Regular approximations of singular Sturm-Liouville problems

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Abstract

Given any self-adjoint realization S of a singular Sturm-Liouville (S-L) problem, it is possible to construct a sequence $\{S_r\}$ of regular S-L problems with the properties

(i) every point of the spectrum of S is the limit of a sequence of eigenvalues from the spectrum of the individual members of $\{S_r\}$

(ii) in the case when S is regular or limit-circle at each endpoint, a convergent sequence of eigenvalues from the individual members of $\{S_r\}$ has to converge to an eigenvalue of S

(iii) in the general case when S is bounded below, property (ii) holds for all eigenvalues below the essential spectrum of S.

1 Introduction

This paper is a sequel to [1] in which the problem of the numerical computation of eigenvalues of singular limit-circle Sturm-Liouville (S-L) problems is discussed. Here we study the general problem of the approximation of the spectrum of singular S-L problems with eigenvalues of regular S-L problems. In particular, we provide here a proof of [1, Theorem 4.1] which is contained in the proof of Theorem 5.1 given below.

In this paper no limitation is placed on the classification of the end-points of the given S-L problem; either end-point may be regular or singular and, if singular, may be of limit-circle (LC) or limit-point (LP) type.

Given any self-adjoint realization S of a S-L problem, regular or singular, our main result consists of an explicit construction of a sequence of regular S-L problems $\{S_r\}$, with discrete eigenvalues $\{\lambda_n(S_r)\}$, with the following properties:

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(i) If S is semi-bounded from below, then the eigenvalues $\{\lambda_n(S)\}$ below the essential spectrum of S are exactly the limits of the corresponding eigenvalues $\{\lambda_n(S_r)\}$, for each fixed index n, as $r \to \infty$; furthermore, the corresponding eigenprojections converge not only strongly but in norm.

(ii) If S is quasi-regular (LC at both end-points) then all the eigenvalues $\{\lambda_n(S)\}$ of S are the limits of convergent sequences $\{\lambda_{n(r)}(S_r)\}$ as $r \to \infty$.

(iii) All results which we prove without using semi-boundedness of S can be proved in the same way for Dirac systems.

In Section 2 we state and prove certain required results and properties of S-L differential operators and boundary-value problems. The properties of abstract operator convergence required to establish the approximation results are given in Section 3. The case when the endpoints are either regular or LC is discussed in Section 4; with both endpoints LP in Section 5; and with mixed endpoint classification in Section 6.

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2 Differential operators

Let I denote any interval of the real line $\mathbb{R}$ with end points a and b, $-\infty \le a < b \le \infty$. A compact interval is denoted by $[a, b]$. Let M denote the differential expression defined by

$$My = [-(py')' + qy] \text{ on } I, \tag{2.1}$$

where

$$p, q, w : I \to \mathbb{R},\ 1/p, q, w \in L_{loc}(I),\ p \ge 0, w > 0\ a.e.. \tag{2.2}$$

For any subinterval J of I let $AC_{loc}(J)$ denote the set of complex valued functions on J which are absolutely continuous on all compact subintervals of J. In the Hilbert space $L^2(J, w)$ of Lebesgue measurable functions which are square integrable with weight w and with inner product given by

$$(f, g) = \int_J f(t)\bar{g(t)}w(t)dt, \tag{2.3}$$

we define the operator $T(M, J)$ by

$$D(M, J) = \{f \in L^2(J, w) : f, pf' \in AC_{\text{loc}}(J) \text{ and } w^{-1}Mf \in L^2(J, w)\}. \tag{2.4}$$

$$T(M,J)f = w^{-1}Mf, \quad f \in D(M,J). \tag{2.5}$$

The operator $T(M,J)$ is called the *maximal operator* of M on J and its domain $D(M,J)$ is called the *maximal domain* of M on J. It is known [7, Ch. 3] that $D(M,J)$ is dense in $L^2(J,w)$. Thus the adjoint of $T(M,J)$ is well defined. Let

$$T_0(M,J) = T^*(M,J), \qquad D_0(M,J) = D(T_0(M,J)). \tag{2.6}$$

It is also known [7, Ch. 3] that $T_0(M,J)$ is densely defined and

$$T_0^*(M,J) = T(M,J); \tag{2.7}$$

$T_0(M,J)$ is the closure of the symmetric operator $T_0'(M,J)$ defined by restricting $w^{-1}M$ to the domain $D_0'(M,J) = \{f \in D(M,J) : f \text{ has compact support in } int(J)\}$. The operator $T_0'(M,J)$ is called the $pre-minimal$ $operator$ of M on J and $D_0'(M,J)$ the $pre-minimal$ $domain$ of M on J. For the case when $J = I$ we will denote $D(M,I)$ by D or by $D(M)$ when we want to emphasize its dependence on M or by $D(I)$ to emphasize its dependence on I. Similarly for D_0, T_0, T.

For the basic theory of Sturm-Liouville problems including the definition of the Lagrange sesquilinear form, the limit-point (LP) / limit-circle (LC) classification, etc. the reader is referred to Naimark [4] or Weidmann [7]. The deficiency index d of $T_0 = T_0(M,I)$ is 0, 1, or 2. Any self-adjoint extension of T_0 is a restriction of T and conversely, i.e.

$$T_0 \subset S = S^* \subset T. \tag{2.8}$$

Consider the differential equation

$$-(py')' + qy = \lambda wy \ \text{ on } I. \tag{2.9}$$

Proposition 2.1 *Let (2.2) hold; let λ be in $\mathbb{R}$; let $a < a' \leq b' < b$. All self-adjoint realizations of M can be constructed as follows:*

> **Case 1.** *Assume $d = 0$; this occurs if and only if both endpoints are LP. In this case, $T_0(M,I) = T(M,I) = T_0^*(M,I)$ is the only self-adjoint realization of M on I ; thus there are no proper self-adjoint extensions of T_0 or restrictions of T. No boundary conditions are required at either endpoint.*
>
> **Case 2.** *Assume $d = 1$; this case occurs precisely when one endpoint is LP and the other either regular or LC.*

(a) Suppose a is LP and b is regular or LC. If the operator S with domain $D(S)$ satisfies (2.8), then there exists a $\psi \in D(S) \subset D$ such that

(i) ψ is not in D_0

(ii) $[\psi, \psi](b) = 0$

(iii) $D(S) = \{f \in D : [f, \psi](b) = 0\}$

(iv) Furthermore, ψ can be taken to be a real non-trivial solution of (2.9) on the interval $[b', b)$ for some real λ.

(b) Suppose a is regular or LC and b is LP. If the operator S with domain $D(S)$ satisfies (2.8), then there exists a $\psi \in D(S) \subset D$ such that

(i) ψ is not in D_0

(ii) $[\psi, \psi](a) = 0$

(iii) $D(S) = \{f \in D : [f, \psi](a) = 0\}$

(iv) Furthermore ψ can be taken to be a real non-trivial solution of (2.9) on the interval $(a, a']$ for some real λ.

Conversely, in either case (a) or (b), given a ψ in D satisfying (i) and (ii) but not necessarily (iv), the set $D(S)$ defined by (iii) is a self-adjoint domain.

Case 3. *Assume $d = 2$. In this case, each endpoint is either regular or LC. If the operator S with domain $D(S)$ satisfies (2.8), then there exist ψ_1, ψ_2 in $D(S) \subset D$ such that*

(i) ψ_1, ψ_2 are linearly independent modulo D_0;

(ii) $[\psi_j, \psi_k](b) - [\psi_j, \psi_k](a) = 0,\ j, k = 1, 2$;

(iii) $D(S) = \{y \in D : [y, \psi_j](b) - [y, \psi_j](a) = 0,\ j = 1, 2\}$;

(iv) furthermore, $\{\psi_j\}$ can be taken to be solutions of (2.9) for some λ in $\mathbb{R}$ on the interval $(a, a']$ and on the interval $[b', b)$. Note that ψ_j need not be the same solution on $(a, a']$ as on $[b', b)$ and need not be a solution on the whole interval (a', b').

Given $\psi_1, \psi_2 \in D$ satisfying (i) and (ii) but not necessarily (iv), the set $D(S)$ defined by (iii) is a self-adjoint domain. Conversely, given a self-adjoint domain $D(S)$ there exist $\psi_1, \psi_2 \in D(S)$ satisfying (i) and (ii) such that $D(S)$ is defined by (iii).

Proof: Without condition (iv) a proof for all cases can be found in [4, Section 18.1] and in [7, Ch. 4]. That condition (iv) can be added without loss of generality was shown in [3, Theorem 2]. ∎

In general (iii) represents "coupled" boundary conditions. We get "separated" boundary conditions when $\psi_1 = 0$ close to b and $\psi_2 = 0$ close to a. All separated boundary conditions may be written in the form $[y, \psi_1](a) = 0 = [y, \psi_2](b)$ with non-trivial real solutions ψ_j of (2.9) for some real λ.

Let $I_r = (a_r, b_r)$ with $-\infty \leq a \leq a_r < b_r \leq b \leq \infty$, $r \in \mathbb{N} = \{1, 2, 3, \ldots\}$. We are interested in approximating a given self-adjoint realization of M in the space $L^2(I, w)$ with self-adjoint realizations of M (more precisely of the restriction of M to I_r) acting in the spaces $L^2(I_r, w)$. An explicit construction of these approximation operators is given and a new concept called "induced restriction" is introduced.

In all that follows we take

(i)

$$a_{r+1} \leq a_r \text{ and } b_r \leq b_{r+1} \qquad r \in \mathbb{N} \tag{2.10}$$

(ii)

$$\lim_{r\to\infty} a_r = a \text{ and } \lim_{r\to\infty} b_r = b. \tag{2.11}$$

Definition 2.2 Induced restriction operators. *Let S satisfy (2.8). According to Proposition 2.1, the characterization of the domain of $S, D(S)$, depends on the value of the deficiency index d. For this reason, we consider the three cases $d = 0, 1,$ or 2 separately. Let*

$$H = L^2(I; w) \text{ and } H_r = L^2(I_r, w),\ r \in \mathbb{N}.$$

We define the induced restriction operator S_r in H_r for each case as follows:

Case 1. $d = 0$. *In this case both endpoints a and b are LP. To define the induced restrictions on the intervals I_r we consider subcases depending on whether neither, one, or both endpoints of I_r are the same as those of I.*

(i) $a_r = a$ *and* $b_r = b$. *Then* $S_r = S$ *is the induced restriction.*

(ii) $a < a_r$ *and* $b_r = b$. *Choose any ψ_r in $D(M, I_r)$ which is not in $D_0(M, I_r)$ and satisfies $[\psi_r, \psi_r](a_r) = 0$. Let $D(S_r) = \{f \in D(M, I_r) : [f, \psi_r](a_r) = 0\}$. By Proposition 2.1 case 2, the operator $S_r = T(M, I_r)$ restricted to $D(S_r)$ is a self-adjoint operator in H_r. We call S_r an induced restriction of S in H_r. Note that the notation $\psi_r\,(r \in \mathbb{N})$ indicates that the required separated boundary condition at a_r may be chosen arbitrarily for each $r \in \mathbb{N}$.*

(iii) $a = a_r$ *and* $b_r < b$. *Choose any* ψ_r *in* $D(M, I_r)$ *which is not in* $D_0(M, I_r)$ *and satisfies* $[\psi_r, \psi_r](b_r) = 0$. *Define* $D(S_r) = \{f \in D(M, I_r) : [f, \psi_r](b_r) = 0\}$. *By Proposition 2.1, case 2,* $D(S_r)$ *is a self-adjoint domain; the associated operator* S_r *is called an induced restriction of* S *on* I_r *or in* H_r.

(iv) $a < a_r$ *and* $b_r < b$. *In this case we need two "boundary condition functions"* $\psi_{r,j}$, $j = 1, 2$. *Let* $\psi_{r,1}$ *and* $\psi_{r,2}$ *be in* $D(M, I_r)$ *satisfying conditions (i) and (ii) of case 3 of Proposition 2.1 with* $\psi_j = \psi_{r,j}$, $j = 1, 2$. *Define* $D(S_r)$ *by (iii) of case 3 of Proposition 2.1. Then by this Proposition, the maximal operator of* M *on* I_r *restricted to* $D(S_r)$*, which we denote by* S_r*, is self-adjoint in* H_r*; it is called an induced restriction of* S *in* H_r.

Case 2. $d = 1$. *In this case, one endpoint of* I *is LP, the other either regular or LC.*

Assume a *is regular or LC and* b *is LP. By Proposition 2.1 there exists a* ψ *in* $D(M, I)$ *satisfying (i), (ii), (iv) of case 2(b), such that* $D(S) = \{f \in D(M, I) : [f, \psi](a) = 0\}$. *If* $a \le a_r < b_r < b$, *choose* ψ_r *in* $D(M, I_r)$ *to be any non-trivial real solution of (2.9) for a real* λ. *Then define* $D(S_r) = \{f \in D(M, I_r) : [f, \psi](a_r) = 0 = [f, \psi_r](b_r)\}$.

When a *is LP and* b *is regular or LC we proceed similarly. Now* $D(S) = \{f \in D(M, I) : [f, \psi](b) = 0\}$, *where* $\psi \in D(M, I)$ *satisfies (i), (ii) and (iv) of case 2(a) and* ψ_r *is chosen as before: define* $D(S_r) = \{f \in D(M, I_r) : [f, \psi_r](a_r) = 0 = [f, \psi](b_r)\}$.

Note that in both of the above cases, a *LP or* b *LP, (iv) of case 2 of Proposition 2.1 assures that* $[f, \psi](a_r) = 0$ *or* $[f, \psi](b_r) = 0$ *is a non-trivial self-adjoint boundary condition for all* r *sufficiently large. It then follows that the boundary condition near the LP endpoint can be any separated boundary condition with real coefficients. Such conditions have the form*

$$[f, \psi_r](a_r) = 0 \quad \text{or} \quad [f, \psi_r](b_r) = 0.$$

Here we have used a subscript r *on* ψ_r *to emphasize that these boundary condition functions can change with* r.

It is interesting to note that near the LP endpoint, the boundary condition is arbitrary for each r *(unrelated to each other for different* r*'s) but near the non-LP endpoint, the boundary condition is determined by the function* ψ. *In this way the separated boundary condition near the non-LP endpoint is inherited from the condition at the non-LP endpoint through the function* ψ*, whereas the separated boundary condition near the LP endpoint is an arbitrary self-adjoint condition.*

Case 3. $d = 2$. *In this case each endpoint is either regular or LC and the boundary conditions determining the induced operator S_r for all r sufficiently large are inherited from those of S as follows: Let*

$$D(S) = \{f \in D(M, I) : [f, \psi_j](b) - [f, \psi_j](a) = 0,\ j = 1, 2\},$$

where ψ_1, ψ_2 satisfy (i), (ii) and (iv) of case 3 of Proposition 2.1. For this case the induced operators S_r in the spaces H_r are defined by

$$\begin{aligned} D(S_r) &= \{f \in D(M, I_r) : [f, \psi_j](b_r) - [f, \psi_j](a_r) = 0,\ j = 1, 2\}, \\ S_r f &= T(M, I_r)f, \qquad f \in D(S_r), \qquad r \in \mathbb{N}. \end{aligned}$$

Below we study to what extent the spectrum $\sigma(S)$ of S and the spectral projections $E(S, \cdot)$ of S can be approximated by the spectrum $\sigma(S_r)$ of S_r and the spectral projections $E(S_r, \cdot)$ of S_r, respectively. It is clear that when $a < a_r < b_r < b$, so that S_r is regular for $r \in N$, no single operator S_r can be a "good" approximation of S, except for a severely restricted class of operators S, e.g. when S itself is regular or when S is singular but each singular endpoint is LC and nonoscillatory. (In this case $\sigma(S)$ is discrete and bounded below as in the regular case.) We will show below that for any singular operator S satisfying (2.8), a sequence of regular operators $\{S_r\}$ in H_r can be constructed which approximates S. In particular, such that the sequence of spectra $\sigma(S_r)$ "converges" to $\sigma(S)$ as $r \to \infty$.

Proposition 2.3 *Let the assumptions and notation of Proposition 2.1 hold. Then for any interval $I_r = (a_r, b_r)$ with $a \le a_r < a' < b' \le b_r \le b$, and for all cases $d = 0, 1$ or 2, the induced restriction S_r of S defined in Definition 2.2 is a self-adjoint operator in the space H_r satisfying $T_0(M, I_r) \subset S_r = S_r^* \subset T(M, I_r)$.*

Proof: This follows from Definition 2.2 and the characterization of self-adjoint realizations of M given in Proposition 2.1. In cases 2 and 3 and with ψ and $\{\psi_r\}$ given the symmetry property (ii) may hold on (a, b) but not on (a_r, b_r). Similarly, the linear independence property (i) may hold on (a, b) but not on (a_r, b_r). With property (iv) added (without loss of generality) to (i) and (ii), it follows that (i) holds on (a_r, b_r) if it holds on (a, b), and also (ii) holds on (a_r, b_r) if it holds on (a, b); both for r sufficiently large so that $a \le a_r \le a'$ and $b' \le b_r \le b$.

The detailed verification of this Proposition for all three cases $d = 0, d = 1$, and $d = 2$ is straightforward and hence omitted. ∎

The induced operators S_r act in the space $H_r = L^2(I_r, w)$, where we identify H_r with the subspace of $H = L^2(I, w)$ obtained by extending the functions in H_r to be zero on $I \setminus I_r$. With each one of these operators S_r we associate an operator S_r' in the space H as follows

$$S_r' = S_r \oplus \Theta_r = S_r P_r \tag{2.12}$$

where P_r is the orthogonal projection of H onto H_r, Θ_r is the zero operator in the space

$$H_r^{\perp} = L^2((a, a_r), w) \oplus L^2((b_r, b), w); \tag{2.13}$$

it being understood that the first space in (2.13) is $\{0\}$ when $a_r = a$ and the second when $b_r = b$. When both $a_r = a$ and $b_r = b$, we take $S_r = S_r' = S$. In (2.12) and (2.13) the circled plus symbol denotes the orthogonal sum.

Note that the operators S_r' are self-adjoint operators in the space H with dense domains given by

$$D(S_r') = D(S_r) \oplus H_r^{\perp}. \tag{2.14}$$

It is the operators S_r' rather than S_r that will be shown to converge to S in a certain sense (cf.section 3). We call S_r' the *induced restriction* of S in H.

Note that the extension of S_r in H_r to S_r' in H is not without complication if I_r is strictly contained in I; for then S_r' has zero as an eigenvalue of infinite multiplicity whether or not zero is in the spectrum of S_r.

3 Abstract operator convergence

In this section we consider certain results on the convergence of sequences of bounded and unbounded operators in an abstract separable Hilbert space. Our main references for the results quoted here are the books by Kato [2], Reed and Simon [5] and Weidmann [6].

Let H be a complex separable Hilbert space; let T_r ($r \in \mathbb{N}$) and T denote self-adjoint operators in H. These operators may be bounded or unbounded but in the context of this paper are best thought of as unbounded in H.

Definition 3.1 *The sequence $\{T_r : r \in \mathbb{N}\}$ is strong resolvent convergent (SRC) to T in H if for some $z \in \mathbb{C} \setminus \mathbb{R}$*

$$\{(T_r - zI)^{-1} f : r \in \mathbb{N}\} \xrightarrow{H} (T - zI)^{-1} f \text{ as } r \to \infty \tag{3.1}$$

for all $f \in H$.

A sufficient condition for a sequence $\{T_r : r \in \mathbb{N}\}$ to be SRC to T is given in

Theorem 3.2 *Let H, $\{T_r : r \in \mathbb{N}\}$ and T be given as above. Suppose there exists a linear manifold $C(T)$ of H such that*

(i) $C(T)$ is a core of T

(ii) if $f \in C(T)$, then there exists $r_0 \in \mathbb{N}$ (where r_0 may depend upon f) such that $f \in D(T_r)$ for all $r \geq r_0$

(iii) $\{T_r f : r \in \mathbb{N}\} \overset{H}{\to} Tf$ for all $f \in C(T)$;

then $\{T_r : r \in \mathbb{N}\}$ is SRC to T in H.

Proof: See [5, Theorem VIII 25(a)] or [6, 9.16(i)]. ∎

For any self-adjoint operators $\{T_r : r \in \mathbb{N}\}$ in Hilbert spaces H_r and for any self-adjoint T in a Hilbert space H we make the following definition:

Definition 3.3 *(i) The sequence $\{T_r : r \in \mathbb{N}\}$ is spectral included for T if for any λ in $\sigma(T)$ there exists a sequence $\{\lambda_r : r \in \mathbb{N}\}$ with $\lambda_r \in \sigma(T_r)$ $(r \in \mathbb{N})$ such that $\lim_{r\to\infty} \lambda_r = \lambda$.*

(ii) The sequence $\{T_r : r \in \mathbb{N}\}$ is spectral exact for T if it is spectral included and if any limit-point of a sequence $\{\lambda_r : r \in \mathbb{N}\}$, with $\lambda_r \in \sigma(T_r)$ $(r \in \mathbb{N})$, belongs to $\sigma(T)$.

(iii) We say that the sequence $\{T_r : r \in \mathbb{N}\}$ is spectral exact for T on some λ-interval (α, β), $-\infty \leq \alpha < \beta \leq \infty$, if this property holds on (α, β).

We pass now to the spectral implications of the convergence of the sequence $\{T_r : r \in \mathbb{N}\}$ to T in the case of SRC.

Theorem 3.4 *Let the sequence $\{T_r : r \in \mathbb{N}\}$ and T in H be given as above. If $\{T_r : r \in \mathbb{N}\}$ is SRC to T in H, then the sequence $\{T_r : r \in \mathbb{N}\}$ is spectral included for T.*

Proof: See [5, Theorem VIII.24(a)]. ∎

For any self-adjoint operator T in H let $\{E(T, \lambda) : \lambda \in \mathbb{R}\}$ denote the spectral resolution of the identity for T; see [5, Theorems VIII 5 and 6] or [6, Section 7.3]. We have then

Theorem 3.5 *Let $\{H_r = H : r \in \mathbb{N}\}$. Let the sequence $\{T_r : r \in \mathbb{N}\}$ and T be given as above ; let $\{T_r : r \in \mathbb{N}\}$ be SRC to T. Then if λ is not an eigenvalue of T the sequence of projection operators $\{E(T_r, \lambda) : r \in \mathbb{N}\}$ converges strongly to $E(T, \lambda)$, i.e. for all $f \in H$*

$$\|E(T_r, \lambda)f - E(T, \lambda)f\| \to 0 \text{ as } r \to \infty. \tag{3.2}$$

Proof: See [5, Theorem VIII 24] or [6, Theorem 9.19]. ∎

Let P_r be the orthogonal projection of $H = L^2(I, w)$ onto the subspace $H_r = L^2(I_r, w)$ for all r in $\mathbb{N}$.

Theorem 3.6 *Assume that S is a self-adjoint realization of M on I and S_r are induced restrictions of S in H_r, S_r' the corresponding induced restrictions in H. Assume further that the sequence $\{S_r' : r \in \mathbb{N}\}$ is SRC to S. Then for every real λ which is not an eigenvalue of S we have*

$$E(S_r, \lambda)P_r \xrightarrow{s} E(S, \lambda), \tag{3.3}$$

and the sequence $\{S_r\}$ is spectral included for S.

Proof: The strong convergence result follows from the following identities, which take into account the eigenvalue at 0, of infinite multiplicity, for S_r'

$$E(S_r', \lambda) = E(S_r, \lambda)P_r \text{ for } \lambda < 0, \tag{3.4}$$

$$E(S_r', \lambda) = E(S_r, \lambda)P_r + (I - P_r) \text{ for } \lambda \geq 0, \tag{3.5}$$

and

$$P_r \xrightarrow{s} I. \tag{3.6}$$

The spectral inclusion of $\{S_r : r \in \mathbb{N}\}$ for S follows from the following considerations: Let

$$S_r^{'}(c) = S_r \oplus cI_r \quad \text{for } r \in \mathbb{N} \tag{3.7}$$

(i.e. $S_r^{'} = S_r^{'}(0)$). Then the sequence $\{S_r^{'}(c) : r \in \mathbb{N}\}$ is SRC to S and therefore spectral included for S. This together with

$$\sigma(S_r'(c)) = \sigma(S_r) \cup \{c\}. \tag{3.8}$$

implies that $\{S_r\}$ is spectral included for S. ∎

4 Each endpoint is regular or LC

In this section we assume that M is regular or LC at each endpoint a and b of I. Let S satisfy

$$T_0(M, I) = T_0 \subset S = S^* \subset T = T(M, I), \quad I = (a, b). \tag{4.1}$$

We want to approximate S with operators on subintervals of I. Let $a_r, b_r, r \in \mathbb{N}$, satisfy

$$a \leq a_r < b_r \leq b \tag{4.2}$$

and, see Section 2 above,

$$\{a_r,\, r \in \mathbb{N}\} \text{ converges to } a, \quad \{b_r, r \in \mathbb{N}\} \text{ converges to } b. \tag{4.3}$$

Let

$$H = L^2(I, w) \text{ and } H_r = L^2(I_r, w),\ I_r = (a_r, b_r). \tag{4.4}$$

Theorem 4.1 *(a) Let S satisfy (4.1), let $A_r, r \in \mathbb{N}$, be any self-adjoint extension of $T_0(M, I_r)$ in H_r i.e.*

$$T_0(M, I_r) \subset A_r = A_r^* \subset T(M, I_r), \quad r \in \mathbb{N}; \tag{4.5}$$

and let A_r be extended to $A_r' = A_r'^*$ in H as in (2.12).

Assume the sequence $\{A_r' : r \in \mathbb{N}\}$ is SRC to S in H. Then

- *(i) for any z in $\mathbb{C}\backslash\mathbb{R}$ the operators $(S-zI)^{-1}$ and $(A_r-zI)^{-1}P_r,\ r \in \mathbb{N}$, are Hilbert-Schmidt integral operators in H ;*
- *(ii) the sequence $\{(A_r - zI)^{-1}P_r : r \in \mathbb{N}\}$ converges to $(S - zI)^{-1}$ in the Hilbert-Schmidt norm ;*
- *(iii) the sequence $\{A_r : r \in \mathbb{N}\}$ is spectral exact for S;*
- *(iv) for $\lambda, \mu \in \mathbb{R}$ not eigenvalues of S, the sequence $\{(E(A_r, \lambda) - E(A_r, \mu))P_r : r \in \mathbb{N}\}$ converges to $E(S, \lambda) - E(S, \mu)$ not only strongly (see Theorem 3.5) but in norm.*

(b) Let S be defined by (iii) of case 3 of Proposition 2.1 in section 2 with "boundary condition" functions ψ_1, ψ_2 in D satisfying (i), (ii) and (iv) of this Proposition. Let S_r' on H, $r \in \mathbb{N}$, be the induced restrictions of S as defined in section 2; then S_r' is self-adjoint in H for each r sufficiently large and the sequence $\{S_r' : r \in \mathbb{N}\}$ is SRC to S; hence, by part (a), the sequence $\{S_r : r \in \mathbb{N}\}$ is spectral exact for S.

Proof:

(a) For z in $\mathbb{C}\setminus\mathbb{R}$, let u_1, u_2 be a fundamental set of solutions of (2.9) with $\lambda = z$. By Weidmann [7, ch. 7], the kernels of the resolvents $(S - zI)^{-1}$ and $(A_r - zI)^{-1}$ have the form

$$R(s,t,z) = \begin{cases} \displaystyle\sum_{i,j=1}^{2} c_{ij} u_i(s) u_j(t) \text{ for } a < s \leq t < b \\ \displaystyle\sum_{i,j=1}^{2} d_{ij} u_i(s) u_j(t) \text{ for } a < t < s < b \end{cases}$$

$$R_r(s,t,z) \quad = \quad \begin{cases} \sum_{i,j=1}^{2} c_{ij}^r u_i(s)u_j(t) \text{ for } a_r < s \le t < b_r \\ \sum_{i,j=1}^{2} d_{ij}^r u_i(s)u_j(t) \text{ for } a_r < t < s < b_r \end{cases}$$

for some constants $c_{ij},\ d_{ij},\ c_{ij}^r,\ d_{ij}^r,\ i,j=1,2$, in $\mathbb{C}$.

Let J_1, J_2 be disjoint closed subintervals of (a,b) with J_1 to the left of J_2. Let Q_j be the orthogonal projection of $L^2(I,w)$ onto $L^2(J_j,w)$, $j=1,2$; let χ_j denote the characteristic function of J_j, $j=1,2$. Then for f in $L^2(I,w)$ and r sufficiently large we have

$$\begin{aligned}(Q_1(S-zI)^{-1}Q_2f)(s) \quad &= \quad \chi_1(s)\sum_{i,j=1}^{2} c_{ij}u_i(s)\int_{J_2} u_j(t)f(t)w(t)\,dt, \\ (Q_1(A_r-zI)^{-1}Q_2f)(s) \quad &= \quad \chi_1(s)\sum_{i,j=1}^{2} c_{ij}^r u_i(s)\int_{J_2} u_j(t)f(t)w(t)\,dt,\end{aligned}$$

and

$$Q_1(A_r-zI)^{-1}Q_2f \overset{H}{\to} Q_1(S-zI)^{-1}Q_2f \text{ as } r\to\infty.$$

This implies that

$$c_{ij}^r \to c_{ij} \text{ as } r\to\infty, \qquad i,j=1,2.$$

Similarly we get

$$d_{ij}^r \to d_{ij} \text{ as } r\to\infty, \qquad i,j=1,2.$$

The Hilbert-Schmidt convergence follows. This easily implies the convergence of the spectrum and the desired norm convergence of the spectral projections.

(b) Define (the notation here is that of Section 2 above)

$$\begin{aligned} C(S) = \{f\in D : f = c_1\psi_1 + c_2\psi_2 \text{ in } (a,a'] \text{ and in } [b',b) \\ \text{for some constants } c_1,\ c_2 \text{ which depend on } f\}; \end{aligned} \tag{4.6}$$

then $C(S)$ is a core of S. This follows from the fact that

$$\dim(D(S) \bmod D_0) = 2$$

and

$$C(S) = D_0' + N$$

where $\dim N = 2$, $N \cap D_0 = \{0\}$. Hence the closure of T restricted to $C(S)$ is a symmetric and at least 2-dimensional extension of T_0.

Also $C(S)$ satisfies the criteria (ii) and (iii) of Theorem 3.2: Given f in $C(S)$ we have that $f \in D(S'_r)$ for all sufficiently large r and $S'_r f = \chi_r S f \to Sf$ in H as $r \to \infty$. Therefore $\{S'_r : r \in \mathbb{N}\}$ is SRC to S in H by Theorem 3.2.

This completes the proof of Theorem 4.1. ∎

Remarks

1. Since each endpoint is regular or LC, the spectra of S and S_r ($r \in \mathbb{N}$) are all discrete. All spectra are unbounded above but each spectrum may or may not be bounded below (see [7, Theorem 7.11)

   (i) Suppose the spectrum of S is bounded below; then the spectrum of S_r is bounded below for each $r \in \mathbb{N}$. This occurs if and only if each endpoint of the differential equation (2.9) for some real λ is non-oscillatory; see [1, pages 10 and 11] and [7, ch. 14]. Let

   $$\sigma(S) = \{\lambda_n(S) : n \in \mathbb{N}_0\}, \quad \sigma(S_r) = \{\lambda_n(S_r) : n \in \mathbb{N}_0\} \qquad r \in \mathbb{N}$$

   where $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Let these eigenvalues be ordered in the usual way:

   $$-\infty < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \text{ with } \lambda_n \to \infty \text{ as } n \to \infty.$$

   In this case the spectral convergence of Theorem 4.1(b) is very simple: for each $n \in \mathbb{N}_0$ we have

   $$\{\lambda_n(S_r) : r \in \mathbb{N}\} \to \lambda_n(S) \text{ as } r \to \infty.$$

   (ii) Suppose the spectrum of S is not bounded below; this occurs if and only if the differential equation (2.9) is oscillatory at one or both endpoints a, b for some real λ. Let

   $$\sigma(S) = \{\lambda_n(S) : n \in \mathbb{Z}\} \text{ where } \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

   Let these eigenvalues be ordered in the usual way

   $$\cdots \leq \lambda_{-2} \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

   with $\lambda_n \to -\infty$ as $n \to -\infty$ and $\lambda_n \to +\infty$ as $n \to +\infty$. Choose the endpoints a_r, b_r for all $r \in \mathbb{N}$ such that $\sigma(S_r)$ is bounded below. This can be done by making sure no endpoint of I_r is oscillatory, e.g. let $a < a_r < b_r < b$, $r \in \mathbb{N}$. Let $\sigma(S_r) = \{\lambda_n(S_r); n \in \mathbb{N}_0\}$, $r \in \mathbb{N}$. In this case the spectral convergence of Theorem 4.1(b) is more complicated. For any fixed $n \in \mathbb{N}_0$ we have

   $$\lambda_n(S_r) \to -\infty \text{ as } r \to \infty.$$

Nevertheless, given any eigenvalue $\lambda_n(S)$ for fixed $n \in \mathbb{Z}$ there exists an index sequence $\{n(r) : r \in \mathbb{N}\}$ which depends on n, such that

$$\lambda_{n(r)}(S_r) \to \lambda_n(S) \text{ as } r \to \infty.$$

2. Theorem 4.1 part (b) gives an explicit construction of an approximating sequence $\{S_r : r \in \mathbb{N}\}$ such that the corresponding sequence $\{S'_r : r \in \mathbb{N}\}$ converges to S in the sense of SRC; this for any sequence $\{a_r : r \in \mathbb{N}\}$ converging to a and any sequence of $\{b_r : r \in \mathbb{N}\}$ converging to b. Given such sequences $\{a_r\}$ and $\{b_r\}$ there may be other approximations that converge to S in the sense of SRC as well.

   It is interesting to note that when this construction is applied to the case when both endpoints a, b are regular and $a < a_r < b_r < b$ and S is determined by Dirichlet boundary conditions $y(a) = 0 = y(b)$, the approximating problems S_r constructed above are not, in general, determined by Dirichlet conditions $y(a_r) = 0 = y(b_r)$. However, in this case the method of Theorem 4.1 part (a) could be employed to show that Dirichlet approximations also converge SRC. This can be seen as follows: The resolvents of R_r are given by

$$R_r(s,t,z) = W(y_{b_r}, y_{a_r})^{-1} \begin{cases} y_{a_r}(s) y_{b_r}(t) & \text{for } a_r \le s \le t \le b_r, \\ y_{b_r}(s) y_{a_r}(t) & \text{for } a_r \le t \le s \le b_r\,, \end{cases}$$

   where y_{a_r} and y_{b_r} are the solutions of $My = zwy$ satisfying $y_{a_r}(a_r) = 0$, resp. $y_{b_r}(b_r) = 0$; since these and the Wronskian $W(y_{b_r}, y_{a_r})$ are continuously dependent on a_r, resp. b_r, SRC of S'_r to S follows.

5 The LP case at both endpoints

When one or both endpoints of I is LP, the spectrum of any associated operator S need not be discrete. In general it consists of eigenvalues and of essential spectrum σ_e, see [7, ch. 15]. Some of the eigenvalues may be imbedded in σ_e or lie in gaps of σ_e. In this section and the next we obtain results which yield spectral inclusion for the entire spectrum of S, and spectral exactness for that part of the spectrum of S which lies below σ_e.

Theorem 5.1 *Let p, q, w satisfy (2.2), let M be given by (2.1) and let $T_0 = T_0(M, I)$ and $T = T(M, I)$ be defined as in section 2. Assume both endpoints a, b of I are LP. Let $S = T_0 = T$; let $I_r = (a_r, b_r)$ with $a \le a_r < b_r \le b$ with $\{a_r : r \in \mathbb{N}\}$ converging to a and $\{b_r : r \in \mathbb{N}\}$ converging to b. Let $\{S_r : r \in \mathbb{N}\}$ be any sequence of induced restrictions of S in H_r as constructed in section 2 above, $\{S'_r : r \in \mathbb{N}\}$ the corresponding sequence of induced restrictions in H. Then the sequence* *(i)* $\{S'_r : r \in \mathbb{N}\}$ *is SRC to* S *in* H*,*

(ii) $\{S_r : r \in \mathbb{N}\}$ *is spectral included for* S *but, in general, not spectral exact for* S*.*

Proof: Define

$$C(S) = \{f \in D(M, I) : f \text{ has compact support in } I = (a, b)\}.$$

Note that $C(S)$ is the pre-minimal domain of M on I denoted by $D'_0 = D'_0(M, I)$. Since $S = T_0$ and T_0 is the closure of the pre-minimal operator $T'_0 = w^{-1}M$ restricted to D'_0, it is clear that $C(S)$ is a core of S. For any $f \in C(S)$ there exists an r_0 such that for all $r \geq r_0$ we have $f \in D(S'_r)$ and $S'_r f = Sf$ in H. Hence, by Theorem 3.2, the sequence $\{S'_r : r \in \mathbb{N}\}$ is SRC to S, and by Theorem 3.6 the sequence $\{S_r\}$ is spectral included for S. ∎

The fact that this convergence is, in general, not spectral exact follows from the following example.

Example 5.2 *Let* $p(t) = 1 = w(t)$, $q(t) \geq 0$, $-\infty < t < \infty$, $q \in L_{\text{loc}}(-\infty, \infty)$*. Then* M *given by*

$$My = -y'' + qy \text{ on } I = (-\infty, \infty)$$

is LP at both end-points. Let $S = T = T_0$ *be the unique self-adjoint realization of* M *in* $L^2(-\infty, \infty)$ *and let* S_r *in* $L^2(-r, r)$ *be determined by the boundary conditions*

$$[y, u](-r) = 0, \quad [y, u](r) = 0, \quad r \in \mathbb{N},$$

where u *is any nontrivial real solution of* $-y'' + qy = -y$*. Then* $-1 \in \sigma(S_r)$, $r \in \mathbb{N}$*, and* $\sigma(S) \subset [0, \infty)$*. This in spite of the fact that the sequence* $\{S_r \oplus \Theta_r : r \in \mathbb{N}\}$*, where* Θ_r *is the zero operator in* $L^2(-\infty, -r) \oplus L^2(r, \infty)$*, is SRC to* S *in* $L^2(-\infty, \infty)$*.*

We will show in Theorem 5.3 below that if the boundary conditions in the above example are replaced by

$$y(-r) = 0 \text{ and } y(r) = 0$$

and the new sequence $\{S_r\}$ is defined accordingly, then the sequence $\{S_r\}$ is spectral exact for S in H; in particular a convergent sequence of eigenvalues $\{\lambda_{n(r)}(S_r) : r \in \mathbb{N}\}$ must have a limit which is in $\sigma(S)$.

Remarks

1. Note that, so far, in this LP/LP case the choice of separated or coupled boundary conditions to determine each S_r is entirely arbitrary. This contrasts with the LC/LC case where the boundary conditions to determine S_r are inherited from the boundary conditions of S.

2. Note that by Theorem 3.6 when λ is not an eigenvalue of S, the projections $\{E(S_r, \lambda)P_r\}$ of $\{S_r\}$ converge strongly to the spectral projection $E(S, \lambda)$ of S.

Theorem 5.3 *Let all the hypotheses of Theorem 5.1 hold. In addition assume that*

(i) the symmetric operator T_0 is bounded below in H

(ii) the induced restrictions S_r for $a \leq a_r < b_r \leq b$ are determined as follows: $f(a_r) = 0$ when $a < a_r$, with no condition when $a = a_r$ and $f(b_r) = 0$ when $b_r < b$, with no condition when $b_r = b$.

Then, in addition to the conclusions of Theorem 5.1, the sequence $\{S_r : r \in \mathbb{N}\}$ is spectral exact for S below the essential spectrum σ_e of S. Also, for any λ below σ_e which is not an eigenvalue of S, the projections $\{E(S_r, \lambda)P_r : r \in \mathbb{N}\}$ converge to $E(S, \lambda)$ not only strongly but in norm.

In particular in the case when the spectrum of S is discrete i.e. $\sigma_e(S) = \emptyset$, the sequence $\{S_r : r \in \mathbb{N}\}$ is spectral exact for S; thus

$$\lim_{r \to \infty} \lambda_n(S_r) = \lambda_n(S) \quad (n \in \mathbb{N}_0).$$

Proof: Let $D(Q_r)$, $r \in \mathbb{N}$ and $D(Q)$ denote the form domains of S_r and S, respectively. Since $D(Q_r)$ and $D(Q)$ are the closures of the pre-minimal domains $D_0'(M, I_r)$ and $D_0'(M, I)$, respectively, with respect to the same form norm it follows that $Q_r \subset Q,\ r \in \mathbb{N}$. Since from the spectral theorem it follows easily that the range of $E(S_r, \lambda)$ is a maximal subspace of $D(Q_r)$ consisting of functions f satisfying $Q_r(f, f) \leq |f|^2$, and similarly for Q, we may conclude that

$$\dim E(S_r, \lambda) \leq \dim E(S, \lambda) < \infty$$

for all λ lying below $\sigma_e(S)$. The conclusion of Theorem 5.3 now follows from [2, Lemma VIII 1.24]. ∎

6 One endpoint is LP, the other regular or LC

Without loss of generality we take the endpoint a to be regular or LC and the endpoint b to be LP; the converse situation follows the same analysis with appropriate changes at the endpoints a and b. Firstly we approximate on intervals with endpoints a_r and b where $a_r \geq a\,(r \in \mathbb{N})$; the results are given in Theorem 6.1. Secondly we approximate on intervals with endpoints a_r and b_r where $a_r \geq a$ but $b_r < b\,(r \in \mathbb{N})$; the results are given in Theorems 6.2 and 6.4. In this scheme Theorem 6.1 bears a strong comparison with Theorem 4.1, Theorems 6.2 and 6.4 with the form of Theorems 5.1 and 5.3.

Theorem 6.1 *Let p, q, w satisfy (2.2); let M be defined by (2.1) and let $T_0 = T_0(M, I)$ and $T = T(M, I)$ with $I = (a, b)$ be defined as in section 2. Let S satisfy (4.1).*

Let $I_r = (a_r, b)$ with $a \leq a_r < b$, $(r \in \mathbb{N})$ and let the sequence $\{a_r\}$ converge to a for both parts (a) and (b) which follow:

(a) Let the operators A_r and $A'_r\,(r \in \mathbb{N})$ be defined as in Theorem 4.1 (a), with $A'_r = A'^*_r$ in H. Assume the sequence $\{A'_r : r \in \mathbb{N}\}$ is SRC to S in H. Then all the consequences (i) to (iv) of Theorem 4.1 (a) hold for the sequence $\{A_r : r \in \mathbb{N}\}$ here defined.

(b) Let S be defined by (iii) of case 2 of Proposition 2.1 in section 2 with "boundary condition" function ψ in D satisfying (i), (ii) and (iv) of this Proposition. Let S_r in $H_r(r \in \mathbb{N})$ be the induced restrictions of S, where S is defined in case 2, part (b) of Proposition 2.1 of section 2; then the corresponding sequence $\{S'_r : r \in \mathbb{N}\}$ of induced restrictions in H is SRC to S in H; hence, by part (a),$\{(S_r - zI)^{-1}P_r\}$ converges to $\{(S - zI)^{-1}\}$ in norm and the sequence $\{S_r\}$ is spectral exact for S.

Proof:

(a) This follows as for theorem (4.1)(a). Let $z \in \mathbb{C}\backslash\mathbb{R}$. Let u_a be a solution of $-(py')' + qy = zwy$ on (a, b) which satisfies the boundary condition at a which determines S; let u_b be a solution of $-(py')' + qy = zwy$ on (a, b) which lies in $L^2((b', b), w)$ for some $b', a < b' < b$. Since b is LP, the solution u_b is unique up to constant multiples. Also u_a is unique up to constant multiples. Choose these constants so that $W(u_a, u_b) = 1$. This is possible since u_a and u_b are linearly independent: their linear dependence would imply that u_a is in D and this would mean that the nonreal number z is an eigenvalue of S. From [7, ch. 7] the kernels of the resolvents of $(S - zI)^{-1}$ and $(S_r - zI)^{-1}$ are given by, with $c_r \in \mathbb{C}$ $(r \in \mathbb{N})$

$$R(s,t,z) = \begin{cases} u_a(s)u_b(t) \text{ for } a < s \leq t < b, \\ u_b(s)u_a(t) \text{ for } a < t < s < b, \end{cases}$$

$$R_r(s,t,z) = \begin{cases} (u_a(s) + c_r u_b(s))u_b(t) \text{ for } a_r < s \leq t < b, \\ u_b(s)(u_a(t) + c_r u_b(t)) \text{ for } a_r < t < s < b. \end{cases}$$

Hence

$$R(s,t,z) - R_r(s,t,z) = \begin{cases} u_a(s)u_b(t) & \text{for} \quad a < s < a_r,\ s \leq t < b, \\ -c_r u_b(s)u_b(t) & \text{for} \quad a_r \leq s < t < b, \\ -c_r u_b(s)u_b(t) & \text{for} \quad a_r \leq t < s < b, \\ u_b(s)u_a(t) & \text{for} \quad a < t < a_r,\ t < s < b. \end{cases}$$

From this and the convergence

$$(A_r - zI)^{-1}P_r \xrightarrow{s} (S - zI)^{-1} \text{ as } r \to \infty$$

it follows that

$$\{c_r\} \to 0 \text{ as } r \to \infty.$$

The convergence $(A_r - zI)^{-1}P_r \to (S - zI)^{-1}$ as $r \to \infty$ with respect to the Hilbert-Schmidt norm now follows. It then follows that the sequence $\{A_r : r \in \mathbb{N}\}$ is spectral exact for S.

(b) The proof follows from (a) above as for the proof of Theorem 4.1(b).

This completes the proof of Theorem 6.1. ▮

Theorem 6.2 *Let the basic conditions given in the first paragraph of Theorem 6.1 hold.*

Let $I_r = (a_r, b_r)$ with $a \le a_r < b_r < b$, $(r \in \mathbb{N})$, and let $\{a_r\}$ converge to a, $\{b_r\}$ to b as $r \to \infty$.

Let $\{S_r\}$ be any sequence of induced restrictions of S in H_r as constructed in case 3 of section 2 above and $\{S_r'\}$ the corresponding induced restrictions in H. Then the sequence

(i) $\{S_r' : r \in \mathbb{N}\}$ is SRC to S in H

(ii) $\{S_r : r \in \mathbb{N}\}$ is spectral included for S but, in general, not spectral exact for S.

Proof: Let ψ be the "boundary condition" of Theorem 6.1 part (b). Define

$$\begin{aligned} C(S) \;=\; \{ & f \in D(M, I) : f(t) = c\,\psi(t) \text{ for } a < t \le a'(f), c \in \mathbb{C}, \\ & \text{and } f(t) = 0 \text{ for } b'(f) \le t < b \text{ for some } a'(f),\ b'(f) \text{ depending on } f\}. \end{aligned}$$

Then it is easy to see that $C(S)$ is a core of S which satisfies the criteria for the sequence $\{S_r' : r \in \mathbb{N}\}$ to be SRC to S in H given in Theorem 3.2 above. The spectral inclusion follows from this result and the spectral exactness fails in general as can be seen from the following example. ▮

Example 6.3 *Let $p(t) = 1 = w(t)$, $q(t) \ge 0$, $0 \le t < \infty$, $q \in L_{\text{loc}}[0, \infty)$. Then M given by*

$$My = -y'' + qy \text{ on } I = [0, \infty)$$

is regular at 0 and LP at ∞. Let S be the self-adjoint realization of M in $L^2(0, \infty)$ determined by the boundary condition

$$y(0) = 0$$

and let S_r *in* $L^2(0,r)$ *be determined by the boundary conditions*

$$y(0)=0, \qquad [y,u](r)=0 \text{ on } I_r=[0,r], \qquad r\in\mathbb{N},$$

where u *is the real solution of* $-y''+qy=-y$ *determined by the initial condition* $y(0)=0$, $y'(0)=1$. *Then* $-1\in\sigma(S_r)$, $r\in\mathbb{N}$, *and* $\sigma(S)\subset[0,\infty)$. *This in spite of the fact that the sequence* $\{S_r\oplus\Theta_r : r\in\mathbb{N}\}$, *where* Θ_r *is the zero operator in* $L^2(r,\infty)$, *is SRC to* S *in* $L^2(0,\infty)$.

Theorem 6.4 *Let all the hypotheses of the first two paragraphs of Theorem 6.2 hold. In addition assume that*

(i) the symmetric operator T_0 *is bounded below in* H

(ii) the induced restrictions S_r *are determined by*

$$[f,\psi](a_r)=0 \text{ and } f(b_r)=0, \quad r\in\mathbb{N}$$

where ψ *is the "boundary condition" function which determines* S *according to case 2 part b of Proposition 2.1 of section 2.*

Then, in addition to the conclusion of Theorem 6.2, the sequence $\{S_r : r\in\mathbb{N}\}$ *is spectral exact for* S *below the essential spectrum* σ_e *of* S. *In addition, for any* λ *below* σ_e *which is not an eigenvalue of* S, *the projections* $\{E(S_r,\lambda)P_r : r\in\mathbb{N}\}$ *converge to* $E(S,\lambda)$ *not only strongly but in norm.*

In particular in the case when the spectrum of S *is discrete i.e.* $\sigma_e(S)=\emptyset$, *the sequence* $\{S_r : r\in\mathbb{N}\}$ *is spectral exact for* S; *thus*

$$\lim_{r\to\infty}\lambda_n(S_r)=\lambda_n(S) \quad (n\in\mathbb{N}_0).$$

Proof: Observe that, for all $r\in\mathbb{N}$, the closed form corresponding to S_r of Theorem 6.4 is a restriction of the closed form corresponding to S_r of Theorem 6.1. Therefore we have for λ below $\sigma_e(S)$ that, for r sufficiently large,

$$\dim E(S_r,\lambda) \;\le\; \dim E(S_r \text{ of Theorem 6.1, } \lambda) \;=\; \dim E(S,\lambda)<\infty,$$

and

$$\{E(S_r,\lambda)P_r\} \xrightarrow{s} E(S,\lambda) \text{ as } r\to\infty.$$

Hence

$$\|E(S_r,\lambda)P_r - E(S,\lambda)\| \to 0 \text{ as } r\to\infty.$$

This completes the proof of Theorem 6.4. ∎

In a forthcoming paper [8], J. Weidmann and G. Stolz will study the approximation of arbitrary isolated eigenvalues (including those which lie in gaps of σ_e) by eigenvalues of regular problems. These authors will also study the general case when T_0 is unbounded above and below as well as Dirac systems.

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