P_0 \)-matrix since the (2,2) entry is negative. Also the (2,2) entry is a principal submatrix of \( A \) that is not an \( M_\triangledown \)-matrix.