A New Hybrid Method for Finding Eigenpairs of Symmetric Quadratic Eigenvalue Problem in an Interval

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Overview

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II. Some Applications of Quadratic Eigenvalue Problem
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   ● Jacobi-Davidson Method
   ● Second-Order Arnoldi Method (SOAR)
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Introduction (QEP)

The behavior of the physical phenomenon in different applications can be described by a second-order differential equation as shown,

\[ f(t) = M\ddot{q}(t) + C\dot{q}(t) + Kq(t) \]

\[ q(0) = g \]

\[ \dot{q}(0) = h \]

(1)

where \( f(t) \) is a time dependent external force vector
\( g, h \) are initial condition vectors at time \( t = 0 \)
\( M = mass \ matrix \),
\( C = damping \ matrix \) and
\( K = stiffness \ matrix \).
Introduction (QEP)

\[ 0 = M \ddot{q}(t) + C \dot{q}(t) + Kq(t) \]  

(2)

The solution of the above homogeneous second-order differential equation can be expressed,

\[ q(t) = e^{\lambda t} u, \]

which is the eigensolution of the corresponding quadratic eigenvalue problem (QEP),

\[ (\lambda^2 M + \lambda C + K)u = 0 \]  

(3)

where, \( \lambda = \text{eigenvalue} \), and

\[ u = \text{eigenvector} \] corresponding to the eigenvalue \( \lambda \).
Some Applications

1. Vibration Analysis of Structural Systems
2. Constrained least squares problem
3. Eigenvalue assignment problem for quadratic matrix pencil
Vibration Analysis of Structural Systems

In practice response $q(t)$ of a system

$$M \ddot{q}(t) + C \dot{q}(t) + Kq(t) = f(t)$$

excited by a time harmonic force $f(t) = f_0 e^{i\omega_0 t}$ with some frequency $\omega_0$. As $i\omega_0$ approaches to eigenvalue of the system, response $q(t)$ blows. This case we call it *resonance* condition.
Constrained least squares problem

\[
\min_{x^T x = \alpha^2} \left\{ x^T Ax - 2b^T x \right\}
\]

where \( A = A^T \in \mathbb{R}^{n \times n} \), \( b = (Ax - \lambda x) \in \mathbb{R}^n \).

The solution is \( x = (A - \lambda I)^{-1} b \), where \( \lambda \) is the smallest eigenvalue of

\[
\left( \lambda^2 I + 2\lambda A + (A^2 - \alpha^{-2} bb^T) \right) y = 0,
\]

where \( y = (A - \lambda I)^{-2} b \).
Partial eigenvalue assignment problem for quadratic matrix pencil

This is the problem of reassigning a part of the spectrum of quadratic system by feedback control, leaving the rest of the spectrum invariant.
A Brief Review of the Existing Methods

1. Linearization
2. Jacobi-Davidson Method for QEP
3. Second-Order Arnoldi Method
Linearization

The Quadratic eigenvalue problem of the form 
\((\lambda^2 M + \lambda C + K)u = 0\) is usually solved in two stages.

**Stage I.** Transform QEP into equivalent *Standard Eigenvalue problem*

\[ Ay = \lambda y \]

where

\[ A = \begin{pmatrix} -M^{-1}C & -M^{-1}K \\ I & O \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} \lambda u \\ u \end{pmatrix} \]

\((M \text{ is a non singular } n \times n \text{ matrix})\).
Linearization

Stage II.

Compute the Eigenpairs of the *Standard Eigenvalue Problem*.

Then find the eigenpairs by using *QR* algorithm if $n$ is small and dense, or apply Krylov-subspace-based methods to find few eigenvalues of the *QEP*.
Jacobi-Davidson Method for QEP[SLEIJPEN 8]

This method also consists of two major steps.

**First step**

The projection of the higher dimension QEP to a low dimension subspace, by choosing suitable $V$,

$$\left( \alpha^2 V^* MV + \alpha V^* CV + V^* KV \right) s = 0$$

where $\alpha$ is an eigenvalue and $x = Vs$ is eigenvector corresponding to the eigenvalue $\alpha$. The pair $(\alpha, x)$ is called the **Ritz pair**.

If the norm of the residual $\| Q(\alpha)x \|$ is larger than tolerance then go to step II.
Jacobi-Davidson Method for QEP

Second Step
The expansion of the search subspace, \( \text{span}(V) \).
Compute orthogonal correction vector \( t \) for Ritz vector \( x_k \), so that we have

\[
(\alpha^2 M + \alpha C + K)(x_k + t) = 0.
\]

As \( t \perp x_k \) the operator \( Q(\lambda) \) can be restricted to the subspace orthogonal to \( x_k \) and solve for \( t \),

\[
\left( I - \frac{px_k^*}{x_k^* p} \right) Q(\alpha)(I - x_k x_k^*) t = -r_k
\]

where, \( r_k = Q(\alpha)x_k \) is residual and \( p = Q'(\alpha)x_k \), with \( Q'(\alpha) = 2\alpha M + C \).
Second-Order Arnoldi Method (SOAR) [BAI 11]

Second-order Krylov subspace $G_k(A, B; u)$ induced by a pair of matrices $A, B \in \mathbb{R}^{n \times n}$ and a vector $u \in \mathbb{R}^n$ is defined as,

$$G_k(A, B; u) = \text{span}\{r_0, r_1, \ldots, r_{k-1}\},$$

where

$$r_0 = u,$$
$$r_1 = Ar_0,$$
$$r_j = Ar_{j-1} + Br_{j-2}, \text{ for } j \geq 2,$$

and $A = -M^{-1}C$, $B = -M^{-1}K$

We have,

$$(\alpha^2 M + \alpha C + K)z = 0$$

where $(\alpha, z)$ is an approximate eigenpair.
Second-Order Arnoldi Method

- In $m << n$ steps, SOAR procedure with $A = -M^{-1}C$, $B = -M^{-1}K$ and a starting vector $u$ generates an $n \times m$ matrix $Q_m$ which is an orthonormal basis of second-order krylov subspace $G_m(A, B; u)$.

- compute $M_m = Q_m^T M Q_m$, $C_m = Q_m^T C Q_m$, $K_m = Q_m^T K Q_m$.

Now we have new reduced system of order $m$,

$$ (\alpha^2 M_m + \alpha C_m + K_m)g = 0 $$

where $(\alpha, g)$ is an eigenpair of reduced QEP.

Obtain the Ritz pairs $(\alpha, z)$ where $z = \frac{Q_m g}{||Q_m g||}$

- Check the residual with the new Ritz pairs $(\alpha, z)$. 
Drawback of the Existing Methods

Linearized method

- The generalized eigenvalue problem is twice the dimension of the original QEP.

- One of the drawback of the linearization process is the lost of the positive definiteness. Subsequently the essential spectral properties of QEP are not guaranteed to be preserved.
Jacobi-Davidson Method

- Successes of the Jacobi-Davidson Method strongly depends how to choose the initial vectors.
- This method usually converges to largest eigenpair. Hence it is not suitable to apply this method to find targeted eigenpair in a given interval.

Second-order Arnoldi method

- Second-order Arnoldi method does not give good approximation of eigenpair closer to the lower part of the spectrum.
- It is very difficult to choose size $m$ to find the invariant subspace.
- Again with this method it is not easy to find eigenvalues in a specific interval.
The New Hybrid Method

The Hybrid method has *two* parts.

**First part**
We choose $m \ll n$ sets of random eigenpairs where $\alpha_i \in [a, b]$, and the orthogonal vectors $v_i, \ i = 1, 2, ..., m$, then run few-iterations of Modified Parametrized Newton’s method for each pair.

**Second part**
Use the above eigenvectors and one of the eigenvalue as an initial eigenvectors and shift to run the Jacobi-Davidson method.
The Modified Parametrized Newton (MPN) Method to the Symmetric QEP

For \( f(u, \lambda) = \begin{pmatrix} Q(\lambda)u \\ u^T u - 1 \end{pmatrix} = \begin{pmatrix} (\lambda^2 M + \lambda C + K)u \\ u^T u - 1 \end{pmatrix} \)

Define \( f_s(u, \lambda) = \begin{pmatrix} Q_s(\lambda)u \\ u^T u - 1 \end{pmatrix} \)

\[
= \begin{pmatrix} \lambda(\alpha_1 + \alpha_2)M + C - \alpha_1 \alpha_2 M + K \\ u^T u - 1 \end{pmatrix} u
\]

\[
= \begin{pmatrix} \lambda R_s + T \\ u^T u - 1 \end{pmatrix},
\]

where \( R_s = [(\alpha_1 + \alpha_2)M + C] \) and \( T = [-\alpha_1 \alpha_2 M + K] \).
Let $Q_s(\lambda)u = (\lambda R_s + T)u$.

Jacobian matrix $J_{fs}$ of $f$ which can be calculated as

$$J_{fs}(u, \lambda) = \begin{pmatrix} \lambda R_s + T & R_s u \\ 2u^T & 0 \end{pmatrix}$$

notice $R_s = (\alpha_1 + \alpha_2)M + C = Q'_s(\lambda)$ is the derivative of the quadratic matrix pencil $Q_s(\lambda)$.
MPN Method

MPN iteration is

\[
\begin{pmatrix}
    x^{(i+1)} \\
    \alpha^{(i+1)}
\end{pmatrix} =
\begin{pmatrix}
    x^{(i)} \\
    \alpha^{(i)}
\end{pmatrix} -
\begin{bmatrix}
    Q_s(\alpha^{(i)}) & R_s^{(i)} x^{(i)} \\
    2x^{(i)T} & 0
\end{bmatrix}^{-1}
\begin{pmatrix}
    Q_s(\alpha^{(i)}) x^{(i)} \\
    x^{(i)T} x^{(i)} - 1
\end{pmatrix}
\]

Now choose a parameter \( \tilde{t} > 0 \) and assume \( \alpha^{(i)} \neq 0 \) so that the method takes the form

\[
\begin{pmatrix}
    Q_s(\alpha^{(i)}) & R_s^{(i)} x^{(i)} \\
    2x^{(i)T} & 0
\end{pmatrix} =
\begin{bmatrix}
    I & 0 \\
    0 & \tilde{t}
\end{bmatrix}
\begin{bmatrix}
    0 & R_s^{(i)} x^{(i)} \\
    x^{(i)T} & \frac{1}{\alpha^{(i)}}
\end{bmatrix}
\begin{pmatrix}
    x^{(i)} \\
    \alpha^{(i)}
\end{pmatrix}
\]

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**MPN Method**

**Modified Parametrized Newton’s iteration** for $Q(\lambda)$ now takes the form

\[
\alpha^{(i+1)} = \alpha^{(i)} - \frac{(r^{(i)})^2}{\hat{\beta}^{(i)}} s
\]

\[
x^{(i+1)} = \frac{1}{\hat{\beta}^{(i)}} Q_s^{-1}(\alpha^{(i)}) R_s^{(i)} x^{(i)}
\]

\[
\hat{\beta}^{(i)} = \left\| \left( Q_s^{-1}(\alpha^{(i)}) R_s^{(i)} x^{(i)} \right) \right\|
\]

\[
\beta^{(i)} = x^{(i)^T} Q_s^{-1}(\alpha^{(i)}) R_s^{(i)} x^{(i)} \quad \text{and} \quad r^{(i)} = \frac{\beta^{(i)}}{\hat{\beta}^{(i)}}
\]

choose the value $s$, that reduce the residual at every iteration

where, residual at $i^{th}$ iteration $= Q_s(\alpha^{(i)}) x^{(i)}$
Convergence Criteria

Lemma

\[ \left| \frac{\beta(i)}{\hat{\beta}(i)} \right| \leq 1, \text{ where } \beta(i) = x(i)^T Q_s^{-1}(\alpha(i)) R_s x(i) \text{ and } \hat{\beta}(i) = \left\| \left( Q_s^{-1}(\alpha(i)) R_s x(i) \right) \right\| \]
Convergence Criteria

Theorem

\[
\frac{\|Res^{(i+1)}\|}{\|Res^{(i)}\|} < 1 \text{ if } \tilde{t} = \left(\frac{\beta(i)}{\hat{\beta}(i)}\right)^2 s, \text{ and } 0 \leq s \leq 1.
\]
Numerical experiments on comparison between *The Jacobi-Davidson Method* and *The Hybrid method*. In our experiments, $M = \text{Identity matrix}$, and $C, K = \text{arbitrary symmetric positive definite Toeplitz matrices}$. Order of matrices are 500 and 800.
Example 1

Matrix size \( n = 500 \), Interval \([41.5 \ 43.5]\)

The approximate initial eigenvalue determined by the Parametrized Newton method = 42.0307903.

Exact eigenvalues in interval \([41.5 \ 43.5]\) are:

41.59432, 41.62015, 41.93211, 43.00123.

**TABLE 1: Convergence comparison between the Hybrid method and the Jacobi-Davidson method for Example 1**

<table>
<thead>
<tr>
<th>Methods</th>
<th>Residual</th>
<th>Iteration</th>
<th>Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hybrid</td>
<td>1.9e^{-9}</td>
<td>2</td>
<td>41.93211</td>
</tr>
<tr>
<td>JD</td>
<td>No convergence</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>
Figure: Norm of log of Residual versus Iteration
Example 2

Matrix size(n) = 800, Interval [70 73]
The approximate initial eigenvalue determined by the Parametrized Newton method = 71.063630.
Exact eigenvalues in interval [70 73] are: 71.488815, 72.89726.

**TABLE 2: Convergence comparison between the Hybrid method and the Jacobi-Davidson method for Example 2**

<table>
<thead>
<tr>
<th>Methods</th>
<th>Residual</th>
<th>Iteration</th>
<th>Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hybrid</td>
<td>$6.61e^{-6}$</td>
<td>1</td>
<td>71.488815</td>
</tr>
<tr>
<td>JD</td>
<td>No convergence</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>
Figure: Norm of log of Residual versus Iteration
**Example 3**

Matrix size(n) = 800, Interval [60 62]
The approximate initial eigenvalue determined by the Parametrized Newton method = 60.9958.
Exact eigenvalues in interval [60 62] are:
60.3894, 60.9596, 61.3442, 61.7969, 61.9379.

**TABLE 3: Convergence comparison between the Hybrid method and the Jacobi-Davidson method for Example 3**

<table>
<thead>
<tr>
<th>Methods</th>
<th>Residual</th>
<th>Iteration</th>
<th>Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hybrid</td>
<td>$5.0e^{-6}$</td>
<td>2</td>
<td>60.3894</td>
</tr>
<tr>
<td>JD</td>
<td>No convergence</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>
**Figure:** Norm of log of Residual *verses* Iteration

![Graph showing norm of log of residual versus iteration](image-url)
Observation from the experiment

Numerical experimental results show that the hybrid method converges faster than the Jacobi-Davidson method alone; indeed, in some cases when the Jacobi-Davidson method did not converge at all, the new method worked quite well.
The method is parametric in nature and rate of convergence depends upon the appropriate choice of the parameter. Studies on how to choose it properly to guarantee or accelerate the convergence is currently underway.

- Find efficient way to solve linear system in Modified Parametrized Newton’s method.
- Find the convergence rate of Hybrid method.
- Find all the eigenvalues in the given interval.
Bibliography


THANK YOU