Regular and singular Sturm-Liouville problems with coupled boundary conditions

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Abstract

Eigenvalues of both regular and singular Sturm-Liouville (S-L) problems with general coupled self-adjoint boundary conditions are characterized. This characterization, although elementary, appears to be new even in the regular case. The singular characterization is an exact parallel of the regular one and reduces to it. One application yields inequalities among the eigenvalues of different coupled boundary conditions. This is a far-reaching extension, even in the regular case, of the well-known relationship among the periodic and semi-periodic eigenvalues.

1 Introduction

Let

$$\{\lambda_n(e^{i\alpha}K),\ n \in N_0 = \{0,1,2,\ldots\}\} \tag{1.1}$$

denote the eigenvalues of the classical regular S-L problem consisting of the equation

$$-(py')' + qy = \lambda wy \text{ on } [a,b],\ \ p > 0, w > 0$$

with coupled boundary conditions

$$Y(b) = e^{i\alpha} KY(a), \tag{1.2}$$

$$\textit{where}\ \ -\pi \le \alpha \le \pi,\ Y = \begin{pmatrix} y \\ py' \end{pmatrix} \text{ and } K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix},\ \ k_{ij} \in \mathbb{R},\ \ \det(K) = 1. \tag{1.3}$$

The case when $\alpha = 0$, $K = I$, the identity matrix, and both endpoints are regular is referred to as the periodic case. When $\alpha = 0$ and $K = -I$ we have the classical semi-periodic problem.

A well-known classical result [W1], [E] gives the following inequalities for $-\pi < \alpha < 0$ and $0 < \alpha < \pi$:

$$-\infty < \lambda_0(I) < \lambda_0(e^{i\alpha}I) < \lambda_0(-I) \le \lambda_1(-I) < \lambda_1(e^{i\alpha}I) < \lambda_1(I) \le \lambda_2(I) < \lambda_2(e^{i\alpha}I) < \cdots \tag{1.4}$$

In this paper we

(i) extend (1.4) to the case when I is replaced by K for a wide class of matrices K,

(ii) prove a version of (1.4) for coefficients p which change sign and arbitrary K,

(iii) extend (1.4) to the singular case and for a wide class of K,

A key role in our proofs is played by a singular analogue of the regular boundary conditions (1.2) similar to that used by Krall and Zettl in [KZ]. It is an exact parallel of the regular case and reduces to it.

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2 Notation and basic assumptions

Consider the (quite possibly most widely studied) differential equation

$$-(py')' + qy = \lambda wy \text{ on } (a,b),\ -\infty \leq a < b \leq \infty. \tag{2.1}$$

The Lebesgue measurable functions p, q, w are assumed to satisfy

$$p, q, w, \text{are real valued and } \frac{1}{p},\ q,\ w \in L_{\mathrm{loc}}(a,b), \text{with } w > 0 \text{ a.e. on } (a,b). \tag{2.2}$$

Remark. Note that we do not assume that $p > 0$. Certainly (2.2) can hold for step functions p with both positive and negative steps and also for continuous p which change sign but with "mild" zeros in (a,b). Below, sign conditions will be placed on p only as needed.

Throughout this paper we assume that (2.2) holds and that:

$$\text{Each endpoint } a,\ b \text{ is either regular or limit-circle (LC).} \tag{2.3}$$

For definitions of well-known terms such as limit-circle (LC), oscillatory (O), nonoscillatory (NO), etc., and for basic well-known results on S-L problems, the reader is referred to [W1] and [N].

The maximal domain Δ associated with (2.1) is defined by

$$\Delta = \{y : (a,b) \to \mathbb{C} : y,\ py' \in AC_{\mathrm{loc}}(a,b) \text{ and } y,\ w^{-1}[-(py')' + qy] \in L^2_w(a,b)\}. \tag{2.4}$$

Here AC_{loc} denotes the complex-valued functions on (a,b) which are absolutely continuous on all compact subintervals of (a,b).

The Lagrange sesquilinear form of (2.1) is given by

$$[y,z] = yp\bar{z}' - \bar{z}py',\ \ y,z \in \Delta. \tag{2.5}$$

In what follows, θ and ϕ denote functions in Δ satisfying the following conditions:

1. they are real valued,
2. for some real λ they are solutions in some neighborhood of a and for some real λ they are solutions in some neighborhood of b; they need not be the same solutions near a and near b and they need not be solutions through the interior of (a,b),
3. $[\theta,\phi](a) = \lim_{t\to a^+} [\theta,\phi](t) = 1$,
4. $[\theta,\phi](b) = \lim_{t\to b^-} [\theta,\phi](t) = 1$. (2.6)

Such functions θ and ϕ exist. This follows from "Naimark's Lemma" (see [N, p.63]).

Self-adjoint boundary conditions fall into two mutually exclusive classes: separated and coupled. The former state the conditions separately at each endpoint, the latter connect ("couple") the endpoints together.

Lemma 2.1 All coupled self-adjoint boundary conditions for (2.1), with each endpoint either regular or LC, have the following representation:

$$Y(b) = e^{i\alpha} KY(a), \ i = \sqrt{-1}, \ -\pi \leq \alpha \leq \pi, \tag{2.7}$$

$$\text{where } K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \text{ with } k_{ij} \in \mathbb{R}, \ 1 \leq i,j \leq 2, \ \det(K) = k_{11}k_{22} - k_{12}k_{21} = 1, \tag{2.8}$$

and, for all $y \in \Delta$,

$$Y(t) = \begin{pmatrix} [y,\theta](t) \\ [y,\phi](t) \end{pmatrix} \ (t \in (a,b)). \tag{2.9}$$

Note that $[y,\theta](a) = \lim_{t \to a^+} [y,\theta](t)$ exists and is finite for each $y \in \Delta$; similarly for $[y,\phi](a)$, $[y,\theta](b)$ and $[y,\phi](b)$. This follows from the fact that each end point is regular or LC.

Proof. Although this is not explicitly stated in [KZ, p.427] it follows from the representation of LC boundary conditions given there. ⋄

Lemma 2.2 Fix $\lambda \in \mathbb{C}$. For each $c, d \in \mathbb{C}$, the singular "LC initial value problem" for (2.1):

$$[y,\theta](a) = c \quad \text{and} \quad [y,\phi](a) = d \tag{2.10}$$

has a unique solution y. Similarly at b.

Proof. Let θ and ϕ be solutions of (2.1) on $(a, a+\delta)$ for some $\delta > 0$ and some fixed real λ_0. To show existence choose $y = d\ \theta - c\ \phi$. To establish uniqueness let u and v be solutions of (2.1) for some arbitrary real λ both satisfying (2.10) on $(a, a+\delta)$. Let $\Phi = \begin{pmatrix} \theta & \phi \\ p\theta' & p\phi' \end{pmatrix}$, $U = \begin{pmatrix} u \\ pu' \end{pmatrix}$, $V = \begin{pmatrix} v \\ pv' \end{pmatrix}$, $G = \begin{pmatrix} \theta\phi & \phi\phi \\ -\theta\theta & -\theta\phi \end{pmatrix}$. Set $Y = \begin{pmatrix} y \\ py' \end{pmatrix} = \Phi Z$ and note that

$$Z' = (\lambda - \lambda_0) wGZ.$$

Solving $U = \Phi Z$ by Cramers rule and noting properties (3) and (4) above we get

$$z_1(t) = [u,\phi](t), \quad a < t < a+\delta, \quad z_2(t) = [u,\theta](t), \quad a < t < a+\delta.$$

Similarly, solving $V = \Phi S$ by Cramers rule we get

$$s_1(t) = [v, \phi](t), \quad a < t < a + \delta, \quad s_2(t) = [v, \theta](t), \quad a < t < a + \delta.$$

Since the endpoint a is regular or LC we have that $wG \in L^1(a, a + \delta)$. This and $z_1(a) = s_1(a), z_2(a) = s_2(a)$ implies that $Z = S$ and hence $U = V$. In particular $u = v$. $\diamond$

For each $\lambda \in \mathbb{R}$ determine unique solutions $u = u(\cdot, \lambda)$ and $v = v(\cdot, \lambda)$ by the singular "LC initial conditions":

$$[u, \theta](a, \lambda) = 0, \ [u, \phi](a, \lambda) = 1; \quad [v, \theta](a, \lambda) = 1, \ [v, \phi](a, \lambda) = 0, \ \lambda \in \mathbb{R}. \tag{2.11}$$

Such solutions u and v exist by Lemma 2.2. We can now define the important "discriminant" function $D = D(K, \lambda)$.

Definition. Let K satisfy (2.8) and let u, v be defined by (2.11). For $\lambda \in \mathbb{R}$ define $D(K, \lambda)$ by

$$D(K, \lambda) = k_{11}[u(\cdot, \lambda), \phi](b) + k_{22}[v(\cdot, \lambda), \theta](b) - k_{12}[v(\cdot, \lambda), \phi](b) - k_{21}[u(\cdot, \lambda), \theta](b). \tag{2.12}$$

When a and b are regular end-points, θ and ϕ can be determined by the regular initial conditions:

$$\theta(a) = 0, (p\theta')(a) = 1, \phi(a) = 1, (p\phi')(a) = 0, \tag{2.13}$$

both for some real $\lambda = \lambda_a$. In this case (2.12) reduces to

$$D(K, \lambda) = k_{11}(pu')(b, \lambda) + k_{22}v(b, \lambda) - k_{12}(pv')(b, \lambda) - k_{21}u(b, \lambda), \ \lambda \in \mathbb{R}.$$

Lemma 2.3 Let (2.2), (2.3), (2.7), (2.8) hold; let $-\pi \le \alpha \le \pi$. Then the spectrum $\sigma(e^{i\alpha}K)$ of the self-adjoint boundary value problem (2.1) and (2.7) is discrete. Moreover we have the following two results:

(i) Assume, in addition to (2.2), that

$$p > 0 \text{ a.e. on } (a, b). \tag{2.14}$$

Then $\sigma(e^{i\alpha}K)$ is not bounded above. It is bounded below if and only if both end points a, b are nonoscillatory; this is always the case for regular endpoints a and b. Thus we adopt the following notation:

If each endpoint is either regular or LCNO (limit-circle nonoscillatory), then the spectrum $\sigma(e^{i\alpha}K)$ is bounded below and hence the eigenvalues can be indexed as follows:

$$\sigma(e^{i\alpha}K) = \{\lambda_n(e^{i\alpha}K), n \in N_0 = \{0, 1, 2, \ldots\}\},$$

where

$$-\infty < \lambda_0(e^{i\alpha}K) \le \lambda_1(e^{i\alpha}K) \le \lambda_2(e^{i\alpha}K) \le \cdots$$

with

$$\lambda_n(e^{i\alpha}K) \to +\infty \text{ as } n \to +\infty. \tag{2.15}$$

In (2.15) equality cannot occur in three consecutive terms since no eigenvalue can have multiplicity greater than 2.

If one or both endpoints is oscillatory (this can only happen at a singular endpoint) then the spectrum is unbounded above and below. In this case we let

$$\sigma(e^{i\alpha}K) = \{\lambda_n(e^{i\alpha}K), n \in Z = \{\dots - 2, -1, 0, 1, 2, \dots\}\}$$

where

$$\cdots \leq \lambda_{-2}(e^{i\alpha}K) \leq \lambda_{-1}(e^{i\alpha}K) \leq \lambda_0(e^{i\alpha}K) \leq \lambda_1(e^{i\alpha}K) \leq \lambda_2(e^{i\alpha}K) \leq \cdots$$

and

$$\lambda_n(e^{i\alpha}K) \to -\infty \text{ as } n \to -\infty, \quad \lambda_n(e^{i\alpha}K) \to +\infty \text{ as } n \to +\infty. \tag{2.16}$$

Equality cannot occur in three consecutive terms. We further adopt the convention, in this case, that λ_0 is the first non-negative eigenvalue. This determines the indexing scheme uniquely.

The case when $p < 0$ a.e. on (a, b) is entirely similar with the positive and negative directions interchanged.

(ii) Assume that p is positive on a set of positive Lebesgue measure and p is negative on a set of positive Lebesgue measure. In this case we say that "p changes sign" on (a, b). Then the spectrum is unbounded above and below regardless of whether or not the endpoints are oscillatory. Thus we have, in this case, the same indexing scheme and corresponding properties as in (2.16) above.

Proof. Part (ii) is proved by M. Möller in [M]. The rest is well known - see [W2]. ⋄

Lemma 2.4 (The Plücker identity.) Let θ and ϕ be defined as above. Then

$$[y, z] = \det \begin{pmatrix} [y, \theta] & [y, \phi] \\ [\bar{z}, \theta] & [\bar{z}, \phi] \end{pmatrix}, \ y, z \in \Delta. \tag{2.17}$$

Proof. See [KZ, Lemma 1, p. 429]. ⋄

3 Results and proofs

Theorem 1 gives an elementary and quite useful characterization of the eigenvalues of regular or singular Sturm-Liouville problems with coupled boundary conditions.

Theorem 1 *Let (2.2), (2.3), (2.7), (2.8), (2.9), (2.11), (2.12) hold. Then for any* α, $-\pi \le \alpha \le \pi$, *the number* λ *is an eigenvalue of the (S-L) boundary value problem (2.1), (2.7) if and only if*

$$D(K,\lambda) = 2\cos\alpha. \tag{3.1}$$

Proof. Let $y = y(\cdot,\lambda) = cu(\cdot,\lambda) + dv(\cdot,\lambda) = cu + dv$. Then

$$Y(b) = \begin{pmatrix} [y,\theta](b) \\ [y,\phi](b) \end{pmatrix} = \begin{pmatrix} c[u,\theta](b) + d[v,\theta](b) \\ c[u,\phi](b) + d[v,\phi](b) \end{pmatrix},$$

and, using (2.11), we have

$$e^{i\alpha}KY(a) = e^{i\alpha}K\begin{pmatrix} d \\ c \end{pmatrix} = e^{i\alpha}\begin{pmatrix} k_{11}d + k_{12}c \\ k_{21}d + k_{22}c \end{pmatrix}.$$

Hence (2.7) holds if and only if

$$\begin{cases} \{[u,\theta](b) - e^{i\alpha}k_{12}\}c + \{[v,\theta](b) - e^{i\alpha}k_{11}\}d = 0 \\ \{[u,\phi](b) - e^{i\alpha}k_{22}\}c + \{[v,\phi](b) - e^{i\alpha}k_{21}\}d = 0. \end{cases} \tag{3.2}$$

The system (3.2) has a nontrivial solution for the pair (c,d) if and only if

$$\begin{aligned} 0 = \det &\begin{pmatrix} [u(\cdot,\lambda),\theta](b) - e^{i\alpha}k_{12} & [v(\cdot,\lambda)\theta](b) - e^{i\alpha}k_{11} \\ [u(\cdot,\lambda),\phi](b) - e^{i\alpha}k_{22} & [v(\cdot,\lambda)\phi](b) - e^{i\alpha}k_{21} \end{pmatrix} \\ &= \{[u(\cdot,\lambda),\theta](b) - e^{i\alpha}k_{12}\}\{[v(\cdot,\lambda),\phi](b) - e^{i\alpha}k_{21}\} \\ &\quad -\{[v(\cdot,\lambda),\theta](b) - e^{i\alpha}k_{11}\}\{[u(\cdot,\lambda),\phi](b) - e^{i\alpha}k_{22}\} \\ &= [u(\cdot,\lambda),\theta](b)[v(\cdot,\lambda),\phi](b) - [v(\cdot,\lambda),\theta](b)[u(\cdot,\lambda),\phi](b) \\ &\quad -e^{i\alpha}k_{12}[v(\cdot,\lambda),\phi](b) - e^{i\alpha}k_{21}[u(\cdot,\lambda),\theta](b) + e^{2i\alpha}k_{12}k_{21} \\ &\quad +e^{i\alpha}k_{11}[u(\cdot,\lambda),\phi](b) + e^{i\alpha}k_{22}[v(\cdot,\lambda),\theta](b) - e^{2i\alpha}k_{11}k_{22} \\ &= -1 + e^{i\alpha}\Big(k_{11}[u(\cdot,\lambda),\phi](b) + k_{22}[v(\cdot,\lambda),\theta](b) \\ &\quad -k_{12}[v(\cdot,\lambda),\phi](b) - k_{21}[u(\cdot,\lambda),\theta](b)\Big) - e^{2i\alpha}(k_{11}k_{22} - k_{12}k_{21}). \end{aligned} \tag{3.3}$$

Here we used Lemma 2.4 in the last step.

Now (3.1) follows from (3.3), (2.7), (2.12) and $e^{-i\alpha} + e^{i\alpha} = 2\cos\alpha$. ⋄

Theorem 2 *Assume the hypotheses and notation of Theorem 1 hold. Let* $\sigma(e^{i\alpha}K) = \{\lambda_n(e^{i\alpha}K), n \in N_0$ *or* $n \in Z\}$. *Then*

(i) $\quad \lambda_n(e^{-i\alpha}K) = \lambda_n(e^{i\alpha}K),\ 0 \le \alpha \le \pi, \quad (3.4)$

(ii) $\quad \lambda_n(e^{i\alpha}K) \ne \lambda_m(e^{i\beta}K),\ 0 \le \alpha \ne \beta \le \pi,\ n,m \in Z. \quad (3.5)$

In case (i) the corresponding eigenfunctions are complex conjugates of each other.

Proof. Since $\cos(-\alpha) = \cos(\alpha)$, (3.4) follows directly from (3.3). The statement about the eigenfunctions follows directly from (2.1) and (2.7), and (3.5) is a consequence of (3.1). $\diamond$

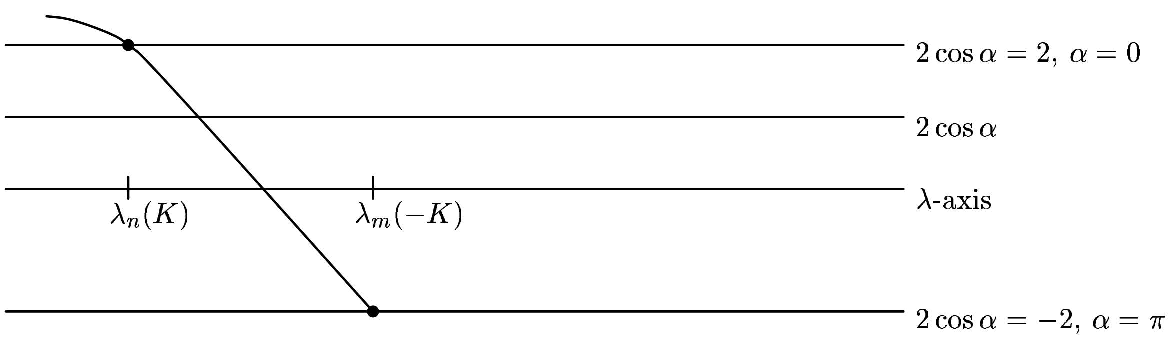
Theorem 3 *Let the notation and hypotheses of Theorem 1 hold. For given K and α, $-\pi \le \alpha \le \pi$, a number λ is an eigenvalue of multiplicity two if and only if all four of the following equations are satisfied:*

$$[u(\cdot,\lambda),\theta](b) = e^{i\alpha}k_{12}, \ [v(\cdot,\lambda),\theta](b) = e^{i\alpha}k_{11}, \ [u(\cdot,\lambda),\phi](b) = e^{i\alpha}k_{22}, \ [v(\cdot,\lambda),\phi](b) = e^{i\alpha}k_{21}.$$

Proof. This follows directly from (3.2). $\diamond$

Theorem 4 *Let (2.2), (2.3), (2.7), (2.9), (2.11), and (2.12) hold. Let K satisfy (2.8). Then for any α satisfying $0 < \alpha < \pi$ or $-\pi < \alpha < 0$, each eigenvalue of $\sigma(e^{i\alpha}K)$ is simple.*

Proof. Let $D(K,\lambda)$ be given by (2.12) and consider its graph:



By Theorem 1 we have $D(K,\lambda) = 2$ for each $\lambda \in \sigma(K)(\alpha = 0)$ and $D(\lambda,-K) = -2$ for each $\lambda \in \sigma(-K)(\alpha = \pi)$. Consider $\lambda_n(K)$ for some n and let $\lambda_m(-K)$ be the first eigenvalue of $\sigma(-K)$ to the right of $\lambda_n(K)$. Then $D(\lambda,K)$ is either strictly increasing or strictly decreasing in the interval $\lambda \in [\lambda_n(K), \lambda_m(-K)]$. If not then, noting that $D(\lambda,K)$ is continuous in λ and using continuity arguments, it would follow that there exists an α in the open interval $(0,\pi)$ such that the equation

$$D(K,\lambda) = 2\cos\alpha$$

has three solutions for λ in the interval $(\lambda_n(K), \lambda_m(-K))$, i.e., the spectrum $\sigma(e^{i\alpha}K)$ has three (or more) points in the interval $(\lambda_n(K), \lambda_m(-K))$. On the other hand, the spectrum $\sigma(K)$ (or $\sigma(-K)$) contains no point in the interval $(\lambda_n(K), \lambda_m(-K))$. This contradicts [W2, Corollary 1, p. 246], since the minimal operator associated with (2.1) in $L^2_w(a,b)$ has deficiency index (2,2) and hence there is no interval on which one self-adjoint extension has three eigenvalues and another self-adjoint extension has no eigenvalue.

Therefore, $D(K, \lambda)$ is strictly monotone for λ in $[\lambda_n(K), \lambda_m(-K)]$ and hence it follows from Theorem 1 that each eigenvalue of $\sigma(e^{i\alpha}K)$ is simple for $0 < \alpha < \pi$ or $-\pi < \alpha < 0$. This completes the proof of Theorem 4. ⋄

Before stating Theorem 5 we establish another helpful lemma.

Lemma 3.1 Assume $-\infty < a < b < \infty$, $p > 0$ a.e. and q/w is bounded below on $[a, b]$, i.e., for some $\lambda_1 \in \mathbb{R}$,

$$q(t) - \lambda_1 w(t) > 0, \;\; t \in [a, b] \text{ a.e.} \tag{3.6}$$

If y is a solution of (2.1) with $\lambda = \lambda_1$ satisfying

$$y(a) = c \geq 0 \text{ and } (py')(a) = d \geq 0, \;\; c^2 + d^2 > 0, \tag{3.7}$$

then y and py' are increasing on $[a, b]$.

Proof. For some α in (a, b), $y(t) > 0$ on $(a, \alpha]$. Hence $(py')' = (q - \lambda_1 w)y \geq 0$ on $[a, \alpha]$ and py' and y are increasing on $[a, \alpha]$.

If $y(\beta) \leq \alpha$ for some β in $(\alpha, b]$, then y assumes its maximum value on the interval $[\alpha, \beta]$ at some point $\gamma \in (\alpha, \beta)$. Repeating the above argument with the end point a replaced by γ, we find that y is increasing in a right neighborhood of γ. This is a contradiction. Hence, y is increasing on $[a, b]$ and similarly, we get that py' is increasing on $[a, b]$. In particular we have that

$$y(b) > c \text{ and } (py')(b) > d.$$

This completes the proof of Lemma 3.1. ⋄

The general self-adjoint coupled boundary conditions (2.7), (2.8), (2.9) reduce to the familiar form

$$Y(b) = e^{i\alpha} K Y(a), \;\; Y = \begin{pmatrix} y \\ py' \end{pmatrix} \tag{3.8}$$

when a and b are regular endpoints. This can be seen as follows:

Determine solutions θ and ϕ by the initial conditions:

$$\theta(a) = 0, \;\; (p\theta')(a) = 1; \quad \phi(a) = -1, \; (p\phi')(a) = 0,$$

both for some $\lambda = \lambda_a \in \mathbb{R}$. Then $[y, \theta](a) = y(a)$ and $[y, \phi](a) = (py')(a)$. Thus

$$Y(a) = \begin{pmatrix} [y, \theta](a) \\ [y, \phi](a) \end{pmatrix} = \begin{pmatrix} y(a) \\ (py')(a) \end{pmatrix},$$

and $[\theta, \phi](a) = 1$.

Similarly, we can determine (a possibly different pair) of solutions θ and ϕ at b. Then we "connect" these different pairs through the interior of (a, b) with the Naimark Lemma [N, p.63] to obtain maximal domain functions θ, ϕ on (a, b). For such θ and ϕ, (2.7)–(2.9) reduces to (3.8).

Theorem 5 *Let (2.1), (2.2), (2.6), (2.7), (2.9) hold. Assume further that*

$$\textit{Each endpoint } a, b \textit{ is either regular or LCNO (LC nonoscillatory)} \tag{3.9}$$

and

$$p > 0 \ a.e. \ on \ (a,b). \tag{3.10}$$

Suppose that K *satisfies one of the following conditions:*

$$\text{(i)} \qquad K = \begin{pmatrix} c & -c+1/d \\ -d & d \end{pmatrix}, \ or \ K = \begin{pmatrix} c & -d \\ -c+1/d & d \end{pmatrix}, \ c > 0, \ d > 0 \tag{3.11}$$

$$\text{(ii)} \qquad K = \begin{pmatrix} c & h \\ 0 & d \end{pmatrix} \ or \ K = \begin{pmatrix} c & 0 \\ h & d \end{pmatrix} \ c > 0, \ d > 0, \ h \leq 0, \ c\,d = 1. \tag{3.12}$$

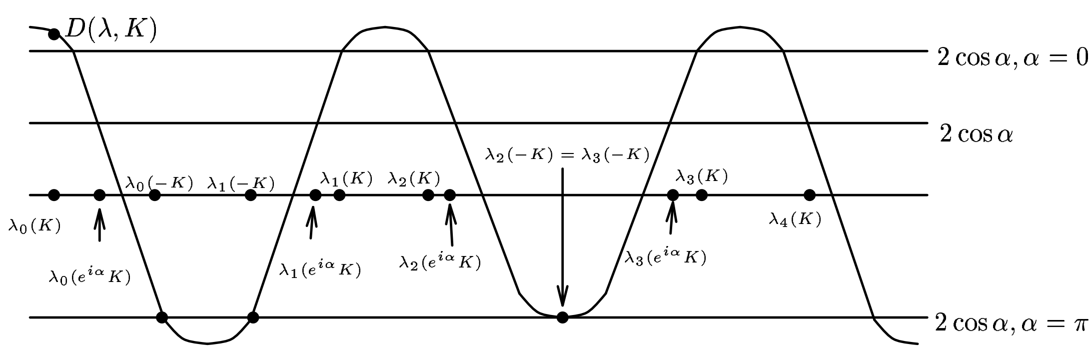
Then the eigenvalues of (2.1), (2.7) are bounded below and the first one $\lambda_0(K)$ *is simple and we have the following ordering for* $-\pi < \alpha < 0$ *and* $0 < \alpha < \pi$*:*

$$-\infty < \lambda_0(K) < \lambda_0(e^{i\alpha}K) < \lambda_0(-K) \leq \lambda_1(-K) < \lambda_1(e^{i\alpha}K) < \lambda_1(K) \leq \lambda_2(K) < \lambda_2(e^{i\alpha}K) < \cdots, \tag{3.13}$$

Furthermore, we have for $0 < \alpha < \beta < \pi$*,*

$$\lambda_0(e^{i\alpha}K) < \lambda_0(e^{i\beta}K) < \lambda_1(e^{i\beta}K) < \lambda_1(e^{i\alpha}K) < \lambda_2(e^{i\alpha}K) < \lambda_2(e^{i\beta}K) < \lambda_3(e^{i\beta}K) < \lambda_3(e^{i\alpha}K) < \cdots \tag{3.14}$$

The inequalities (3.13) and (3.14) are illustrated on the graph:



Note that $D(-K,\lambda) = -D(K,\lambda)$.

Moreover, for general K satisfying (2.8) and (3.11) or (3.12) we have that the smaller of the two eigenvalues $\lambda_0(K)$ and $\lambda_0(-K)$ is simple.

Proof of Theorem 5. The case when one or both endpoints are infinite can be transformed to the case when they are both finite with a transformation that preserves the LCNO class. In the latter case, by results of Niessen and Zettl [NZ, pp.555,561], for any problem with one or both endpoints finite and LCNO there is an equivalent regular problem with exactly the same eigenvalues. Thus it suffices to prove Theorem 5 for the regular case.

First we prove the case when q/w is bounded below on $[a, b]$, say $q(t) \geq -Mw(t)$, a.e. $t \in [a, b]$, $M \in \mathbb{R}$. By Lemma 3.1 we have

$$v(b, \lambda) > 1, \ (pu')(b, \lambda) > 1 \text{ for } \lambda < -M. \tag{3.15}$$

From (3.15) and (2.12) we get

$$D(\lambda, K) > 2 \tag{3.16}$$

for $\lambda < -M$ and K satisfying (3.11) or (3.12).

The inequalities (3.13) now follow for this case from (3.16), Theorem 4 and the proof of Theorem 4 which establishes the strict monotonicity of $D(\lambda, K)$ for λ strictly between the appropriate eigenvalues corresponding to K and $-K$.

If q/w is not bounded below we define, for $M > 0$,

$$q_M(t) = \begin{cases} q(t), & \text{if } q(t) \geq -Mw(t), \\ -Mw(t), & \text{if } q(t) < -Mw(t). \end{cases}$$

Then $q_M \to q$ in $L^1(a, b)$ as $M \to \infty$ by the Lebesgue dominated convergence theorem. Let $y_M = y(\cdot, c, d, \lambda, p, q_m, w)$ be the unique solution of (2.1) determined by the initial conditions $y(a) = c$, $(py')(a) = d$. Then $y_M \to y$ uniformly on $[a, b]$ as $M \to \infty$ by the $L^1(a, b)$ continuous dependence of solutions y of (2.1) on q (see [W1, p.23]).

We know that the equation $D(\lambda, K) = 2$ has a smallest solution, namely $\lambda = \lambda_0(K)$. Thus by continuity, either $D(\lambda, K) > 2$ for all $\lambda < \lambda_0(K)$ or $D(\lambda, K) < 2$ for all $\lambda < \lambda_0(K)$. In the latter case we would have $D(\lambda, K, q) < 2$ for all $\lambda < \lambda_0(K)$ and $D(\lambda, K, q_M) > 2$ for all $\lambda < -M$ and any $M > 0$. This yields a contradiction to $y_M(\cdot, \lambda) \to y(\cdot, \lambda)$ on $[a, b]$. ◇

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