Computing Eigenvalues of Singular Sturm-Liouville Problems

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Abstract

We describe a new algorithm to compute the eigenvalues of singular Sturm-Liouville problems with separated self-adjoint boundary conditions for both the limit-circle nonoscillatory and oscillatory cases. Also described is a numerical code implementing this algorithm and how it compares with SLEIGN. The latter is the only effective general purpose software available for the computation of the eigenvalues of singular Sturm-Liouville problems.

1 Introduction

The code SLEIGN (Sturm-Liouville eigenvalue) was introduced in 1978; see Bailey [3] and [4], and especially Bailey, Gordon and Shampine [7]. It is a general purpose software package designed to compute the eigenvalues of (i) SL problems with regular, separated, self-adjoint boundary conditions and (ii) singular SL problems. In the singular case the code automatically selects a boundary condition. Thus in the singular limit-circle case the user does not have the option of specifying which of the infinitely many self-adjoint singular boundary conditions are to be used.

The purpose of this article is to describe a basic approximation theorem and a new code called SLEIGN2. Although it was inspired by SLEIGN it includes a new algorithm which will be described here. SLEIGN2 is designed to compute the eigenvalues of any SL problem with separated, self-adjoint boundary conditions. These problems may be regular or singular at each endpoint of the underlying interval. The user has complete freedom in specifying the boundary conditions as long as these are formulated appropriately in a manner to be described below.

A Sturm-Liouville (SL) boundary value problem consists of a second order linear ordinary differential equation

$$-(py')' + qy = \lambda wy \text{ on } (a, b) \tag{1.1}$$

and boundary conditions. Here $I = (a, b)$ is a bounded or unbounded open interval of the real line $\mathbb{R}$ i.e. $-\infty \leq a < b \leq \infty$; the coefficients $p, q, w : (a, b)$ into $\mathbb{R}$; $\lambda \in \mathbb{C}$, the complex field. In order that (1.1) have solutions in an appropriate sense certain additional restrictions are placed on p, q and w to be specified in Section 2. The nature of the boundary conditions depends on the classification of the endpoints as regular or singular. These will also be given in Section 2.

In general non-trivial solutions of (1.1) satisfying the imposed boundary conditions only exist for certain distinguished values of the parameter λ; these are the so-called eigenvalues of the problem.

The determination of the eigenvalues of SL problems is of great interest in mathematics and its applications. Their numerical calculation is of considerable importance in numerical analysis and is a meeting ground for analysis, numerical analysis and applied mathematics.

SLEIGN is remarkably successful. For regular problems the only comparable code is DO2KEF in the NAG Library [9]. Both are very effective in the regular case. But in the singular case SLEIGN has no serious competitor. Both codes are based on the Prüfer transformation and on a knowledge of the precise number of zeros of the eigenfunctions. The linear second order differential equation (1.1) is transformed into a nonlinear first order differential equation for the phase function which can be integrated efficiently and dependably. The computation of a specified eigenvalue requires knowledge of the precise number of zeros of its eigenfunction but does not require any knowledge of either preceding or subsequent eigenvalues. Thus, in principle, each eigenvalue is computed independently and to any described accuracy. SLEIGN will make use of an initial guess if one is provided, but otherwise has its own guesstimator written into the program.

The equation for the phase function has the form

$$\theta' = p^{-1}cos^2\theta + (\lambda w - q)sin^2\theta \quad \text{on } (a, b). \tag{1.2}$$

If a is a finite singular endpoint and $p(a) = 0$, it is clear from (1.2) that $\theta'(a)$ becomes infinite for all initial conditions $\theta(a) = \alpha$, $0 \leq \alpha < \pi$, with the possible exception of $\alpha = \pi/2$. Similarly, at a singular endpoint at which w or q is unbounded θ' becomes infinite for all initial conditions $0 \leq \alpha < \pi$ with one possible exception. SLEIGN takes advantage of these exceptional cases when searching for a possible boundary condition. It is clear that for all the nonexceptional cases the obstacles are formidable. This is why a new code SLEIGN2 is being developed.

As previously mentioned SLEIGN is the only available general purpose code which is effective for a wide range of singular SL problems. These occur when at least one of the coefficients p^{-1}, q, w is not integrable up to the endpoint (i.e. is unbounded in a suitably severe way) or if one or both of these endpoints is infinite. Such boundary value problems are termed singular. These present particular difficulties both in the determination of well-posed problems and in the numerical calculation of eigenvalues.

We present a brief account of these technical matters in the next section when we recall the well-established limit-point/limit-circle classification of the endpoints of the interval (a, b); see Titchmarsh [25], Chapter II and Naimark [20], p. 73. It suffices here to say that SLEIGN is effective with many limit-point singularities, when boundary conditions are not required, and with limit-circle singularities. In the latter case the user does not have an option to specify the boundary condition, since SLEIGN automatically selects a "special" one within the code. In many cases of interest in mathematical physics this special boundary condition determines the Friedrichs extension. See the examples in Bailey [[3], p. 27] and Marletta [19]. For a discussion of the Friedrichs extension see Niessen and Zettl [21]. The problem represented by the Friedrichs extension is often of special importance in applications.

SLEIGN2 is a general purpose code for limit-circle boundary conditions to be specified by the user. Both nonoscillatory and oscillatory endpoints are included. The numerical problems in the oscillatory case are particularly difficult. In this case every eigenfunction has an infinite number of zeros in any neighborhood of the end point. Recall that the algorithm on which SLEIGN is based depends on knowing that, when the eigenvalues are enumerated appropriately, the nth eigenfunction has precisely n zeros in the interval (a, b).

It should not be thought that the design of SLEIGN2 to cope with both the non-oscillatory and the oscillatory limit-circle cases is introduced out of a wish for completeness alone. Both cases present Strum-Liouville boundary value problems which are of importance in applied mathematics and mathematical physics. There are many applications for limit-circle non-oscillatory equations and we have quoted some of these in references [2], [8], [11], [12] and [14]. However, the limit-circle oscillatory equations also play a role in mathematical physics; see [17] and its references.

SLEIGN2 does require the user to distinguish between regular and singular endpoints; in the singular case the user must further distinguish between the limit-point and limit-circle cases, and in the latter case to describe the boundary condition appropriately as explained below.

All this is an essential feature of SLEIGN2. These classifications of the endpoints and boundary conditions are not part of the program. They depend on an analytic assessment of the problem. A number of questions for further investigation come to mind. Can SLEIGN be enhanced to handle continuous spectra in the limit point case? While SLEIGN also computes eigenfunctions, SLEIGN2 computes only eigenvalues. How can eigenfunctions be computed? Can the conditions $p > 0$, $w > 0$ a.e. be relaxed? In particular one wants to study the cases when p or w change sign; see in particular [5], [6] and [14]. We plan to pursue these and other questions in subsequent papers.

The contents of the paper are as follows. The introduction in Section 1 is followed by a discussion of SL expressions in Section 2. Boundary conditions are discussed in Section 3. The basic approximation theorem is stated in Section 4; Section 5 is devoted to a brief description of SLEIGN2. Finally a wide range of examples is discussed in detail in Section 6. The given SLEIGN 2 numerical results for these examples are intended to illustrate the capability of SLEIGN 2 to handle a variety of limit-circle problems, including both the nonoscillatory and oscillatory cases, and to allow the user to select boundary conditions.

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2 Sturm-Liouville Differential Expressions

Let I denote any interval of the real line R with endpoints a and b where $-\infty \leq a < b \leq \infty$. A compact, i.e. a bounded and closed interval, is denoted by $[a, b] = \{t \in R : a \leq t \leq b\}$ where $-\infty < a < b < \infty$.

By $L(I)$ or $L^1(I)$ we denote the space of complex valued measurable functions on I for which

$$\int_I |y(t)|dt = \int_I |y| < \infty. \tag{2.1}$$

The notation $L_{loc}(I)$ is used to denote the space of functions y satisfying $y \in L[\alpha, \beta]$ for all compact subintervals $[\alpha, \beta]$ of I.

Likewise $L^2(I)$ denotes the space (of equivalence classes) of functions y such that

$$\int_I |y(t)|^2 dt < \infty;$$

and if w is a positive measurable function on I then $L^2(I; w)$ represents the weighted space of functions y satisfying

$$\int_I |y(t)|^2 w(t) dt < \infty. \tag{2.2}$$

This is a Hilbert space if vectors in the space are determined by equivalence classes of functions and the inner product is given by

$$(y, z) = \int_I w(t)y(t)\overline{z}(t)dt. \tag{2.3}$$

The bar over z denotes the complex conjugate.

Throughout this paper we assume that the coefficients p, q, w satisfy:

$$p, q, w : I \to R \tag{2.4}$$

$$p^{-1}, q, w \in L_{loc}(I) \tag{2.5}$$

$$p(t) > 0, w(t) > 0, \text{ almost everywhere on } I. \tag{2.6}$$

When additional conditions are needed they will be specified.

The standard existence theorems for solutions of the differential equation

$$-(py')' + qy = \lambda wy \text{ on } I, \tag{2.7}$$

under the conditions (2.4), (2.5), (2.6), can be found in Naimark [20]. By a solution of (2.7) we mean a function y such that y and py' are both absolutely continuous on all compact subintervals of I (so that the left hand side of (2.7) is defined a.e. on I) and (2.7) holds a.e. Note that the quasi-derivative py' is required to be absolutely continuous, the classical derivative y' may not be.

Definition 2.1 *The endpoint a of I is said to be regular if a is finite and*

$$p^{-1}, q, w \in L[a, \alpha] \text{ for some } \alpha \text{ in} (a, b). \tag{2.8}$$

Similarly the endpoint b is regular if it is finite and

$$p^{-1}, q, w \in L[\beta, b] \text{ for some } \beta \text{ in} (a, b). \tag{2.9}$$

An endpoint is called singular if it is not regular. Thus an endpoint is singular if it is infinite, or the endpoint is finite but at least one of p^{-1}, q, w is not integrable in any neighborhood of the endpoint.

The singular endpoint a is classified as limit-circle (LC for short) if all solutions of equation (2.7) are in $L^2((a,\alpha);w)$ for some α in I; similarly b is LC if all solutions of (2.7) are in $L^2((\beta,b);\, w)$ for some β in I. An endpoint which is not LC is called limit-point or LP for short. It is well known [20, 25] that this LC/LP classification is independent of λ in C, see Weidmann [26, Sections B, 14].

In Section 4 we will need a further subclassification of the LC endpoints into oscillatory and nonoscillatory cases. It follows from condition (2.5) and (2.6) that a nontrivial solution of (1.1) with λ real can only have isolated zeros in (a,b). Thus the zeros of nontrivial solutions of (1.1) can accumulate only at an endpoint. The endpoint a is called oscillatory (O) if (1.1) has a nontrivial solution which has a zero in every interval (a,α), $a<\alpha<b$. Otherwise the endpoint a is called nonoscillatory (NO). Similar definitions are made for b. In the LP case this classification depends on λ. However, given the LC case the O or NO classification is independent of λ in R, see Weidmann [26, Section B, 14].

3 Boundary Conditions

In this section and in fact for the rest of this paper we assume for the differential equation

$$-(py')' + qy = \lambda wy \text{ on } I = (a,b), -\infty \le a < b \le \infty,$$

that each endpoint a, b is either regular or LC. If one or both endpoints are LP either SLEIGN or SLEIGN2 may be used.

Define the quasi-differential expression M by

$$My = -(py')' + qy \text{ on } I$$

Define the "maximal domain" Δ by $\Delta = \{y : I \to C : y, py'$ are absolutely continuous on all compact subintervals of I, and $y,\ w^{-1}My \in L^2(I;w)\}$.

For all functions y, z in Δ we have the well known Green's formula

$$\int_\alpha^\beta \{\overline{z}My - y\overline{Mz}\} = [y,z]_\alpha^\beta,\ \alpha,\beta \in I, \tag{3.1}$$

where

$$[y,z](t) = (y(p\overline{z}') - \overline{z}(py'))(t) \ \ all \ \ t \in I. \tag{3.2}$$

It is clear from (3.1) that both limits

$$[y,z](a) = \lim_{t\to a^+}[y,z](t) \text{ and } [y,z](b) = \lim_{t\to b^-}[y,z](t), \tag{3.3}$$

exist and are finite for all y, z in Δ.

Choose real valued functions u, v in Δ with the following properties

$$[u,v](a) \ne 0 \ne [u,v](b). \tag{3.4}$$

Such a choice is always possible. For example u and v can be any two real linearly independent solutions of (1.1) for any real value of λ.

As shown in Naimark [20] all separated self-adjoint boundary conditions for (1.1) can be formulated as follows (for y in Δ):

$$A_1[y,u](a) + A_2[y,v](a) = 0 \tag{3.5}$$

$$B_1[y,u](b) + B_2[y,v](b) = 0 \tag{3.6}$$

where A_1, A_2, B_1, B_2 are all real and $A_1^2 + A_2^2 > 0,\ B_1^2 + B_2^2 > 0$.

If a is regular and we choose u, v to be real solutions of (1.1) for any particular λ in R, e.g. $\lambda = 0$ satisfying

$$u(a) = 0,\ (pu')(a) = 1,\ v(a) = 1,\ (pv')(a) = 0,$$

then (3.5) reduces to the more familiar form:

$$A_1 y(a) + A_2(py')(a) = 0. \tag{3.7}$$

Similarly, if b is regular we can choose u and v so that (3.6) reduces to

$$B_1 y(b) + B_2(py')(b) = 0. \tag{3.8}$$

Also it can be shown that if both a and b are regular we can choose u and v so that both (3.5) and (3.6) reduce to (3.8) and (3.9). Thus (3.5) and (3.6) can be viewed as the analogues of (3.8) and (3.9) in the singular LC case.

Another equivalent form of (3.7) and (3.8) is given by

$$y(a) cos\ \alpha + (py')(a) sin\ \alpha = 0, 0 \leq \alpha < \pi, \tag{3.9}$$

$$y(b) cos\ \beta + (py')(b) sin\ \beta = 0,\ 0 < \beta \leq \pi. \tag{3.10}$$

Remark 3.1 If a is a regular endpoint of the open interval $I = (a,b)$ then for any solution y of (1.1) with λ in C, $y(a)$ and $(py')(a)$ can be defined by

$$y(a) = \lim_{t\to a} y(t),\ (py')(a) = \lim_{t\to a}(py')(t). \tag{3.11}$$

These limits exist and are finite.

4 The basic approximation theorem

Throughout this section we assume that each endpoint a or b of the interval I is either regular or limit-circle.

The differential equation

$$-(py')' + qy = \lambda wy \text{ on } I = (a,b),\ -\infty \leq a < b \leq \infty, \tag{4.1}$$

together with the boundary conditions

$$A_1[y,u](a) + A_2[y,v](a) = 0,\ A_1,\ A_2 \in R,\ A_1^2 + A_2^2 > 0, \tag{4.2}$$

$$B_1[y,u](b) + B_2[y,v](b) = 0,\ B_1,\ B_2 \in R, B_1^2 + B_2^2 > 0, \tag{4.3}$$

where u and v are real valued functions in Δ satisfying (3.4), is called a separated Sturm-Liouville (SL) boundary value problem (BVP). (The boundary conditions (4.2), (4.3) are separated; there are also coupled, i.e. nonseparated conditions which could be used but we will consider only boundary conditions of type (4.2) and (4.3) in this paper.)

A complex number λ is called an eigenvalue of the BVP if there exists a nontrivial solution of (4.1) which satisfies (4.2) and (4.3). Such a solution is called an eigenfunction belonging to λ or associated with λ. It is well known [see Naimark [20]] that under the conditions (2.4), (2.5) and (2.6) and the assumption that each endpoint is either regular or LC there exist an infinite number of eigenvalues. These are all real, countable, isolated and each eigenfunction is unique up to constant multiples.

If each endpoint is in the NO case (this automatically holds at a regular endpoint while it may or may not hold at an LC endpoint) then, in addition, the eigenvalues are bounded below. Thus they can be indexed such that

$$-\infty < \lambda_0 < \lambda_2 < ... \text{ and } \lambda_n \to \infty \text{ as } n \to \infty. \tag{4.4}$$

Furthermore, if ϕ_n denotes an eigenfunction corresponding to λ_n, $n \in N_0 = \{0, 1, 2, ...\}$, then ϕ_n has exactly n zeros in the open interval (a, b), see Atkinson [1] and Weidmann [26]. This fact is of critical importance in the numerical calculation of the eigenvalues by SLEIGN.

If one or both endpoints is oscillatory, then the eigenvalues are not bounded below. With λ_n, $n \in Z = \{-2, -1, 0, 1, 2, ...\}$ denoting the eigenvalues and ϕ_n the corresponding eigenfunctions we have, in this case,

$$... < \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2..., \tag{4.5}$$

$$\lambda_n \to -\infty \text{ as } n \to -\infty \text{ and } \lambda_n \to +\infty \text{ as } n \to \infty, \tag{4.6}$$

and each eigenfunction ϕ_n, $n \in Z$, has infinitely many zeros in (a, b), see again [26]. Note that in this case when Z is the index set the indexing scheme for λ_n is rather arbitrary: λ_1 in one set might be λ_{83} in another set. To get a definite procedure one can denote the first nonnegative eigenvalue by λ_0. The others are then determined by (4.5).

How can the eigenvalues λ_n be computed numerically?

Let $P(a, b)$ represent the BVP consisting of (4.1), (4.2) and (4.3). Recall that u, v are real members of Δ satisfying (3.4), p, q, w satisfy (2.4), (2.5) and (2.6) and each endpoint is assumed to be either regular or LC. For α, β in I with $a < \alpha < \beta < b$ let $P[\alpha, \beta]$ represent the regular BVP on $[\alpha, \beta]$ given by

$$-(py')' + qy = \lambda wy \text{ on } [\alpha, \beta] \tag{4.7}$$

$$A_1[y,u](\alpha) + A_2[y,v](\alpha) = A_1\{y(\alpha)(pu')(\alpha) - u(\alpha)(py')(\alpha)\}$$
$$+A_2\{y(\alpha)(pv')(\alpha) - v(\alpha)(py')(\alpha) = A_1(\alpha)y(\alpha) + A_2(\alpha)(py')(\alpha) = 0 \tag{4.8}$$

with

$$A_1(\alpha) = A_1(pu')(\alpha) + A_2(pv')(\alpha), A_2(\alpha) = -A_1u(\alpha) - A_2v(\alpha), \tag{4.9}$$

and

$$B_1[y,u](\beta) + B_2[y,v](\beta) = B_1(\beta)y(\beta) + B_2(\beta)(py')(\beta) = 0 \tag{4.10}$$

with

$$B_1(\beta) = B_1(pu')(\beta) + B_2(pv')(\beta),\ B_2(\beta) = -B_1u(\beta) - B_2v(\beta). \tag{4.11}$$

A basic point of this paper is to claim that the regular BVP $P[\alpha,\beta]$ is a good approximation of the BVP $P(a,b)$ even if the latter is singular LC at one or both endpoints, when α is "close" to a and β is "close" to b. The approximation of $P(a,b)$ by $P[\alpha,\beta]$ is the basis of our algorithm. However, in our numerical implementation when one or both endpoints are oscillatory we restrict u and v to be real linearly independent solutions of (1.1) for some real value of λ.

Denote the eigenvalues of $P(a,b)$ and $P[\alpha,\beta]$ by $\lambda_n(a,b)$, $\lambda_n[\alpha,\beta]$.

If each endpoint $a,\ b$ is nonoscillatory then the convergence of $\{\lambda_n[\alpha,\beta]\}$ to $\lambda_n(a,b)$ is straightforward. However, if a is oscillatory then this "convergence" is much more complicated since in this case we have that for each n in N_0

$$\lambda_n[\alpha,\beta] \to -\infty \text{ as } \alpha \to a \text{ and } \beta \to b. \tag{4.12}$$

Yet in spite of (4.12) the eigenvalues of oscillatory problems can be approximated by eigenvalues from regular problems!

Theorem 4.1 *Let (2.4), (2.5), (2.6) hold, let $a < \alpha_k < \beta_k < b$, $k = 1,2,3,...$, and let $\alpha_k \to a$ and $\beta_k \to b$ with α_k decreasing and β_k increasing. Assume each of a and b is either regular or LC. Suppose $\lambda = \lambda_n(a,b)$ is an eigenvalue of problem $P(a,b)$. Then each open interval containing λ contains an eigenvalue of the regular problem $P[\alpha_k,\beta_k]$ for all sufficiently large integers k.*

Proof: This will be given in a subsequent paper. It is a consequence of an abstract result in operator theory; see Kato [[16], p. 371]. To see that this abstract result applies we construct a sequence of self-adjoint operators A_k corresponding to problem $P[\alpha_k,\beta_k]$ and a self-adjoint operator A corresponding to problem $P(a,b)$ all in the Hilbert space $L^2(I,w)$ such that the sequence A_k converges to A in the sense of norm resolvent convergence. ∎

Theorem 4.1 has the following consequence: Given an eigenvalue λ of the (singular) problem $P(a,b)$ there exists a sequence of eigenvalues chosen from the regular problems $P[\alpha_k,\beta_k]$ which converges to λ. Since this sequence is constructed differently in the oscillatory and nonoscillatory cases we treat these cases separately below. In the oscillatory case, as we will see in the statement of the construction below, the index of the approximating eigenvalue from the regular problem $P[\alpha_k,\beta_k]$ changes with k.

Construction of an approximating sequence Assume the hypotheses and the notation of Theorem 4.1 hold.

(a) Suppose both endpoints a,b are nonoscillatory. Choose u,v in Δ satisfying (3.4). Denote the eigenvalues of problems $P(a,b)$ and $P[\alpha_k,\beta_k]$ by $\lambda_n(a,b)$ and $\lambda_n[\alpha_k,\beta_k]$, respectively, with the index n so chosen that in all cases we have

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < ..., \lambda_n \to \infty \text{ as } n \to \infty. \tag{4.13}$$

Then for any $n \in N_0$ we have

$$\lambda_n[\alpha_k, \beta_k] \to \lambda_n(a, b) \text{ as } k \to \infty. \tag{4.14}$$

(b) Assume at least one of the endpoints a, b is oscillatory. For any fixed but arbitrary λ in R let u, v be real linearly independent solutions of (1.1). (Note that in this case the eigenvalues $\lambda_n(a, b)$ of the singular problem $P(a, b)$ are indexed with $n \in Z$ while those of the regular problems $P[\alpha_k, \beta_k]$ i.e. $\lambda_n[\alpha_k, \beta_k]$ are indexed with $n \in N_0$, $k = 1, 2, 3, ...$ In place of (4.13) we have

$$... < \lambda_{-2}(a, b) < \lambda_{-1}(a, b) < \lambda_0(a, b) < \lambda_1(a, b) < \lambda_2(a, b) < ...$$

with

$$\lambda_n(a, b) \to -\infty \text{ as } n \to -\infty; \ \lambda_n(a, b) \to \infty \text{ as } n \to \infty. \tag{4.15}$$

Of course we still have

$$-\infty < \lambda_0[\alpha_k, \beta_k] < \lambda_1[\alpha_k, \beta_k] < \lambda_2[\alpha_k, \beta_k] < ... \tag{4.16}$$

with $\lambda_n[\alpha_k, \beta_k] \to \infty$ as $n \to \infty$ for each $k = 1, 2, 3, ...$ In this case there is no simple approximation comparable to (4.14). Instead we have the following: Given any $\lambda_n(a, b)$, $n \in Z$, there exists a sequence of positive integers m_k satisfying $m_1 \leq m_2 \leq m_3 \leq ...$ with $m_k \to \infty$ as $k \to \infty$ such that

$$\lambda_{m_k}[\alpha_k, \beta_k] \to \lambda_n(a, b). \tag{4.17}$$

The sequence m_k depends on the index n in Z.

The existence of index sequence m_k such that (4.17) holds follows immediately from Theorem 4.1. However the construction of this sequence for a given $\lambda_n(a, b)$ is not clear from Theorem 4.1 or its proof. We now describe a procedure which can be used to construct an index sequence m_k. Find an integer k which is large enough so that among the eigenvalues $\lambda_r[\alpha_k, \beta_k]$, $r \in N_0$ of problem $P[\alpha_k, \beta_k]$ there is one which is "close" to $\lambda_n(a, b)$. Let m_1 be the index of this eigenvalue of the regular problem $P[\alpha_k, \beta_k]$. Count the number of zeros l the "boundary condition function" $A_1 u + A_2 v$ has in the interval (α_{k+1}, α_k) and count the number of zeros r the function $B_1 u + B_2 v$ has in the interval (β_k, β_{k+1}). Now let $m_2 = m_1 + l + r$, and repeat this process. This procedure is based on the fact that as α_k moves toward α_{k+1}, each time it passes through a zero of the function $A_1 u + A_2 v$ in the interval (α_{k+1}, α_k) the problem $P[\alpha_{k+1}, \beta_k]$ "gives birth" to a new eigenvalue to the left of $\lambda_0[\alpha_k, \beta_k]$. For example if $A_1 u + A_2 v$ has three zeros in the interval (α_{k+1}, α_k) then the eigenvalues $\lambda_0[\alpha_{k+1}, \beta_k]$ $\lambda_1[\alpha_{k+1}, \beta_k]$, $\lambda_2[\alpha_{k+1}, \beta_k]$ are all considerably to the left of $\lambda_0[\alpha_k, \beta_k]$ and $\lambda_3[\alpha_{k+1}, \beta_k]$ is "close" to $\lambda_0[\alpha_k, \beta_k]$. In this case $l = 3$ in the above scheme. Thus the index "jumps" by 3 due to the three zeros of the boundary condition function $A_1 u + A_2 v$ in the interval (α_{k+1}, α_k). A similar jump in the index is produced by each zero of the boundary condition function $B_1 u + B_2 v$ in the interval (β_k, β_{k+1}).

This construction of the index sequence can be simplified if only one end point, say a, is oscillatory. Then for k sufficiently large all the β_k are to the right of the right-most zero of the function $B_1 u + B_2 v$ in the interval (a,b). For such k we have $m_{k+1} = m_k +$ the number of zeros of the function $A_1 u + A_2 v$ in the interval (α_{k+1}, α_k). Similarly for the case when a is NO and b is O.

In the above construction of m_1, m_2, $m_3, ...$ once m_1 is selected m_k is determined for all $k \geq 2$ by the above zero counting scheme.

In a subsequent communication, in collaboration with J. Weidmann, it will be shown that if a sequence of eigenvalues of the regular problems $P[\alpha_k, \beta_k]$ converges as $k \to \infty$ then the limit must be an eigenvalue of the singular problems $P(a, b)$.

Remark 4.2 If one endpoint, say a, is regular we can take $\alpha_k = a$ for each $k = 1, 2, 3, ...$ However, although this is true "theoretically" there is a practical problem in the "weakly regular" case. This is the regular case when one or more of the coefficients p^{-1}, q, w although integrable near a is unbounded near a. More precisely, a is weakly regular if a is finite,

$$p^{-1}, q, w \in L(a, \alpha) \text{ for some } \alpha \text{ in}(a, b)$$

but at least one of p^{-1}, q, w is not bounded in (a, α). A similar definition is made at b.

In the case of weakly regular interior points Theorem 4.1 and hence (4.14) is valid but in this case there are additional complexities in computing eigenvalues of the regular problems.

Remark 4.3 Given the singular BVP $P(a, b)$ consisting of (4.1), (4.2) and (4.3) a key feature of this paper is the construction of the regular BVP $P[\alpha, \beta]$ consisting of (4.7) to (4.11) to approximate $P(a, b)$. Notice that this approximation is driven by the "boundary condition functions" u and v. It is these functions that determine by (4.8) and (4.10) which regular boundary conditions approximate the singular ones (4.2) and (4.3). Note that for each regular endpoint "near" the singular one a specific regular boundary condition is determined. This boundary condition changes every time the regular endpoint is moved closer to the singular one. It is this point which seems to have been missed in the approximation by regular to singular boundary value problem in the existing SL literature. The common approach is to fix a regular boundary condition, e.g. a Dirichlet or Neumann condition at a regular endpoint near the singular one, and then to keep the regular condition unchanged as the regular endpoint is moved closer to the singular one. This approach is likely to fail in all but a few exceptional cases.

Remark 4.4 The reason we choose solutions u, v in part (b) rather than general maximal domain functions as in part (a) is that of the index sequence m_k in (4.14) is based on the zeros of these solutions u, v.

As mentioned above we plan to publish a subsequent paper in which we will give a detailed proof of the convergence of the algorithm stated in Theorem 4.1 for both the oscillatory and nonoscillatory cases.

5 A Brief Description of SLEIGN2.

Here we indicate how SLEIGN2 can be used to compute an eigenvalue $\lambda_n(a, b)$ of a regular or LC boundary value problem. As an illustration we consider the limit-circle oscillatory case.

For simplicity, suppose that the problem is (4.7) with regular end point a and LCO endpoint b, and with boundary conditions

$$y(a) = 0, \quad [y, u](b) = 0,$$

where u is a solution of (4.7) for some fixed real λ.

Let $b_T \in (a,b)$ be close enough to b that the problem on $[a,b_T]$ is expected to approximate the given problem on $[a,b)$, so far as the desired eigenvalue $\lambda_n(a,b)$ is concerned.

As in SLEIGN, make the change of variables

$$y = \rho sin\ \theta, \quad py' = \rho cos\ \theta$$

from the unknowns y, py' to new unknowns ρ, θ. (This is the so called Prüfer transformation.) Then one finds that θ satisfies the differential equation, see (1.2),

$$\theta' = \frac{1}{p} cos^2\theta + (\lambda w - q) sin^2\theta. \tag{5.1}$$

The boundary condition for y at $x = a$ translates into

$$\theta(a) = 0, \tag{5.2}$$

while the boundary condition at $x = b_T$ becomes

$$\tan\theta(b_T) = \frac{u}{pu'}(b_T). \tag{5.3}$$

It is not hard to see that if a value of λ is found for which the differential equation (5.1) has a solution θ satisfying both (5.2) and (5.3), then that value of λ is an eigenvalue of the approximating problem on $[a,b_T]$. And to find such a value of λ numerically one can simply try different real numbers λ in (5.1), integrate numerically this equation from the initial point a starting with (5.2), and see what value $\theta(b_T)$ turns out to be. When a solution θ is found such that (5.3) is satisfied then the chosen value of λ is an eigenvalue of $P[a,b_T]$.

Obviously there will be many different values of λ such that (5.3) is satisfied, and we want to find that one which is closest to $\lambda_n(a,b)$. So, bearing in mind that, according to our notation, $\lambda_0(a,b)$ is the least positive eigenvalue of $P(a,b)$, one can integrate (5.1), (5.2) with $\lambda = 0$ to obtain $\theta_0(b_T)$, say. Then the next higher value of λ for which (5.3) is satisfied is our approximation to λ_0, provided that the corresponding value of $\theta(b_T)$ satisfies

$$\theta_0(b_T) < \theta(b_T) < \theta_0(b_T) + \pi.$$

(This follows from the well known fact that $\theta(x)$, which depends upon λ as well as x, is monotone increasing in λ at any value of x for which $\theta(x)$ is equal to an integral multiple of π.) Likewise our approximation to $\lambda_n(a,b)$ is that value of λ for which $\theta(b_T)$ satisfies (5.3) and also

$$\theta_0(b_T) + n\pi < \theta(b_T) < \theta_0(b_T) + (n+1)\pi.$$

In the above procedure the accuracy of the computed eigenvalue is affected by the choice of b_T. Some experimentation with this choice may be needed to achieve the required accuracy.

Without going into the details of how a scheme such as has been sketched above might be implemented, we simply remark here that the program SLEIGN2 is constructed along these lines, and some of the eigenvalues of a wide range of significant difficult examples, listed below in Section 6, were computed using this program. Some of these examples can be solved analytically, in the sense that a transcendental equation can be found whose roots are the desired eigenvalues. In those cases the eigenvalues obtained in the two independent ways - by SLEIGN2 and by the transcendental equation - are shown for comparison.

Of necessity in this paper we have given only a very brief outline of the method used by SLEIGN2 to compute eigenvalues. The complete SLEIGN2 program will be published in due course in an appropriate mathematical software periodical.

6 Examples

In this section we discuss and present numerical results from a wide range of examples. The notation from the earlier section will be followed. The endpoints of the interval $I = (a, b)$ are classified as R (regular), LP (limit-point), LCNO (limit-circle and nonoscillatory) and LC0 (limit-circle and oscillatory); the boundary condition functions will be denoted by u, v. In most of the examples these are real solutions for some real value of the parameter λ; when solutions are not available in an appropriate explicit form for any real value of λ; u and v are taken from the maximal domain Δ. In all cases it is essential to choose u, v so that $[u, v](a) \neq 0$ at endpoint a; similarly for the endpoint b. Boundary conditions at regular (R) endpoints are given in the more usual form e.g. $y(a) = 0$ or $(py')(b) = 0$.

In most of the examples a transcendental equation is given which provides an alternative method to determine numerical values of the eigenvalues; this gives a valuable independent check on the SLEIGN2 computations. However we have to make clear the fact that the values obtained from the transcendental equations are not used in SLEIGN2 for estimation purposes; the computer program SLEIGN2 stands by itself and is quite independent of such alternative estimates.

In some cases the roots of the transcendental equation $\Phi(s) = 0$ were computed with the help of the comptuer programs COULCC devised by Thompson and Barnett [23] and [24]; we acknowledge here our indebtedness to these programs.

In many of the examples it is convenient to introduce the complex parameter

$$s = \sqrt{\lambda},$$

for which the following notation and conventions are used:

$$\lambda = \mu + i\nu, \quad s = \sigma + i\tau$$

$$0 \leq arg(\lambda) \leq 2\pi, \quad 0 \leq arg(s) \leq \pi$$
$$\lambda = re^{i\theta}, \quad s = \sqrt{\lambda} = r^{1/2}e^{i\theta/2}.$$

If the transcendental equation is of the form

$$\Phi(s) = 0$$

then the roots of this equation which give the (necessarily real) eigenvalues are found either on
(i) the line $s = \sigma \geq 0$

or on
(ii) the line $s = i\tau$ with $\tau \geq 0$.

Note that (i) gives the nonnegative eigenvalues $\lambda = \sigma^2 \geq 0$ and (ii) gives the negative eigenvalues $\lambda = -\tau^2$. Thus a search for the roots of $\Phi(s) = 0$ has to be made both on the positive real axis and on the positive imaginary axis.

We give here the notation we have adopted, in all examples given below, to determine the indexing of the eigenvalues, viz

Non-oscillatory case (LCNO): λ_0 is the first eigenvalue of the problem with eigenvalues $\{\lambda_n : n \in N_0\}$ Oscillatory case (LCO): λ_0 is chosen to be the first non-negative eigenvalue in the sequence of

eigenvalues $\{\lambda_n : n \in \mathbb{Z}\}$.

Example 1. The Legendre equation. This is the equation

$$-((1-x^2)y'(x))' + (1/4)y(x) = \lambda y(x), -1 < x < 1, \ i.e. \ \ I = (-1,1). \tag{6.1}$$

Each endpoint -1 and +1 is LCNO. For $\lambda = 1/4$ two linearly independent solutions are

$$u(x) = 1, \ v(x) = ln((1+x)/(1-x)), \ x \in I. \tag{6.2}$$

Boundary conditions.

$$(i) \qquad [y,u](-1) = 0 = [y,u](1) \tag{6.3}$$

The BVP (6.1), (6.2), (6.3) is the classical case whose eigenfunctions are the classical Legendre polynomials and whose eigenvalues are known to be:

$$\lambda_n = n(n+1) + 1/4, \ n = 0,1,2,3,... \tag{6.4}$$

Eigenvalues computed with SLEIGN2:

$$\begin{aligned} \lambda_0 &= 0.25000 \\ \lambda_1 &= 2.25000 \\ \lambda_2 &= 6.25000 \end{aligned}$$

$$(ii) \qquad I = [0,1), \quad y(0) = 0, \quad [y,v](1) \equiv lim_{x\to 1}[y,v](x) = 0. \tag{6.5}$$

The transcendental equation for the BVP (6.1), (6.2), (6.5) is

$$\Phi(s) \equiv \pi^{-1}cos(s\pi)[\Psi(s+1/2) + \Psi(-s+1/2) + 2\gamma - ln(2)] + 1 = 0 \tag{6.6}$$

where $\Psi(s) = \Gamma'(s)/\Gamma(s)$ with Γ the classical gamma function and γ is Euler's number.

For a derivation of (6.6) the reader is referred to Erdelyi [10], Fulton [13], Kamke [15] and Titchmarsh [25].

Computed eigenvalues of the BVP (6.1), (6.2), (6.5) with

$SLEIGN2$	Transcendental equation
$\lambda_0 : 0.2500$	0.25
$\lambda_1 : 9.13773$	9.13774
$\lambda_2 : 26.07673$	26.0767

Example 2. The Bessel equation. This is the equation

$$-y''(x) + \frac{c}{x^2}y(x) = \lambda y(x), \qquad 0 < x \leq 1, \quad I = (0,1], \; c \in R. \tag{6.7}$$

The endpoint 1 is regular and 0 is a singular endpoint for all $c \neq 0$. Let

$$c = \nu^2 - 1/4. \tag{6.8}$$

For $\lambda = 0$,

$$u(t) = t^{\nu+1/2}, \; v(t) = t^{-\nu+1/2} \tag{6.9}$$

are solutions. These are linearly independent except when $\nu = 0$. In that case

$$u(t) = t^{1/2}, \; v(t) = t^{1/2} ln \; t \tag{6.10}$$

are linearly independent solutions.

¿From (6.9) and (6.10) it follows that the singular endpoint 0 is
(i) LP for $c \geq 3/4$
(ii) LCNO for $-1/4 \leq c < 3/4, \; c \neq 0$
(iii) LCO for $c < -1/4$.

To illustrate the use of SLEIGN2 we consider two BVP's for (6.7), both in the nonoscillatory case.

$$(i) \qquad \nu = 3/4, \; I = (0,1], \quad [y,u](0) = 0, \;\; y(1) = 0. \tag{6.11}$$

The transcendental equation for the BVP (6.7), (6.9), (6.11) is given by

$$\Phi(s) = J_\nu(s) = 0 \tag{6.12}$$

where J_ν is the Bessel function of order ν. For a discussion of the special functions needed in the derivation of (6.12) the reader is referred to [10].

Computed eigenvalues of the BVP (6.7), (6.9, (6.11) with

$SLEIGN2$	Transcendental equation
$\lambda_0 : 12.18714$	12.1871
$\lambda_1 : 44.25755$	44.2576
$\lambda_2 : 96.07161$	96.0716

$$(ii) \qquad I = (0,1], \quad \nu = 3/4, \;\; [y,v](0) = 0, \;\; y(1) = 0. \tag{6.13}$$

In this case the transcendental equation is

$$\Phi(s) \equiv J_{-\nu}(s) = 0. \tag{6.14}$$

For a discussion of the special functions needed to derive (6.14) see [10].

Computed eigenvalues of the BVP (6.7), (6.9), (6.13) with

$SLEIGN2$	Transcendental equation
$\lambda_0 : 1.12055$	1.12044
$\lambda_1 : 18.3540$	18.3531
$\lambda_2 : 55.3622$	55.3603 .

Example 3. The Boyd equation. This is the equation, see [8],

$$-y''(x) - \frac{1}{x}y(x) = \lambda y(x),\ 0 < x \leq 1,\ I = (0,1]. \tag{6.15}$$

The endpoint 1 is regular, and the endpoint 0 is singular and in the LCNO case. General solutions can be obtained in terms of the confluent hypergeometric functions and are, see [8], [11],

$$M_{k,1/2}(x/k) \qquad W_{k,1/2}(x/k)$$

where $k = k(\lambda) = (2is)^{-1}$, and M, W are the Whittaker functions. The solution $kM_{k,1/2}(x/k)$ is analytic in both x and λ in all of C; hence (6.15) is nonoscillatory at 0. In this case it is easier to use maximal domain functions to determine the boundary conditions at zero rather than solutions. It can be checked that

$$u(x) = x,\ \ v(x) = 1 - x\, ln(x),\ \ x \in I, \tag{6.16}$$

are in Δ, the maximal domain and that

$$[u, v](x) \to -1 \ \text{ as } \ x \to 0^+. \tag{6.17}$$

¿From this it follows [20] that 0 is a LC endpoint. Thus 0 is LCNO.

Boundary conditions.

$$(i) \qquad I = (0,1],\ \ [y,u](0) = 0, \qquad y(1) = 0. \tag{6.18}$$

The transcendental equation for the BVP (6.15), (6.16), (6.18) is

$$\Phi(s) = (2is)^{-1} M_{(2is)^{-1},2^{-1}}(2is) = 0. \tag{6.19}$$

There are infinitely many roots of (6.19) on the positive real axis $s = \sigma > 0$ ($s = 0$ is not an eigenvalue; this is the reason for the presence of the factor $(2is)^{-1}$) and possibly a finite number on the line $s = i\tau$ with $\tau > 0$.

Computed eigenvalues of the BVP (6.15), (6.16), (6.18) are

$SLEIGN2$	Transcendental equation
$\lambda_0 : 7.37399$	7.3740
$\lambda_1 : 36.33602$	36.3360
$\lambda_2 : 85.29258$	85.2925
$\lambda_3 : 154.09862$	154.099
$\lambda_4 : 242.70555$	242.705

$$(ii) \qquad I = (0,1], \quad [y,v](0) = 0 \quad y(1) = 0\ . \tag{6.20}$$

The transcendental equation for the BVP (6.15), (6.16), (6.20) is

$$\Phi(s) = \Gamma(1 - (2is)^{-1}) W_{(2is)^{-1},1/2}(2is) = 0. \tag{6.21}$$

Again, there are infinitely many roots of (6.21) on the positive real axis $s = \sigma > 0$; and possibly a finite number on the positive imaginary axis $s = i\tau$ with $\tau > 0$.

Computed eigenvalues of BVP (6.15), (6.16), (6.20) are

$SLEIGN2$	Transcendental equation
$\lambda_0 : 0.984323$	0.98432
$\lambda_1 : 21.97069$	21.9707
$\lambda_2 : 62.12795$	62.1281
$\lambda_3 : 121.81338$	121.813
$\lambda_4 : 201.11951$	201.120

In this example there is another method for obtaining the results of both BVP's (i) and (ii) above.

It is shown in [2, Section 3] that the singularity at 0 of the Boyd equation (6.15) can be regularized at the expense of using quasi-derivatives and a weight function. Details of this regularizing transformation are mentioned in [2, Section 4] with references to other relevant papers. The BVP (6.15), (6.16), (6.18) has the same eigenvalues as the following regular S-L problem in the weighted Hilbert space $L^2((0,1);w)$

$$-(py')' + qy = \lambda wy \ \ on \ \ [0,1] = I \tag{6.22}$$

$$(i)' \qquad y(0) = 0 = y(1) \tag{6.23}$$

where

$$p = r^2, \quad w = r^2, \quad q = -r^2 ln^2, \tag{6.24}$$

with

$$r(x) = exp(-(xln(x) - x)) \qquad x \in [0,1]. \tag{6.25}$$

Note that p and w are continuous and positive on [0,1]; q is not bounded on [0,1] but $q \in L(0,1)$. Hence (6.22), (6.23), (6.24) is a regular S-L problem. (Although the eigenvalues of (6.22), (6.23), (6.24) are the same as those of (6.15), (6.16), (6.18) the eigenfunctions are not. However, the eigenfunctions of the two problems are related to each other by a simple transformation, namely multiplication by a function.)

This example, and others like it, which are regular according to the minimal conditions imposed by Definition 2.1 above, and yet have one or more coefficients which are unbounded at an endpoint, makes it desirable to have a computer eigenvalue routine which is effective in such cases. For this reason SLEIGN2 is designed to handle such problems, which in this paper are designated by "weakly regular" problems. Note that a weakly regular problem is a regular problem in the sense of Definition 2.1.

When applied to the weakly regular problem (6.22) and (6.23) SLEIGN2 computed the following values

λ_0 :	7.37418
λ_1 :	36.33650
λ_2 :	85.29278
λ_3 :	154.09987
λ_4 :	242.70728

which are seen to compare favorably with those values obtained in the singular problem (6.15) to (6.18).

Similarly the singular problem (6.15), (6.16), (6.20) has the same eigenvalues as the regular problem (6.22), (6.24) with the boundary conditions

$$(ii)' \qquad y(0) + (py')(0) = 0, \qquad y(1) = 0. \tag{6.26}$$

Again applying SLEIGN2 to this weakly regular problem yields the values

λ_0 :	0.9847
λ_1 :	21.97142
λ_2 :	62.12913
λ_3 :	121.81473
λ_4 :	201.12811

which should be compared with those values obtained for the singular problem (6.15), (6.16) and (6.20).

Example 4. The Sears-Titchmarsh equation. This is given by

$$-y''(x) - e^{2x}y(x) = \lambda y(x), \quad 0 \le x < \infty, \quad I = [0, \infty). \tag{6.27}$$

See [25,] [22].

The substitution $t = e^x$ transforms (6.27) to

$$-(ty'(t))' - ty(t) = \lambda t^{-1}y(t), \quad 1 \le t < \infty, \quad I = [1, \infty). \tag{6.28}$$

In this form the equation (6.28) is a transformed version of Bessel's equation (see Kamke [15, 1(a) of section C2.162 with the choice of parameters $a = 1$, $m = 2$, $b = 1$ and $c = \lambda$]). Solutions of (6.28) can be given in the form

$$J_{is}(t) \text{ and } J_{-is}(t).$$

For $\lambda = -1/4$ these solutions $J_{\pm 1/2}(t)$ reduce to the elementary solutions

$$t^{-1/2} \cos t \text{ and } t^{-1/2} \sin t \text{ on } I.$$

It now follows that on $[1, \infty)$ (6.28) has the properties that the endpoint 1 is regular and the endpoint ∞ is LCO in $L^2((1, \infty; t^{-1})$. We choose boundary condition functions u, v given by

$$u(t) = t^{-1/2}(cos\, t + sin\, t),\; v(t) = t^{-1/2}(cos\, t - sin\, t). \tag{6.29}$$

Boundary conditions.

$$(i) \qquad I = [1, \infty), \quad y(1) = 0, \quad [y, u](\infty) = 0. \tag{6.30}$$

The transcendental equation for the BVP (6.28), (6.29), (6.30) is given by

$$\Phi(s) = \frac{J_{is}(1) + J_{-is}(1)}{2\, cos(is\pi/2)} = 0. \tag{6.31}$$

The factor in the denominator of (6.31) prevents the spurious roots

$$s = i(2r + 1), \quad r \in N_0$$

from appearing. (Recall that $J_{-n}(z) = (-1)^n J_n(z)$. The numbers $(i(2r+1))^2 = -(2r+1)^2$, $r \in N_0$ are NOT eigenvalues). Since the endpoint at ∞ is oscillatory the eigenvalues of this problem are unbounded below as well as above. Thus the equation (6.31) has infinitely many roots on the nonnegative real axis $s = \sigma \geq 0$ and infinitely many roots on the positive imaginary axis $s = i\tau$ with $\tau > 0$.

Here we define λ_0 to be the first nonnegative eigenvalue and the others such that

$$...\lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < ... \quad .$$

(i) Computed eigenvalues of the BVP (6.28), (6.29) and (6.30) are

	$SLEIGN2$	Transcendental equation
λ_{-2}	-12.2449	-12.2476
λ_{-1}	-1.69345	-1.69336
λ_0	6.76010	6.76091
λ_1	17.83053	17.8354

$$(ii) \qquad I = [1, \infty) \quad y(1) = 0 \quad [y, v](\infty) = 0. \tag{6.32}$$

The transcendental equation for (6.28), (6.29), (6.32) is given by

$$\Phi(s) = \frac{J_{is}(1) - J_{-is}(1)}{2 sin(is\pi/2)} = 0. \tag{6.33}$$

Again, the factor in the denominator prevents the roots

$$s = i2r, \quad r \in N_0$$

from occurring; the numbers

$$(i2r)^2 = -4r^2, \quad r \in N_0$$

are NOT eigenvalues; there are infinitely many positive and infinitely many negative eigenvalues.

Computed eigenvalues of (6.28), (6.29), (6.32) are

$SLEIGN2$	Transcendental equation
$\lambda_{-2} : -20.2499$	-20.2499
$\lambda_{-1} : -6.19488$	-6.19343
$\lambda_0 : 2.29382$	2.29404
$\lambda_1 : 11.96032$	11.9682

Example 5. The BEZ equation. This is

$$-(xy'(x))' - \frac{1}{x}y(x) = \lambda y(x), \quad 0 < x \leq 1, \; I \; (0, 1]. \tag{6.34}$$

The equation (6.34) is also a special case of a (transformed) version of the Bessel equation. See Kamke [15, (1a) of section C2.162 with choice of parameters $a = c = m = 1$, $b = \lambda$] for details. Solutions of (6.34) can be expressed in terms of the Bessel functions

$$J_{+2i}(2x^{1/2}s) \text{ and } J_{-2i}(2x^{1/2}s).$$

For $\lambda = 0$ these solutions reduce to the form

$$x^i \; and \; x^{-i}$$

and from these we obtain the elementary real solutions

$$u(x) = cos(ln(x)) \text{ and } v(x) = sin(ln(x)), \quad 0 < x \leq 1. \tag{6.35}$$

Thus the endpoint 0 is LCO and the endpoint 1 is regular.

Boundary value problems.

$$(i) \qquad I = (0, 1], \quad [y, u](0) = 0, \quad y(1) = 0. \tag{6.36}$$

The eigenvalues of the BVP (6.34), (6.35), (6.36) are characterized by the roots of the equation

$$\Phi(s) = \Gamma(1 + 2i)s^{-2i}J_{2i}(2s) + \Gamma(1 - 2i)s^{2i}J_{-2i}(2s) = 0. \tag{6.37}$$

There are infinitely many roots of (6.37) of the form $s = \sigma \geq 0$ and infinitely many of the form $s = i\tau$ with $\tau > 0$; $s = 0$ is not a root.

Computed eigenvalues of (6.34), (6.35), (6.36) are

$SLEIGN2$	Transcendental equation
$\lambda_{-2} : -126.73$	-126.727
$\lambda_{-1} : -5.42637$	-5.4264
$\lambda_0 : 4.39738$	4.3976
$\lambda_1 : 18.11879$	18.1188
$\lambda_2 : 37.91803$	37.9181
$\lambda_3 : 63.46119$	63.4664

$$(ii) \qquad I = (0, 1], \quad [y, v](0) = 0 \;\; y(1) = 0. \tag{6.38}$$

The transcendental equation for the BVP (6.34), (6.35), (6.38) is given by

$$\Phi(s) = \Gamma(1 + 2i)s^{-2i}J_{2i}(2s) - \Gamma(1 - 2i)s^{2i}J_{-2i}(2s) = 0. \tag{6.39}$$

This time $s = 0$ is a root of (6.39) and there are infinitely many positive roots $s = \sigma > 0$ and infinitely many roots of the form $s = i\tau$ with $\tau > 0$.

Computed eigenvalues of (6.34), (6.35), (6.38) are

$SLEIGN2$	Transcendental equation
$\lambda_{-3} :$	$-19,461.1$
$\lambda_{-2} :$	-609.62
$\lambda_{-1} : -26.335$	-26.3430
$\lambda_0 : 0.0000$	0.0000
$\lambda_1 : 10.44857$	10.4481
$\lambda_2 : 27.28691$	27.2868
$\lambda_3 : 50.1425$	50.0150
$\lambda_4 : 78.36476$	78.364
$\lambda_5 : 112.19733$	112.196

In this example SLEIGN2 could not reach λ_{-2} and λ_{-3} but it is likely that this difficulty was due to machine capacity and could be removed with a larger computing facility.

Example 6. The BEZ^2 equation. If we take the BEZ equation (6.34) and move the LC singularity from the origin 0 to the endpoints +1 and −1, and then bring the two transformed equations together we obtain

$$-((1 - |x|)y'(x))' - \frac{1}{1 - |x|}y(x) = \lambda y(x) \quad -1 < x < 1. \tag{6.40}$$

Solutions of this equation are

$$J_{+2i}(2(1 - |x|)^{1/2}s) \quad \text{and} \quad J_{-2i}(2(1 - |x|)^{1/2}s);$$

all solutions y and their quasi-derivatives py' (where $p(x) = 1 - |x| \;\; (x \in (-1, 1))$) are continuous at the origin 0; the classical derivative y' may or may not be continuous at 0.

This example is LCO at both endpoints +1 and −1. Separated boundary conditons can be imposed at these endpoints, as in Example 5, by using the solutions when $\lambda = 0$ i.e.

$$u(x) = \cos(ln(1-|x|) \qquad v(x) = \sin(ln(1-|x|)) \quad (x \in (-1,1)). \tag{6.41}$$

To illustrate the effectiveness of SLEIGN2 in the case of boundary value problems when both endpoints are LCO, two sets of boundary conditions are considered. The eigenvalues have been computed by SLEIGN2 and are given below; we have not made use of the equivalent transcendental equation but since the coefficients of the equation (6.40) are even about the origin 0 of $(-1,1)$, the choice of both boundary conditions gives all eigenfunctions as even or odd. The odd eigenfunctions satisfy $y(0) = 0$ and so also appear in one or other of the two boundary value problems (i) and (ii) of the BEZ equation in Example 5. Thus to obtain an independent check on the results for the BEZ^2 equation reference should be made to the results of the previous example.

Boundary value problems

$$(i) \qquad I = (-1,1) \quad [y,v](-1) = [y,v](1) = 0 \tag{6.42}$$

Computed eigenvalues are:

$SLEIGN2$	Transcendental equation
$\lambda_{-1} : -5.42635$	-5.42636
$\lambda_0 : 0.0000$	
$\lambda_1 : 4.39755$	4.39755
$\lambda_2 : 10.16808$	
$\lambda_3 : 18.11874$	18.1188

$$(ii) \qquad I = (-1,1) \quad [y,v](-1) = [y,v](1) = 0 \tag{6.43}$$

Computed eigenvalues are:

$SLEIGN2$	Transcendental equation
$\lambda_{-2} : -26.366$	-26.3440
$\lambda_{-1} : -1.75422$	
$\lambda_0 : 0.0000$	0.000
$\lambda_1 : 4.10963$	
$\lambda_2 : 10.44811$	10.4481

Example 7: The Latzko equation This is the differential equation

$$-((1-x^7)y'(x))' = \lambda x^7 y(x) \quad 0 \le x < 1 \tag{6.44}$$

which has a celebrated history; it is associated with a heat conduction problem first studied by Latzko in 1921. For full details to the literature concerned with the study of this equation and an associated boundary value problem see Fichera [12, Pages 43 to 45 and detailed references].

The differential equation (6.44) has a strong comparison with the Legendre equation (6.1). The endpoint 0 is clearly regular; the endpoint 1 is singular since $\int_0^1 (1-x^7)^{-1}dx = \infty$. To determine the classification of 1 let $\lambda = 0$ when solutions are

$$1 \text{ and } \int_0^x \frac{1}{1-t^7}dt \quad (x \in [0,1)).$$

Now $\int_0^x (1-t^7)^{-1}dx \sim K \;\; ln(1/(1-x))$ as $x \to 1-$ for some $K > 0$. Thus both solutions are in $L^2(0,1)$ and hence in the weighted space $L^2(0,1;\; x^7)$; hence endpoint 1 is in the LC case. Since 1 is a solution on $[0,1)$ this endpoint is LCNO.

For boundary condition functions we can take

$$u(x) = 1 \qquad v(x) = ln(1/(1-x)) \quad (x \in [0,1)) \tag{6.45}$$

noting that u is a solution, but v is a maximal domain function which is not a solution. A calculation shows that $[u,v](1) = 7 \neq 0$ so that the pair $\{u,v\}$ form a bases for boundary conditions at 1.

The boundary value problem considered by Fichera [12, Page 43] is equivalent to

$$(i) \qquad I = [0,1) \quad y(0) = 0 \quad [y,u](1) = 0 \tag{6.46}$$

This is not immediately clear since the Fichera problem [12, (1.11.12)] requires that $\lim_{x\to 1-} y(x)$ exist and is finite at 1. However it may be shown that this requirement is equivalent to the boundary condition $[y,u](1) = 0$ of (6.46), i.e. $y \in \Delta$ and $[y,u](1) = 0$ is equivalent to $y' \in L(0,1)$ and is equivalent to $y \in AC[0,1]$. This is a typical case in which a boundary value problem in applied mathematics has to have a boundary condition interpreted in the Glazman-Krein-Naimark form; see [20, Chapter V].

Extended efforts have been made to find numerical approximations to the first few eigenvalues of the problem (6.46); see [12, Page 44]. Here we report only on the results of Scarpini [12, references 95, 96]

$$\begin{aligned} 8.721575 &< \lambda_0 < 8.727471 \\ 128.2512 &< \lambda_1 < 152.4231 \\ 208.3475 &< \lambda_2 < 435.0634 \end{aligned} \tag{6.47}$$

and the results of Durfee [12, reference 90]

$$\begin{aligned} \lambda_0 &= 8.72747 \\ \lambda_1 &= 152.423 \\ \lambda_2 &= 435.06 \\ \lambda_3 &= 855.68 \\ \lambda_4 &= 1414.1 \end{aligned} \tag{6.48}$$

When SLEIGN2 is applied to (6.46) the following results are obtained

$$\begin{aligned} \lambda_0 &= 8.7274702 \\ \lambda_1 &= 152.423014 \\ \lambda_2 &= 435.060768 \\ \lambda_3 &= 855.6817 \\ \lambda_4 &= 1414.12619. \end{aligned} \tag{6.49}$$

Two comments are called for:

(a) The comparison between the results (6.48) and (6.49) is impressive and consolidates both the early work of Durfee and the effectiveness of SLEIGN2 .

(b) Equally impressive is the comparison between (6.47) and (6.49). The upper bounds in (6.47) are obtained by the method of orthogonal invariants, see [12], and it now appears likely that these upper bounds are in fact numerical approximations in themselves to the respective eigenvalues. It is of interest to ask if this result can be proved in general from the analysis in Fichera [12].

The second boundary value problem discussed here, but one not considered in Fichera [12] is (see (6.45) for boundary condition function v)

$$(ii) \qquad I = [0,1) \quad y(0) = 0 \quad [y,v](1) = 0. \tag{6.50}$$

When SLEIGN2 is applied to (6.50) the following results are obtained

$$\begin{aligned} \lambda_0 &= 11.44499 \\ \lambda_1 &= 110.55933 \\ \lambda_2 &= 1324.5611 \end{aligned} \tag{6.51}$$

It would be of great interest to compare these results with the corresponding result from the method of orthogonal invariants.

Example 8: The Laplace tidal wave equation This important differential equation is discussed by Homer [14]; it has a long history as reflected in the references given in [14].

The complete form of the equation is

$$\begin{aligned} -(\frac{1-\mu^2}{\mu^2-\tau^2}y'(\mu))' &+ \frac{1}{\mu^2-\tau^2}[\frac{s}{\tau}(\frac{\mu^2+\tau^2}{\mu^2-\tau^2}) + \frac{s^2}{1-\mu^2}]y(\mu) \\ &= \lambda y(\mu) \quad (-1 < \mu < 1) \end{aligned} \tag{6.52}$$

where the physical parameters s and τ satisfy

$$s \in \mathbb{Z}\backslash\{0\} \qquad 0 < \tau < 1. \tag{6.53}$$

It is important to note that the equation (6.52) not only has singular endpoints but also interior singularities at $\pm\tau$. There is the additional interest, both for the analysis and for the application to tidal wave theory, that the leading coefficient $(1-\mu^2)/(\mu^2-\tau^2)$ changes sign at the interior points $\pm\tau$, in view of (6.53).

There are no known explicit solutions of (6.52) in terms of special functions or contour integrals. In this example then there is no transcendental equation for the calculation of eigenvalues. Nevertheless in view of the importance of the tidal wave equation for applied mathematics a considerable effort has been extended to obtain numerical values for the eigenfunctions and eigenvalues of (6.52), when the equation is considered in the space $L^2(-1,1)$. For details see [14] and the references given in that paper.

Here we do not attempt to apply SLEIGN2 to the tidal wave equation in its complete form, since some limit-point singularities are involved, but we consider a particular differential equation which represents the nature of the interior limit-circle singularities of the general equation.

The differential equations is, see [14, Equations (3.3) and (3.4)]

$$-(\frac{1}{x}y'(x))' + (\frac{k}{x^2} + \frac{k^2}{x})y(x) = \lambda y(x) \quad (x \in (0,1)) \tag{6.54}$$

where the parameter $k \in \mathbb{R}$ and $k \neq 0$, i.e. $k^2 > 0$. For convenience, as in [14], we have placed the singularity at the origin.

With no explicit solutions for this equation we classify the singularity at 0 by looking at elements of the maximal domain Δ, see [14, Pages 165 to 168]. A calculation show that

$$(i) \qquad x^2 \text{ and } x - \frac{1}{k} \in \Delta \tag{6.55}$$

$$(ii) \qquad [x^2, x-\frac{1}{k}] = \frac{1}{x}(x^2 - 2x(x-\frac{1}{k})) \to \frac{2}{k} \neq 0 \quad \text{as } x \to 0+.$$

Thus the equation is LC at 0. Additionally a Frobenius analysis at 0 shows that there are solutions of the form

$$\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad x^2 \sum_{n=0}^{\infty} b_n x^n$$

in a neighborhood of the origin; see [14, Page 165]. Hence the equation is LCNO in $L^2(0,1)$ and a basis for the boundary condition functions is

$$u(x) = x^2 \quad v(x) = x - \frac{1}{k} \quad (x \in (0,1)). \tag{6.56}$$

The outcome of an application of SLEIGN2 to the following two boundary values problems is given below:

$$(i) \quad I = (0,1] \quad [y,u](0) = 0 \qquad y(1) = 0 \tag{6.57}$$

$$(ii) \quad I = (0,1] \quad [y,v](0) = 0 \qquad y(1)0 \tag{6.58}$$

(i)	$k = 1$	$\lambda_0 :$	30.39575
		$\lambda_1 :$	102.4418
	$k = -0.5$	$\lambda_0 :$ 24.4595	
		λ_1 94.1055	
(ii)	$k = 1$	$\lambda_0 :$	-1.58500
		$\lambda_1 :$	42.1899
		$\lambda_2 :$	127.6009
	$k = -0.5$	*Not able to obtain numbers reliable to more than one digit.*	

This tidal wave example has presented SLEIGN2 with a number of unusual difficulties which are under investigation. We hope to report further on this example in a subsequent publication.

Comments on the examples.

1. Two boundary value problems are considered for each equation. For Examples 1, 2, and 3 there is a "special" boundary condition. This is the condition which determines the Friedrichs extension. It is condition (6.3) for Example 1, (6.11) for Example 2 and (6.18) for Example 3. For the first two examples this follows from the well known fact (see Niessen and Zettl [21]) that the Friedrichs extension is determined by the principal solution boundary condition. In Example 3 this follows from the fact that the maximal domain function u in (6.16) is asymptotically the same as the principle solution $M_{k,1/2}(x/k)$. In these three examples both endpoints are nonoscillatory and consequently (see [21]) the minimal operator is bounded below and hence has a Friedrichs extension.

   In Examples 4 and 5 one endpoint is oscillatory; Example 6 has both endpoints oscillatory. Consequently ([21]) the minimal operator is not bounded below. Therefore there is no Friedrichs extension in these three cases, i.e. there is no "special" ("ausgezeichnete" is the word Friedrichs used) boundary condition.

   Examples 7 and 8 follow the same pattern as Examples 1, 2 and 3; the special boundary conditions are (6.46) and (6.57) respectively.

2. In Examples 1, 2, 7 and 8, SLEIGN rather than SLEIGN2 could be used to compute the eigenvalues for the Friedrichs boundary condition, but not for the other boundary conditions. For Examples 4, 5 and 6, SLEIGN2 must be used since SLEIGN is not designed to handle general limit-circle boundary conditions.

3. We believe SLEIGN2 is the first general purpose S-L code capable of computing eigenvalues in the oscillatory limit-circle case.

4. In giving the numerical results from SLEIGN2 for Examples 1 to 8 we have not attempted to obtain precision results of the kind given in papers relating to SLEIGN; see for example Bailey [3], Bailey, Garbow, Kaper, Zettl [5] and Marletta [19]. We have been concerned in this general paper to show that it is possible to offer an automatic computing code for Sturm-Liouville boundary value problems in the limit-circle case <u>and</u> allow the user to select boundary conditions. The results obtained and the comparison with earlier results or results from transcendental equations, confirm our belief that SLEIGN2 is capable of fulfilling our objectives. When SLEIGN2 is made available in the public domain we shall also make available more detailed numerical results which will indicate not only the scope, but also the precision of this code.

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