A Catalogue of Sturm-Liouville differential equations

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Dedicated to all scientists who, down the long years, have contributed to Sturm-Liouville theory.

Abstract. The idea for this catalogue follows from the conference entitled:

Bicentenaire de Charles François Sturm

held at the University of Geneva, Switzerland from 15 to 19 September 2003. One of the main interests for this meeting involved the historical development of the theory of Sturm-Liouville differential equations. This theory began with the original work of Sturm from 1829 to 1836 and was then followed by the short but significant joint paper of Sturm and Liouville in 1837, on second-order linear ordinary differential equations with an eigenvalue parameter.

The details of the early development of Sturm-Liouville theory, from the beginnings about 1830, are given in a historical survey paper of Jesper Lützen (1984), in which paper a complete set of references may be found to the relevant work of both Sturm and Liouville.

The catalogue commences with sections devoted to a brief summary of Sturm-Liouville theory including some details of differential expressions and equations, Hilbert function spaces, differential operators, classification of interval endpoints, boundary condition functions and the Liouville transform.

There follows a collection of more than 50 examples of Sturm-Liouville differential equations; many of these examples are connected with well-known special functions, and with problems in mathematical physics and applied mathematics.

For most of these examples the interval endpoints are classified within the relevant Hilbert function space, and boundary condition functions are given to determine the domains of the relevant differential operators. In many cases the spectra of these operators are given.

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Contents

1. Introduction

The idea for this paper follows from the conference entitled:

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held at the University of Geneva, Switzerland from 15 to 19 September 2003. One of the main interests for this meeting involved the development of the theory of Sturm-Liouville differential equations. This theory began with the original work of Sturm from 1829 to 1836 and then followed by the short but significant joint paper of Sturm and Liouville in 1837, on second-order linear ordinary differential equations with an eigenvalue parameter. Details for the 1837 paper is given as reference [56] in this paper; for a complete set of historical references see the historical survey paper [59] of Lützen.

This present catalogue of examples of Sturm-Liouville differential equations is based on four main sources:

1. The list of 32 examples prepared by Bailey, Everitt and Zettl in the year 2001 for the final version of the computer program SLEIGN2; this list is to be found within the LaTeX file xamples.tex contained in the package associated with the publication [11, Data base file xamples.tex]; all these 32 examples are contained within this catalogue.
2. A selection from the set of 59 examples prepared by Pryce and published in 1993 in the text [69, Appendix B.2]; see also [70].
3. A selection from the set of 217 examples prepared by Pruess, Fulton and Xie in the report [68].
4. A selection drawn up from a general appeal, made in October 2003, for examples but with the request relayed in the following terms; examples to be included should satisfy one or more of the following criteria:
   (i) The solutions of the differential equation are given explicitly in terms of special functions; see for example Abramowitz and Stegun [1], the Erdélyi *at al* Bateman volumes [27], the recent text of Slavyanov and Lay [77] and the earlier text of Bell [16].
   (ii) Examples with special connections to applied mathematics and mathematical physics.
   (iii) Examples with special connections to numerical analysis; see the work of Zettl [81] and [82].

The overall aim was to be content with about 50 examples, as now to be seen in the list given below.

The naming of these examples of Sturm-Liouville differential equations is somewhat arbitrary; where named special functions are concerned the chosen name is clear; in certain other cases the name has been chosen to reflect one or more of the authors concerned.

2. Notations

The real and complex fields are represented by $\mathbb{R}$ and $\mathbb{C}$ respectively; a general interval of $\mathbb{R}$ is represented by I; compact and open intervals of $\mathbb{R}$ are represented by $[a,b]$ and (a,b) respectively. The prime symbol $'$ denotes classical differentiation on the real line $\mathbb{R}$.

Lebesgue integration on $\mathbb{R}$ is denoted by L, and $L^1(I)$ denotes the Lebesgue integration space of complex-valued functions defined on the interval I. The local integration space $L^1_{\mathrm{loc}}(I)$ is the set of all complex-valued functions on I which are Lebesgue integrable on all compact sub-intervals $[a,b] \subseteq I$; if I is compact then $L^1(I) \equiv L^1_{\mathrm{loc}}(I)$.

Absolute continuity, with respect to Lebesgue measure, is denoted by AC; the space of all complex-valued functions defined on I which are absolutely continuous on all compact sub-intervals of I, is denoted by $AC_{\mathrm{loc}}(I)$.

A weight function w on I is a Lebesgue measurable function $w : I \to \mathbb{R}$ satisfying $w(x) > 0$ for almost all $x \in I$.

Given an interval I and a weight function w the space $L^2(I;w)$ is defined as the set of all complex-valued, Lebesgue measurable functions $f : I \to \mathbb{C}$ such that

$$\int_I |f(x)|^2 \, w(x) \; dx < +\infty.$$

Taking equivalent classes into account $L^2(I;w)$ is a Hilbert function space with inner product

$$(f,g)_w := \int_I f(x)\overline{g}(x)w(x) \; dx \text{ for all } f,g \in L^2(I;w).$$

3. Sturm-Liouville differential expressions and equations

Given the interval (a,b), then a set of Sturm-Liouville coefficients $\{p,q,w\}$ has to satisfy the minimal conditions

$$\begin{array}{rl} (i) & p,q,w : (a,b) \to \mathbb{R} \\ (ii) & p^{-1},q,w \in L^1_{\mathrm{loc}}(a,b) \\ (iii) & w \text{ is a weight function on } (a,b). \end{array}$$

Note that in general there is no sign restriction on the leading coefficient p.

Given the interval (a,b) and the set of Sturm-Liouville coefficients $\{p,q,w\}$ the associated Sturm-Liouville differential expression $M(p,q) \equiv M[\cdot]$ is the linear operator defined by

$$\begin{array}{rl} (i) & \text{domain } D(M) := \{f : (a,b) \to \mathbb{C} : f, pf' \in AC_{\mathrm{loc}}(a,b)\} \\ (ii) & \begin{cases} M[f](x) := -(p(x)f(x)')' + q(x)f(x) \text{ for all } f \in D(M) \\ \text{and almost all } x \in (a,b). \end{cases} \end{array}$$

We note that $M[f] \in L^1_{\mathrm{loc}}(I)$ for all $f \in D(M)$; it is shown in [64, Chapter V, Section 17] that $D(M)$ is dense in the Banach space $L^1(a,b)$.

Given the interval (a,b) and the set of Sturm-Liouville coefficients $\{p,q,w\}$ the associated Sturm-Liouville differential equation is the second-order linear ordinary differential equation

$$M[y](x) \equiv -(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \text{ for all } x \in (a,b),$$

where $\lambda \in \mathbb{C}$ is a complex-valued spectral parameter.

The above minimal conditions on the set of coefficients $\{p,q,w\}$ imply that the Sturm-Liouville differential equation has a solution to any initial value problem at a point $c \in (a,b)$; see the existence theorem in [64, Chapter V, Section 15], *i.e.* given two complex numbers $\xi, \eta \in \mathbb{C}$ and any value of the parameter $\lambda \in \mathbb{C}$, there exists a unique solution of the differential equation, say $y(\cdot,\lambda) : (a,b) \to \mathbb{C}$, with the properties:

$$\begin{array}{rl} (i) & y(\cdot,\lambda) \text{ and } (py')(\cdot,\lambda) \in AC_{\text{loc}}(a,b) \\ (ii) & y(c,\lambda) = \xi \text{ and } (py')(c,\lambda) = \eta \\ (iii) & y(x,\cdot) \text{ and } (py')(x,\cdot) \text{ are holomorphic on } \mathbb{C}. \end{array}$$

4. Operator theory

Full details of the following quoted operator theoretic results are to be found in [64, Chapter V, Section 17] and [34, Sections I, IV and V].

The Green's formula for the differential expression M is, for any compact interval $[\alpha,\beta] \subset (a,b)$,

$$\int_\alpha^\beta \left\{\overline{g}(x)M[f](x) - f(x)\overline{M[g]}(x)\right\} dx = [f,g](\beta) - [f,g](\alpha) \text{ for all } f,g \in D(M),$$

where the symplectic form $[\cdot,\cdot](\cdot) : D(M) \times D(M) \times (a,b) \to \mathbb{C}$ is defined by

$$[f,g](x) := f(x)(p\overline{g}')(x) - (pf')(x)\overline{g}(x).$$

Incorporating now the weight function w and the Hilbert function space $L^2((a,b);w)$, the maximal operator T_1 generated from M is defined by

$$\begin{array}{rl} (i) & T_1 : D(T_1) \subset L^2((a,b);w) \to L^2((a,b);w) \\ (ii) & D(T_1) := \{f \in D(M) : f, w^{-1}M[f] \in L^2((a,b);w)\} \\ (iii) & T_1 f := w^{-1}M[f] \text{ for all } f \in D(T_1). \end{array}$$

We note that, from the Green's formula, the symplectic form of M has the property that the following limits

$$[f,g](a) := \lim_{x \to a^+} [f,g](x) \text{ and } [f,g](b) := \lim_{x \to b^-} [f,g](x)$$

both exist and are finite in $\mathbb{C}$.

The minimal operator T_0 generated by M is defined by

$$\begin{array}{rl} (i) & T_0 : D(T_0) \subset L^2((a,b);w) \to L^2((a,b);w) \\ (ii) & D(T_0) := \{f \in D(T_1) : [f,g](b) = [f,g](a) = 0 \text{ for all } g \in D(T_1) \\ (iii) & T_0 f := w^{-1}M[f] \text{ for all } f \in D(T_0). \end{array}$$

With these definitions the following properties hold for T_0 and T_1, and their adjoint operators

- (i) $T_0 \subseteq T_1$
- (ii) T_0 is closed and symmetric in $L^2((a,b);w)$
- (iii) $T_0^* = T_1$ and $T_1^* = T_0$
- (iv) T_1 is closed in $L^2((a,b);w)$
- (v) T_0 has equal deficiency indices (d,d) with $0 \leq d \leq 2$.

Self-adjoint extensions T of T_0 exist and satisfy

$$T_0 \subseteq T \subseteq T_1$$

where the domain $D(T)$ is determined, as a restriction of the domain $D(T_1)$, by applying symmetric boundary conditions to the elements of the maximal domain $D(T_1)$.

5. Endpoint classification

A detailed account of the classification of the endpoints a and b of the interval (a,b), given the coefficients $\{p,q,w\}$, is provided in the SLEIGN2 paper [10, Section 4]. Here we give a shorter account to cover all the examples selected for this paper.

Given the interval (a,b) and the coefficients $\{p,q,w\}$ the endpoint a is classified, independently, as *regular* (notation R), *limit-point* (notation LP), *limit-circle* (notation LC), as follows:

1. The endpoint a is R if $a \in \mathbb{R}$ and for $c \in (a,b)$ the coefficients satisfy
$$p^{-1}, q, w \in L^1(a,c]$$
2. The endpoint a is LC if a is not R and there exist elements $f,g \in D(T_1)$ such that
$$[f,g](a) \neq 0$$
3. The endpoint a is LP if for all elements $f,g \in D(T_1)$
$$[f,g](a) = 0.$$

Remark 5.1. We note

1. There is a similar classification into R, LC and LP for the endpoint b of the interval (a,b).
2. The classification of both endpoints a and b depends only on the coefficients $\{p,q,w\}$ and not on the spectral parameter λ.
3. The endpoint classifications for a and b are analytically connected to the number of solutions of the Sturm-Liouville differential equation $M[y] = \lambda wy$ on (a,b) in the space $L^2((a,b);w)$; see the more detailed account in [10, Section 4].
4. For any endpoint of a Sturm-Liouville differential expression the three classifications R, LC and LP are mutually exclusive.

5. If a is R then the classical initial value problem, see the end of Section 3 above, can be solved at this point; this property does not hold if a is LC or LP.
6. See also the account of endpoint classification in the text [69, Chapter 7], where the additional classification LPO is introduced, but not used in this present account.

Remark 5.2. The LC classification at any endpoint is further divided into two sub-cases as follows:

(i) The limit-circle non-oscillatory case (notation LCNO)
(ii) The limit-circle oscillatory case (notation LCO).

This additional classification is connected with the oscillation properties of the solutions of the Sturm-Liouville differential equation $M[y] = \lambda wy$ on (a, b); for details see [10, Section 4]. As with the initial endpoint classification this additional sub-classification depends only on the coefficients $\{p, q, w\}$ and not on the spectral parameter λ.

6. Endpoint boundary condition functions

We suppose given the interval (a, b) of $\mathbb{R}$ and a set of coefficients $\{p, q, w\}$ to create a Sturm-Liouville differential equation, with classified endpoints.

There is a very complete account of separated and coupled boundary conditions for the associated Sturm-Liouville boundary value problems, in the paper [10, Section 5].

Here, for use in cataloguing the Sturm-Liouville examples, we give information concerning the use of boundary condition functions at any endpoint in the LC classification. The use of these boundary condition functions takes the same form in both LCNO and LCO cases.

Let a be R; then a separated boundary condition at this endpoint, for a solution y of the Sturm-Liouville differential equation $M[y] = \lambda wy$ on (a, b), takes the form, where $A_1, A_2 \in \mathbb{R}$ with $A_1^2 + A_2^2 > 0$,

$$A_1 y(a) + A_2 (py')(a) = 0.$$

If b is R the there is a similar form for a separated boundary condition

$$B_1 y(b) + B_2 (py')(b) = 0.$$

Let a be LC; then a separated boundary condition at this endpoint, for a solution $y \in D(T_1)$ of the Sturm-Liouville differential equation $M[y] = \lambda wy$ on (a, b), takes the form,

$$A_1 [y, u](a) + A_2 [y, v](a) = 0$$

where

$$\begin{array}{rl} (i) & A_1, A_2 \in \mathbb{R} \text{ with } A_1^2 + A_2^2 > 0 \\ (ii) & u, v : (a, b) \to \mathbb{R} \\ (iii) & u, v \in D(T_1) \\ (iv) & [u, v](a) \neq 0. \end{array}$$

Such pairs $\{u, v\}$ of elements from the maximal domain $D(T_1)$ always exist under the LC classification on the endpoint a, see [10, Section 5].

If b is LC then there is a similar form for a separated boundary condition involving a pair $\{u, v\}$ of boundary condition functions, in general a different pair from the pair required for the endpoint a, to give

$$B_1[y, u](b) + B_2[y, v](b) = 0.$$

For any given particular Sturm-Liouville differential equation the search for pairs of such boundary condition functions may start with a study of the solutions of the differential equation $M[y] = \lambda wy$ on (a, b), and also with a direct search within the elements of the maximal domain $D(T_1)$.

For the examples given in the catalogue a suitable choice of these boundary condition functions is given, for endpoints in the LC case.

Remark 6.1. In practice it is sufficient to determine the pair $\{u, v\}$ in a neighbourhood $(a, c]$ of a, or $[c, b)$ of b, so that they are locally in the maximal domain $D(T_1)$; this practice is adopted in many of the examples given in this catalogue.

7. The Liouville transformation

The named Liouville transformation, see [30, Section 4.3] and [17, Chapter 10, Section 10] for details, of the general Sturm-Liouville differential equation

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \text{ for all } x \in (a, b)$$

provides a means, under additional conditions on the coefficients $\{p, q, w\}$, to yield a simpler Sturm-Liouville form of the differential equation

$$-Y''(X) + Q(X)Y(X) = \lambda Y(X) \text{ for all } X \in (A, B).$$

The minimal additional conditions required, see [30, Section 4.3], are

$$\begin{array}{rl} (i) & p \text{ and } p' \in AC_{\mathrm{loc}}(a, b), \text{ and } p(x) > 0 \text{ for all } x \in (a, b) \\ (ii) & w \text{ and } w' \in AC_{\mathrm{loc}}(a, b), \text{ and } w(x) > 0 \text{ for all } x \in (a, b). \end{array}$$

The Liouville transformation changes the variables x and y to X and Y as follows, see [30, Section 4.3]:

(i) For $k \in (a,b)$ and $K \in \mathbb{R}$ the mapping $X(\cdot) : (a,b) \to (A,B)$ defines a new independent variable $X(\cdot)$ by

$$X(x) = l(x) := K + \int_k^x \{w(t)/p(t)\}^{1/2}\, dt \text{ for all } x \in (a,b)$$

$$A := K - \int_a^k \{w(t)/p(t)\}^{1/2}\, dt \text{ and } B := K + \int_k^b \{w(t)/p(t)\}^{1/2}\, dt$$

where $-\infty \le A < B \le +\infty$; there is then an inverse mapping $L(\cdot) : (A,B) \to (a,b)$.

(ii) Define the new dependent variable $Y(\cdot)$ by

$$\begin{aligned} Y(X) &:= \{p(x)w(x)\}^{1/4} y(x) \text{ for all } x \in (a,b) \\ &:= \{p(L(X))w(L(X))\}^{1/4} y(L(X)) \text{ for all } X \in (A,B). \end{aligned}$$

The new coefficient Q is given by

$$Q(X) = w(x)^{-1}q(x) - \{w(x)^{-3}p(x)\}^{1/4}(p(x)(\{p(x)w(x)\}^{-1/4})')' \text{ for all } x \in (a,b).$$

An example of this Liouville transformation is worked in Section 11 for one form of the Bessel equation.

8. Fourier equation

This is the classical Sturm-Liouville differential equation, see [78, Chapter I, and Chapter IV, Section 4.1],

$$-y''(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty)$$

with solutions

$$\cos(x\sqrt{\lambda}) \text{ and } \sin(x\sqrt{\lambda}).$$

Endpoint classification in $L^2(-\infty,+\infty)$:

Endpoint	Classification
$-\infty$	LP
0	R
$+\infty$	LP

This is a simple constant coefficient equation; for any self-adjoint boundary value problem on a compact interval the eigenvalues can be characterized in terms of the solutions of a transcendental equation involving only trigonometric functions.

For a study of boundary value problems on the half-line $[0,\infty)$ or the whole line $(-\infty,\infty)$ see [78, Chapter IV, Section 4.1] and [2, Volume **II**, Appendix 2, Section 132, Part 2].

9. Hypergeometric equation

The standard form for this differential equation is, see [46, Chapter 4, Section 3], [80, Chapter XIV, Section 14.2], [16, Chapter 9, Section 9.2], [27, Chapter II, Section 2.1.1], [1, Chapter 15, Section 15.5] and [78, Chapter IV, Sections 4.18 to 4.20],

$$z(1-z)y''(z) + [c-(a+b+1)z]y'(z) - aby(z) = 0 \text{ for all } z \in \mathbb{C}$$

where, in general $a, b, c \in \mathbb{C}$. In terms of the hypergeometric function ${}_2F_1$, solutions of this equation are, with certain restrictions on the parameters and the independent variable z,

$${}_2F_1(a, b; c; z) \text{ and } z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; z).$$

For consideration of this hypergeometric equation in Sturm-Liouville form we replace the variable z by the real variable $x \in (0,1)$. Thereafter, on multiplying by the factor $x^\alpha(1-x)^\beta$ and rearranging the terms gives the Sturm-Liouville equation, for all $\alpha, \beta \in \mathbb{R}$,

$$-\left(x^{\alpha+1}(1-x)^{\beta+1}y'(x)\right)' = \lambda x^\alpha(1-x)^\beta y(x) \text{ for all } x \in (0,1).$$

In this form the relationship between the parameters $\{a, b, c\}$ and $\{\alpha, \beta, \lambda\}$ is

$$c = \alpha + 1 \qquad a + b = \alpha + \beta + 1 \qquad ab = -\lambda;$$

these equations can be solved for $\{a, b, c\}$ in terms of $\{\alpha, \beta, \lambda\}$ as in [78, Chapter IV, Section 4.18].

Given $\alpha, \beta \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ the solutions of this Sturm-Liouville equation can then be represented in terms of the hypergeometric function ${}_2F_1$, as above.

For the case when $\lambda = 0$ the general solution of this differential equation takes the form, for $c \in (0,1)$,

$$y(x) = k\int_c^x \frac{1}{t^{\alpha+1}(1-t)^{\beta+1}}\, dt + l \text{ for all } x \in (0,1)$$

where the numbers $k, l \in \mathbb{C}$. From this representation it may be shown that the following classifications, in the space $L^2((0,1); x^\alpha(1-x)^\beta)$, of the endpoints 0 and 1 hold:

Endpoint	Parameters α, β	Classification
0	For $\alpha \in (-1,0)$ and all $\beta \in \mathbb{R}$	R
0	For $\alpha \in [0,1)$ and all $\beta \in \mathbb{R}$	LCNO
0	For $\alpha \in (-\infty,-1] \cup [1,\infty)$ and all $\beta \in \mathbb{R}$	LP
1	For $\beta \in (-1,0)$ and all $\alpha \in \mathbb{R}$	R
1	For $\beta \in [0,1)$ and all $\alpha \in \mathbb{R}$	LCNO
1	For $\beta \in (-\infty,-1] \cup [1,\infty)$ and all $\alpha \in \mathbb{R}$	LP

For the endpoint 0, for $\alpha \in [0,1)$ and for all $\beta \in \mathbb{R}$ the LCNO boundary condition functions u, v take the form, for all $x \in (0,1)$,

Parameter	u	v
$\alpha = 0$	1	$\ln(x)$
$\alpha \in (0,1)$	1	$x^{-\alpha}$

For the endpoint 1, for $\beta \in [0,1)$ and for all $\alpha \in \mathbb{R}$ the LCNO boundary condition functions u, v take the form, for all $x \in (0,1)$,

Parameter	u	v
$\beta = 0$	1	$\ln(1-x)$
$\beta \in (0,1)$	1	$(1-x)^{-\beta}$

Another form of the hypergeometric differential equation is obtained if in the original equation above the independent variable z is replaced by $-z$ to give

$$z(1+z)y''(z) + [c + (a+b+1)z]y'(z) + aby(z) = 0 \text{ for all } z \in \mathbb{C}$$

with general solutions

$${}_2F_1(a,b;c;-z) \text{ and } z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; -z);$$

see the account in [78, Chapter IV, Section 4.18].

For consideration of this hypergeometric equation in Sturm-Liouville form we replace the variable z by the real variable $x \in (0,\infty)$. Thereafter, on multiplying by the factor $x^\alpha(1+x)^\beta$ and re-arranging the terms gives the Sturm-Liouville equation, for all $\alpha, \beta \in \mathbb{R}$,

$$-\left(x^{\alpha+1}(1+x)^{\beta+1}y'(x)\right)' = \lambda x^\alpha (1+x)^\beta y(x) \text{ for all } x \in (0,\infty).$$

In this form the relationship between the parameters $\{a,b,c\}$ and $\{\alpha,\beta,\lambda\}$ is

$$c = \alpha + 1 \qquad a + b = \alpha + \beta + 1 \qquad ab = \lambda;$$

these equations can be solved for $\{a,b,c\}$ in terms of $\{\alpha,\beta,\lambda\}$ as in [78, Chapter IV, Section 4.18].

Given $\alpha, \beta \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ the solutions of this Sturm-Liouville equation can then be represented in terms of the hypergeometric function ${}_2F_1$, as above.

For the case when $\lambda = 0$ the general solution of this differential equation takes the form, for $c \in (0,\infty)$,

$$y(x) = k\int_c^x \frac{1}{t^{\alpha+1}(1+t)^{\beta+1}}\,dt + l \text{ for all } x \in (0,\infty)$$

where the numbers $k, l \in \mathbb{C}$. From this representation it may be shown that the following classifications, in the space $L^2((0,\infty); x^\alpha(1+x)^\beta)$, of the endpoints 0 and $+\infty$ hold:

Endpoint	Parameters α, β	Classification
0	For $\alpha \in (-1, 0)$ and all $\beta \in \mathbb{R}$	R
0	For $\alpha \in [0, 1)$ and all $\beta \in \mathbb{R}$	LCNO
0	For $\alpha \in (-\infty, -1] \cup [1, \infty)$ and all $\beta \in \mathbb{R}$	LP
$+\infty$	For all $\alpha, \beta \in \mathbb{R}$	LP

For the endpoint 0, for $\alpha \in [0, 1)$ and for all $\beta \in \mathbb{R}$ the LCNO boundary condition functions u, v take the form, for all $x \in (0, 1)$,

Parameter	u	v
$\alpha = 0$	1	$\ln(x)$
$\alpha \in (0, 1)$	1	$x^{-\alpha}$

The spectral properties of these hypergeometric Sturm-Liouville differential equations seem not to have been studied in detail; however there are a number of very interesting special cases, together with their spectral properties, considered in [78, Chapter IV, Sections 4.18 to 4.20].

10. Kummer equation

The Kummer differential equation is a special case of the confluent hypergeometric differential equation

$$zw''(z) + (b - z)y'(z) - aw(z) = 0 \text{ for } z \in \mathbb{C}.$$

Taking the parameter $b \in \mathbb{R}$ to be real-valued, putting $\lambda = -a \in \mathbb{C}$, replacing the independent variable z by $x \in \mathbb{R}$, and then writing the resulting differential equation in Lagrange symmetric form, gives the Sturm-Liouville example

$$-(x^b \exp(-x)y'(x))' = \lambda x^{b-1} \exp(-x)y(x) \text{ for all } x \in (0, \infty).$$

Solutions of this Sturm-Liouville differential equation are given in the form, using the Kummer functions M and U,

$$M(-\lambda, b, x) \text{ and } U(-\lambda, b, x) \text{ for all } x \in (0, \infty), b \in \mathbb{R} \text{ and } \lambda \in \mathbb{C};$$

see[1, Chapter 13, Section 13.1].

Endpoint classification in $L^2((0, +\infty); x^{b-1} \exp(-x))$:

Endpoint	Parameter b	Classification
0	For $b \leq 0$	LP
0	For $0 < b < 1$	R
0	For $1 \leq b < 2$	LCNO
0	For $b \geq 2$	LP
∞	For all $b \in \mathbb{R}$	LP

For the endpoint 0 and then for $b \in (0,2)$, the LCNO boundary condition functions u, v take the form, for all $x \in (0,\infty)$,

Parameter	u	v
$b = 1$	1	$\ln(x)$
$1 < b < 2$	1	x^{1-b}

See also the Laguerre differential equation given in Section 27 below.

11. Bessel equation

The Bessel differential equation has many different forms, see [79, Chapter IV], [1, Chapters 9 and 10], [27, Volume **II**, Chapter VII], [46, Chapter 8], [16, Chapter 4]; see in particular [52, Part C, Section 2.162].

One elegant Sturm-Liouville form, see [33, Section 1], of this differential equation is, where the parameter $\alpha \in \mathbb{R}$,

$$-(x^{2\alpha+1}y'(x))' = \lambda x^{2\alpha+1}y(x) \text{ for all } x \in (0,\infty).$$

Solutions of this differential equation are, for all $\alpha \in \mathbb{R}$,

$$x^{-\alpha}J_\alpha(x\sqrt{\lambda}) \text{ and } x^{-\alpha}Y_\alpha(x\sqrt{\lambda}) \text{ for all } x \in (0,\infty)$$

where J_α and Y_α are the classical Bessel functions, and the power $x^{-\alpha}$ is defined by $x^{-\alpha} := \exp(-\alpha\ln(x))$ for all $x \in (0,\infty)$.

For the case when $\lambda = 0$ the general solution of this differential equation takes the form, for $c \in (0,\infty)$,

$$y(x) = k\int_c^x \frac{1}{t^{2\alpha+1}}\,dt + l \text{ for all } x \in (0,\infty)$$

where the numbers $k, l \in \mathbb{C}$. From this representation it may be shown that the following classifications, in the space $L^2((0,\infty);x^{2\alpha+1})$, of the endpoints 0 and $+\infty$ hold:

Endpoint	Parameter α	Classification
0	For $\alpha \in (-1,1)$	LCNO
0	For $\alpha \in (-\infty,-1] \cup [1,\infty)$	LP
∞	For all $\alpha \in \mathbb{R}$	LP

For the endpoint 0 and then for $\alpha \in (-1,1)$, the LCNO boundary condition functions u, v take the form, for all $x \in (0,\infty)$,

Parameter	u	v
$\alpha = 0$	1	$\ln(x)$
$\alpha \in (-1,0) \cup (0,1)$	1	$x^{-2\alpha}$

As an example of the Liouville transformation, see Section 7 above, let $k = 1$ and $K = 1$ to give, for the form of the Bessel differential equation above,

$$X(x) = 1 + \int_1^x dt = x \text{ for all } x \in (0,\infty);$$

a computation then shows that

$$Q(X) = (\alpha^2 - 1/4)x^{-2} = (\alpha^2 - 1/4)X^{-2} \text{ for all } X \in (0, \infty).$$

Thus the Liouville form of this Bessel differential equation is

$$-Y''(X) + (\alpha^2 - 1/4)X^{-2}Y(X) = \lambda Y(X) \text{ for all } X \in (0, \infty),$$

where we can now take the parameter $\alpha \in [0, \infty)$.

12. Bessel equation: Liouville form

In the Liouville normal form, see Sections 7 and 11 above, the Bessel differential equation appears as

$$-y''(x) + \left(\nu^2 - 1/4\right) x^{-2} y(x) = \lambda y(x) \text{ for all } x \in (0, +\infty),$$

with the parameter $\nu \in [0, +\infty)$; this differential equation is extensively studied in [78, Chapter IV, Sections 4.8 to 4.15]; see also [2, Volume **II**, Appendix 2, Section 132, Part 5]. In this form the equation has solutions

$$x^{1/2} J_\nu(x\sqrt{\lambda}) \text{ and } x^{1/2} Y_\nu(x\sqrt{\lambda}).$$

Endpoint classification in $L^2(0, +\infty)$:

Endpoint	Parameter ν	Classification
0	For $\nu = 1/2$	R
0	For all $\nu \in [0, 1)$ but $\nu \neq 1/2$	LCNO
0	For all $\nu \in [1, \infty)$	LP
$+\infty$	For all $\nu \in [0, \infty)$	LP

For endpoint 0 and $\nu \in (0, 1)$ but $\nu \neq 1/2$, the LCNO boundary condition functions u, v are determined by, for all $x \in (0, +\infty)$,

Parameter	u	v
$\nu \in (0, 1)$ but $\nu \neq 1/2$	$x^{\nu+1/2}$	$x^{-\nu+1/2}$
$\nu = 0$	$x^{1/2}$	$x^{1/2} \ln(x)$

(a) Problems on $(0, 1]$ with $y(1) = 0$:

For $0 \leq \nu < 1, \nu \neq \frac{1}{2}$: the Friedrichs case: $A1 = 1, A2 = 0$ yields the classical Fourier-Bessel series; here $\lambda_n = j_{\nu,n}^2$ where $\{j_{\nu,n} : n = 0, 1, 2, \ldots\}$ are the zeros (positive) of the Bessel function $J_\nu(\cdot)$.

For $\nu \geq 1$; LP at 0 so that there is a unique boundary value problem with $\lambda_n = j_{\nu,n}^2$ as before.

(b) Problems on $[1, \infty)$ all have continuous spectrum on $[0, \infty)$:

For Dirichlet and Neumann boundary conditions there are no eigenvalues.

For $A1 = A2 = 1$ at 1 there is one isolated negative eigenvalue.

(c) Problems on $(0, \infty)$ all have continuous spectrum on $[0, \infty)$:

For $\nu \geq 1$ there are no eigenvalues.

For $0 \leq \nu < 1$ the Friedrichs case is given by $A1 = 1, A2 = 0$; there are no eigenvalues.

For $\nu = 0.45$ and $A1 = 10, A2 = -1$ there is one isolated eigenvalue near to the value -175.57.

One of the interesting features of this Liouville form of the Bessel equation is that it is possible to choose purely imaginary values of the order ν of the Bessel function solutions. If $\nu = ik$, with $k \in \mathbb{R}$, then the Liouville form of the equation becomes

$$-y''(x) - \left(k^2 + 1/4\right) x^{-2} y(x) = \lambda y(x) \text{ for all } x \in (0, +\infty)$$

with solutions

$$x^{1/2} J_{ik}(x\sqrt{\lambda}) \text{ and } x^{1/2} Y_{ik}(x\sqrt{\lambda}).$$

This differential equation is considered below in Section 44 under the name the Krall equation

13. Bessel equation: form 1

This special case of the Bessel equation is

$$-y''(x) - xy(x) = \lambda y(x) \text{ for all } x \in [0, \infty).$$

This differential equation has explicit solutions in terms of Bessel functions of order 1/3; see [1, Chapter 10, Section 10.4], [28, Section 3], [29, Section 4] and [78, Chapter IV, Section 4.13].

Endpoint classification in $L^2(0, +\infty)$:

Endpoint	Classification
0	R
$+\infty$	LP

14. Bessel equation: form 2

This special case of the Bessel equation is

$$-(x^\beta y'(x))' = \lambda x^\alpha y(x) \text{ for all } x \in (0, \infty)$$

with the parameters $\alpha > -1$ and $\beta < 1$. This differential equation has solutions of the form, see [52, Section C, Equation 2.162 (1a)] and [35, Section 2.3],

$$y(x, \lambda) = x^{\frac{1}{2}(1-\beta)} Z_\nu \left(k^{-1} x^k \sqrt{\lambda}\right) \text{ for all } x \in (0, \infty) \text{ and all } \lambda \in \mathbb{C}$$

where the real parameters ν and k are defined by

$$\nu := (1 - \beta)/(\alpha - \beta + 2) \text{ and } k := \tfrac{1}{2}(\alpha - \beta + 2),$$

and Z_ν is any Bessel function, $J_\nu, Y_\nu, H_\nu^{(1)}, H_\nu^{(2)}$, of order ν. A calculation shows that with the given restrictions on α and β we have

$$0 < \nu < 1 \text{ and } k > 0.$$

Endpoint classification in $L^2(0,+\infty;x^\alpha)$, for all α and β as above,

Endpoint	Classification
0	R
$+\infty$	LP

15. Bessel equation: form 3

This special case of the Bessel equation is

$$-(x^\tau y'(x))' = \lambda y(x) \text{ for all } x \in [1,\infty).$$

where the real parameter $\tau \in (-\infty,\infty)$. This differential equation has solutions of the form, see [52, Section C, Equation 2.162] and [32, Section 5],

$$y(x,\lambda) = x^{\frac{1}{2}(1-\tau)} Z_v\left(2(2-\tau)^{-1} x^{\frac{1}{2}(2-\tau)}\sqrt{\lambda}\right) \text{ for all } x \in [1,\infty) \text{ and all } \lambda \in \mathbb{C}$$

where the real parameter ν is defined by

$$\nu := (1-\tau)/(2-\tau)\ ,$$

and where Z_ν is any Bessel function, $J_\nu, Y_\nu, H_\nu^{(1)}, H_\nu^{(2)}$, of order ν.

Endpoint classification in $L^2(1,\infty)$, for all $\tau \in (-\infty,\infty)$,

Endpoint	Classification
1	R
$+\infty$	LP

16. Bessel equation: form 4

This special case of the Bessel equation is, with $a > 0$,

$$-y''(x) + (\nu^2 - 1/4)x^{-2}y(x) = \lambda y(x) \text{ for all } x \in [a,\infty).$$

This equation is a special case of the Liouville form of the Bessel differential equation, see Section 12 above, with the parameter $\nu \geq 0$ and considered on the interval $[a,\infty)$ to avoid the singularity at the endpoint 0. The reason for this choice of endpoint is to relate to the Weber integral transform as considered in [78, Chapter IV, Section 4.10]. As in Section 12 above this equation has solutions

$$x^{1/2}J_\nu(x\sqrt{\lambda}) \text{ and } x^{1/2}Y_\nu(x\sqrt{\lambda}) \text{ for all } x \in [a,\infty)$$

but now the endpoint classification in $L^2[a,\infty)$ is

Endpoint	Classification
a	R
$+\infty$	LP

17. Bessel equation: modified form

The modified Bessel functions, notation I_ν and K_ν, are best defined, on the real line $\mathbb{R}$, in terms of the classical Bessel functions J_ν and Y_ν by, see [16, Chapter 4, Section 4.7],

$$I_\nu(x) := i^{-\nu}J_\nu(ix) \text{ and } K_\nu(x) := \frac{\pi}{2}i^{\nu+1}\{J_\nu(ix) + iY_\nu(ix)\} \text{ for all } x \in \mathbb{R}.$$

The properties of these special functions are considered in [16, Chapter 4, Section 4.7 to 4.9].

With careful attention to the branch definition of the powers of the factors i^ν it may be shown that

$$I_\nu(\cdot) : \mathbb{R} \to \mathbb{R} \text{ and } K_\nu(\cdot) : \mathbb{R} \to \mathbb{R}.$$

The functions $I_\nu(x\sqrt{\lambda})$ and $K_\nu(x\sqrt{\lambda})$ form an independent basis of solutions for the differential equation

$$(xy'(x))' - \nu^2x^{-1}y(x) = \lambda xy(x) \text{ for all } x \in (0,\infty)$$

and have properties similar to the classical Bessel functions $J_\nu(x\sqrt{\lambda})$ and $Y_\nu(x\sqrt{\lambda})$, respectively, when $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$.

If the Liouville transformation is applied to this last equation, or in the Bessel Liouville differential equation, see Section 12 above, the formal transformation $x \longmapsto ix$ is applied, then the resulting differential equation has the form

$$y''(x) - \left(\nu^2 - 1/4\right)x^{-2}y(x) = \lambda y(x) \text{ for all } x \in (0,+\infty).$$

This gives one interesting property of the Liouville form of the differential equation for the modified Bessel functions; in the standard Sturm-Liouville form given in Section 2 above the leading coefficient p has to be taken as negative valued on the interval $(0,\infty)$, *i.e.*

$$p(x) = -1 \qquad q(x) = -\left(\nu^2 - 1/4\right)x^{-2} \qquad w(x) = 1 \text{ for all } x \in (0,\infty).$$

The independent solutions of this Liouville form are

$$x^{1/2}I_\nu(x\sqrt{\lambda}) \text{ and } x^{1/2}K_\nu(x\sqrt{\lambda}) \text{ for all } x \in (0,\infty).$$

Endpoint classification in $L^2(0,+\infty)$:

Endpoint	Parameter ν	Classification
0	For $\nu = 1/2$	R
0	For all $\nu \in [0,1)$ but $\nu \neq 1/2$	LCNO
0	For all $\nu \in [1,\infty)$	LP
$+\infty$	For all $\nu \in [0,\infty)$	LP

For endpoint 0 and $\nu \in (0,1)$ but $\nu \neq 1/2$, the LCNO boundary condition functions u, v are determined by, for all $x \in (0,+\infty)$,

Parameter	u	v
$\nu \in (0,1)$ but $\nu \neq 1/2$	$x^{\nu+1/2}$	$x^{-\nu+1/2}$
$\nu = 0$	$x^{1/2}$	$x^{1/2}\ln(x)$

18. Airy equation

The Airy differential equation, in Sturm-Liouville form, is

$$-y''(x) + xy(x) = \lambda y(x) \text{ for all } x \in \mathbb{R}.$$

The solutions of this equation can be expressed in terms of the Bessel functions $J_{1/3}$ and $J_{-1/3}$, or in terms of the Airy functions $\mathrm{Ai}(\cdot)$ and $\mathrm{Bi}(\cdot)$. For a detailed study of the properties of these functions see [1, Chapter 10, Section 10.4]; see also the results in [29, Section 5].

Endpoint classification in $L^2(-\infty, \infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

The spectrum of the boundary value problem on the interval $(-\infty, \infty)$ has no eigenvalues and is continuous on the real line in $\mathbb{C}$; the spectrum for any problem on the interval $[0, \infty)$ is discrete.

19. Legendre equation

The standard form for this differential equation is, see [80, Chapter XV],

$$-\left(\left(1 - x^2\right) y'(x)\right)' + \tfrac{1}{4} y(x) = \lambda y(x) \text{ for all } x \in (-1, +1);$$

see also [1, Chapter 8], [27, Volume **I**, Chapter III], [16, Chapter 3] and [2, Volume **II**, Appendix 2, Section 132, Part 3].

Endpoint classification in $L^2(-1, +1)$:

Endpoint	Classification
-1	LCNO
$+1$	LCNO

For both endpoints the boundary condition functions u, v are given by (note that u and v are solutions of the Legendre equation for $\lambda = 1/4$)

$$u(x) = 1 \qquad v(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \text{ for all } x \in (-1, +1).$$

(i) The Legendre polynomials are obtained by taking the principal (Friedrichs) boundary condition at both endpoints ± 1 : enter $A1 = 1, A2 = 0,\ B1 = 1, B2 = 0$; *i.e.* take the boundary condition function u at ± 1; eigenvalues: $\lambda_n = (n + 1/2)^2$; $n = 0, 1, 2, \cdots$; eigenfunctions: Legendre polynomials $P_n(x)$.

(ii) Enter $A1 = 0,\ A2 = 1,\ B1 = 0,\ B2 = 1$, *i.e.* use the boundary condition function v at ± 1; eigenvalues: μ_n; $n = 0, 1, 2, \cdots$ but no explicit formula is available; eigenfunctions are logarithmically unbounded at ± 1.

(iii) Observe that $\mu_n < \lambda_n < \mu_{n+1}$; $n = 0, 1, 2 \cdots$.

The Liouville normal form of the Legendre differential equation is

$$-y''(x)+\tfrac{1}{4}\sec^2(x)y(x)=\lambda y(x) \text{ for all } x\in\left(-\tfrac{1}{2}\pi,\tfrac{1}{2}\pi\right);$$

this form of the equation is studied in detail in [78, Chapter IV, Sections 4.5 to 4.7].

20. Legendre equation: associated form

This Sturm-Liouville differential equation is an extension of the classical Legendre equation of Section 19:

$$-\left(\left(1-x^2\right)y'(x)\right)'+\frac{\mu^2}{1-x^2}y(x)=\lambda y(x) \text{ for all } x\in(-1,+1)$$

where the parameter $\mu\in[0,\infty)$;see [1, Chapter 8], [27, Volume **I**, Chapter III], [16, Chapter 3, Section 3.9] and [78, Chapter IV, Section 4.3].

Endpoint classification in $L^2(-1,+1)$:

Endpoint	Parameter	Classification
-1	$0\le\mu<1$	LCNO
-1	$1\le\mu$	LP

Endpoint	Parameter	Classification
$+1$	$0\le\mu<1$	LCNO
$+1$	$1\le\mu$	LP

For the endpoint -1 and for the LCNO cases the boundary condition functions u,v are determined by

Parameter	u	v
$\mu=0$	1	$\ln\left(\dfrac{1+x}{1-x}\right)$
$0<\mu<1$	$(1-x^2)^{\mu/2}$	$(1-x^2)^{-\mu/2}$

For the endpoint $+1$ and for the LCNO cases the boundary condition functions u,v are determined by

Parameter	u	v
$\mu=0$	1	$\ln\left(\dfrac{1+x}{1-x}\right)$
$0<\mu<1$	$(1-x^2)^{\mu/2}$	$(1-x^2)^{-\mu/2}$

If the spectral parameter λ is written as $\lambda=\nu(\nu+1)$ then the solutions of this modified Legendre equation are the associated Legendre functions $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ for $x\in(-1,+1)$; see [1, Chapter 8] and [16, Chapter 3, Section 3.9].

21. Hermite equation

The most elegant Sturm-Liouville form for this differential equation is

$$-(\exp(-x^2)y'(x))' = \lambda\exp(-x^2)y(x) \text{ for all } x \in (-\infty,\infty).$$

For all $n \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ and for $\lambda = 2n+1$ this equation has the Hermite polynomials H_n for solutions. These polynomials are orthogonal and complete in the Hilbert function space $L^2((-\infty,\infty);\exp(-x^2))$.

Endpoint classification in $L^2((-\infty,\infty);\exp(-x^2))$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

22. Hermite equation: Liouville form

The Liouville transformation applied to the Hermite differential equation gives

$$-y''(x) + x^2y(x) = \lambda y(x) \text{ for all } x \in (-\infty,+\infty).$$

For all $n \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ and for $\lambda = 2n+1$ this equation has the Hermite functions $\exp(-\frac{1}{2}x^2)H_n$ for solutions. These functions are orthogonal and complete in the Hilbert function space $L^2(-\infty,\infty)$.

Endpoint classification in $L^2(-\infty,+\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

For a classical treatment see [78, Chapter IV, Section 2].

This differential equation is also called the harmonic oscillator equation; see example 15 in the list to be found within the LaTeX file xamples.tex contained in the package associated with the publication [11, Data base file xamples.tex; example 15].

This differential equation is also considered under the name of the parabolic cylinder function; see [1, Chapter 19].

23. Jacobi equation

The general form of the Jacobi differential equation is

$$-\left((1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x)\right)' = \lambda(1-x)^{\alpha}(1+x)^{\beta}y(x) \text{ for all } x \in (-1,+1),$$

where the parameters $\alpha,\beta \in (-\infty,+\infty)$. Apart from an isomorphic transformation of the independent variable this differential equation coincides with the Sturm-Liouville form of the hypergeometric equation considered in Section 9 above.

Endpoint classification in the weighted space $L^2((-1,+1);(1-x)^\alpha(1+x)^\beta))$:

Endpoint	Parameter	Classification
-1	$\beta \leq -1$	LP
-1	$-1 < \beta < 0$	R
-1	$0 \leq \beta < 1$	LCNO
-1	$1 \leq \beta$	LP

Endpoint	Parameter	Classification
$+1$	$\alpha \leq -1$	LP
$+1$	$-1 < \alpha < 0$	R
$+1$	$0 \leq \alpha < 1$	LCNO
$+1$	$1 \leq \alpha$	LP

For the endpoint -1 and for the LCNO cases the boundary condition functions u, v are determined by

Parameter	u	v
$\beta = 0$	1	$\ln\left(\dfrac{1+x}{1-x}\right)$
$0 < \beta < 1$	1	$(1+x)^{-\beta}$

For the endpoint $+1$ and for the LCNO cases the boundary condition functions u, v are determined by

Parameter	u	v
$\alpha = 0$	1	$\ln\left(\dfrac{1+x}{1-x}\right)$
$0 < \alpha < 1$	1	$(1-x)^{-\alpha}$

To obtain the classical Jacobi orthogonal polynomials it is necessary to take $-1 < \alpha,\ \beta$; then note the required boundary conditions:

Endpoint -1:

Parameter	Boundary condition
$-1 < \beta < 0$	$(py')(-1) = 0$ or $[y,v](-1) = 0$
$0 \leq \beta < 1$	$[y,u](-1) = 0$

Endpoint $+1$:

Parameter	Boundary condition
$-1 < \alpha < 0$	$(py')(+1) = 0$ or $[y,v](+1) = 0$
$0 \leq \alpha < 1$	$[y,u](+1) = 0$

For the classical Jacobi orthogonal polynomials the eigenvalues are given by:

$$\lambda_n = n(n+\alpha+\beta+1) \text{ for } n = 0,1,2,\ldots$$

and this explicit formula can be used to give an independent check on the accuracy of the results from the SLEIGN2 code.

It is interesting to note that the required boundary condition for these Jacobi polynomials is the Friedrichs condition in the LCNO cases.

In addition to the cases of the Jacobi equation mentioned in this section, there are other values of the parameters α and β which lead to important Sturm-Liouville differential equations; see the paper [53] and the book [3].

24. Jacobi equation: Liouville form

The Liouville transformation applied to the Jacobi differential equation gives

$$-y''(x) + q(x)y(x) = \lambda y(x) \text{ for all } x \in (-\pi/2, +\pi/2)$$

where the coefficient q is given by, for all $x \in (-\pi/2, +\pi/2)$,

$$\begin{aligned} q(x) &= \frac{\beta^2 - 1/4}{4\tan^2((x+\pi)/2)} + \frac{\alpha^2 - 1/4}{4\tan^2((x-\pi)/2)} - \frac{4\alpha\beta + 4\beta + 4\alpha + 3}{8} \\ &= \frac{\beta^2 - 1/4}{4\sin^2((x+\pi)/2)} + \frac{\alpha^2 - 1/4}{4\sin^2((x-\pi)/2)} - \frac{(\alpha+\beta+1)^2}{4}. \end{aligned}$$

Here the parameters $\alpha, \beta \in (-\infty+, \infty)$.

Endpoint classification in the space $L^2(-\pi/2, +\pi/2)$:

Endpoint	Parameter	Classification
$-\pi/2$	$\beta \le -1$	LP
$-\pi/2$	$-1 < \beta < 1$ but $\beta^2 \ne 1/4$	LCNO
$-\pi/2$	$\beta^2 = 1/4$	R
$-\pi/2$	$1 \le \beta$	LP

Endpoint	Parameter	Classification
$+\pi/2$	$\alpha \le -1$	LP
$+\pi/2$	$-1 < \alpha < 1$ but $\alpha^2 \ne 1/4$	LCNO
$+\pi/2$	$\alpha^2 = 1/4$	R
$+\pi/2$	$1 \le \alpha$	LP

For the endpoint $-\pi/2$ and for LCNO cases the boundary condition functions u, v are determined by, here $b(x) = 2\tan^{-1}(1) + x$ for all $x \in (-\pi/2, +\pi/2)$,

Parameter	u	v
$-1 < \beta < 0$	$b(x)^{\frac{1}{2}-\beta}$	$b(x)^{\frac{1}{2}+\beta}$
$\beta = 0$	$\sqrt{b(x)}$	$\sqrt{b(x)}\ln(b(x))$
$0 < \beta < 1$	$b(x)^{\frac{1}{2}+\beta}$	$b(x)^{\frac{1}{2}-\beta}$

For the endpoint $+\pi/2$ and for LCNO cases the boundary condition functions u, v are determined by, here $a(x) = 2\tan^{-1}(1) - x$ for all $x \in (-\pi/2, +\pi/2)$,

Parameter	u	v
$-1 < \alpha < 0$	$a(x)^{\frac{1}{2}-\alpha}$	$a(x)^{\frac{1}{2}+\alpha}$
$\alpha = 0$	$\sqrt{a(x)}$	$\sqrt{a(x)}\ln(a(x))$
$0 < \alpha < 1$	$a(x)^{\frac{1}{2}+\alpha}$	$a(x)^{\frac{1}{2}-\alpha}$

The classical Jacobi orthogonal polynomials are produced only when both $\alpha, \beta > -1$. For $\alpha, \beta > +1$ the LP condition holds and no boundary condition is required to give the polynomials. If $-1 < \alpha, \beta < 1$ then the LCNO condition holds and boundary conditions are required to produce the Jacobi polynomials; these conditions are as follows:

Endpoint $-\pi/2$

Parameter	Boundary condition
$-1 < \beta < 0$	$[y, v](-\pi/2) = 0$
$0 \leq \beta < 1$	$[y, u](-\pi/2) = 0$

Endpoint $+\pi/2$

Parameter	Boundary condition
$-1 < \alpha < 0$	$[y, v](+\pi/2) = 0$
$0 \leq \alpha < 1$	$[y, u](+\pi/2) = 0$

Recall from Section 23 for the classical orthogonal Jacobi polynomials the eigenvalues are given explicitly by:

$$\lambda_n = n(n + \alpha + \beta + 1) \text{ for } n = 0, 1, 2, \ldots$$

25. Jacobi function equation

This is another Jacobi differential equation which corresponds to the hypergeometric differential equation considered over the half-line $[0, \infty)$, see the second equation in Section 9 above, and the paper [37].

This equation is written in the form

$$-(\omega(x)y'(x))' - \rho^2\omega(x)y(x) = \lambda\omega(x)y(x) \text{ for all } x \in (0, \infty)$$

where

(i) $\alpha \geq \beta \geq -1/2$
(ii) $\rho = \alpha + \beta + 1$
(iii) $\omega(x) \equiv \omega(x)_{\alpha,\beta} = 2^{2\rho}(\sinh(x))^{2\alpha+1}(\cosh(x))^{2\beta+1}$ for all $x \in (0, \infty)$.

Endpoint classification, for all $\beta \in [-1/2, \infty)$, in $L^2((0,\infty);\omega)$:

Endpoint	Parameter α	Classification
0	For $\alpha \in [-1/2, 0)$	R
0	For $\alpha \in [0, 1)$	LCNO
0	For $\alpha \in [1, \infty)$	LP
$+\infty$	For all $\alpha \in [-1/2, \infty)$	LP

For the endpoint 0, for $\alpha \in [0,1)$ and for all $\beta \in [1/2, \infty)$ the LCNO boundary condition functions u, v take the form, for all $x \in (0,1)$,

Parameter	u	v
$\alpha = 0$	1	$\ln(x)$
$\alpha \in (0,1)$	1	$x^{-2\alpha}$

26. Jacobi function equation: Liouville form

In the Liouville normal form, see Sections 7 and 11 above, the Jacobi function differential equation of Section 25 above appears as

$$-y''(x) + q(x)y(x) = \lambda y(x) \text{ for all } x \in (0,\infty),$$

where the coefficient q is determined by, again with $\alpha \geq \beta \geq -1/2$,

$$q(x) = \frac{\alpha^2 - 1/4}{(\sinh(x))^2} - \frac{\beta^2 - 1/4}{(\cosh(x))^2} \text{ for all } x \in (0,\infty).$$

Endpoint classification, for all $\beta \in [-1/2, \infty)$, in $L^2(0,\infty)$:

Endpoint	Parameter α	Classification
0	For $\alpha = -1/2$	R
0	For $\alpha \in (-1/2, 1/2)$	LCNO
0	For $\alpha = 1/2$	R
0	For $\alpha \in (1/2, 1)$	LCNO
0	For $\alpha \in [1, \infty)$	LP
$+\infty$	For all $\alpha \in [-1/2, \infty)$	LP

For the endpoint 0, for $\alpha \in [-1/2, 1)$ but $|\alpha| \neq 1/2$ and for all $\beta \in [-1/2, \infty)$ the LCNO boundary condition functions u, v take the form, for all $x \in (0,1)$,

Parameter	u	v
$\alpha = 0$	$x^{1/2}$	$x^{1/2}\ln(x)$
$\alpha \in [-1/2, 1)$ but $\lvert\alpha\rvert \neq 1/2$	$x^{\lvert\alpha\rvert+1/2}$	$x^{-\lvert\alpha\rvert+1/2}$

27. Laguerre equation

The general form of the Laguerre differential equation is

$$-(x^{\alpha+1}\exp(-x)y'(x))' = \lambda x^{\alpha}\exp(-x)y(x) \text{ for all } x \in (0,+\infty)$$

where the parameter $\alpha \in (-\infty,+\infty)$.

Endpoint classification in the weighted space $L^2((0,+\infty); x^{\alpha}\exp(-x))$:

Endpoint	Parameter	Classification
0	$\alpha \le -1$	LP
0	$-1 < \alpha < 0$	R
0	$0 \le \alpha < 1$	LCNO
0	$1 \le \alpha$	LP
$+\infty$	$\alpha \in (-\infty,+\infty)$	LP

For these LCNO cases the boundary condition functions u, v are given by:

Endpoint	Parameter	u	v
0	$\alpha = 0$	1	$\ln(x)$
0	$0 < \alpha < 1$	1	$x^{-\alpha}$

This is the classical form of the differential equation which for parameter $\alpha > -1$ produces the classical Laguerre polynomials as eigenfunctions; for the boundary condition $[y,1](0) = 0$ at 0, when required, the eigenvalues are then (remarkably!) independent of α and given by $\lambda_n = n$ $(n = 0,1,2,...)$; see [1, Chapter 22, Section 22.6].

See also the Kummer differential equation given in Section 10 above.

28. Laguerre equation: Liouville form

The Liouville transformation applied to the Laguerre differential equation gives

$$-y''(x) + \left(\frac{\alpha^2 - 1/4}{x^2} - \frac{\alpha+1}{2} + \frac{x^2}{16}\right) y(x) = \lambda y(x) \text{ for all } x \in (0,+\infty)$$

where the parameter $\alpha \in (-\infty,+\infty)$.

Endpoint classification in the space $L^2(0,+\infty)$:

Endpoint	Parameter	Classification
0	$\alpha \le -1$	LP
0	$-1 < \alpha < 1$, but $\alpha^2 \ne 1/4$	LCNO
0	$\alpha^2 = 1/4$	R
0	$1 \le \alpha$	LP
$+\infty$	$\alpha \in (-\infty,+\infty)$	LP

For these LCNO cases the boundary condition functions u, v are given by:

Endpoint	Parameter	u	v
0	$-1 < \alpha < 0$ but $\alpha \neq -1/2$	$x^{\frac{1}{2}-\alpha}$	$x^{\frac{1}{2}+\alpha}$
0	$\alpha = -1/2$	x	1
0	$\alpha = 0$	$x^{1/2}$	$x^{1/2}\ln(x)$
0	$0 < \alpha < 1$ but $\alpha \neq 1/2$	$x^{\frac{1}{2}+\alpha}$	$x^{\frac{1}{2}-\alpha}$
0	$\alpha = 1/2$	x	1

The Laguerre polynomials are produced as eigenfunctions only when $\alpha > -1$. For $\alpha \geq 1$ the LP condition holds at 0. For $0 \leq \alpha < 1$ the appropriate boundary condition is the Friedrichs condition: $[y, u](0) = 0$; for $-1 < \alpha < 0$ use the non-Friedrichs condition: $[y,\ v](0) = 0$. In all these cases $\lambda_n = n$ for $n = 0, 1, 2, ...$

29. Heun equation

One Sturm-Liouville form of the general Heun differential equation is

$$-(py')' + qy = \lambda wy \text{ on } (0,1)$$

where the coefficients p, q, w are given explicitly by, for all $x \in (0,1)$,

$$\begin{aligned} p(x) &= x^c(1-x)^d(x+s)^e \\ q(x) &= abx^c(1-x)^{d-1}(x+s)^{e-1} \\ w(x) &= x^{c-1}(1-x)^{d-1}(x+s)^{e-1}. \end{aligned}$$

The parameters a, b, c, d, e and s are all real numbers and satisfy the following two conditions

$$(i)\ s > 0 \text{ and } c \geq 1, d \geq 1, a \geq b$$

and

$$(ii)\ a + b + 1 - c - d - e = 0.$$

From these conditions it follows that

$$a \geq 1, b \geq 1, e \geq 1 \text{ and } a + b - d \geq 1.$$

The differential equation above is a special case of the general Heun equation

$$\frac{d^2w(z)}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right)\frac{dw(z)}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}w(z) = 0$$

with the general parameters $\alpha, \beta, \gamma, \delta, \varepsilon$ replaced by the real numbers a, b, c, d, e, a replaced by $-s$, and q replaced by the spectral parameter λ. For general information concerning the Heun equation see [27, Chapter XV, Section 15.3], [77] and the compendium [74]; for the special form of the Heun equation considered here, and for the connection with confluence of singularities and applications, see the recent paper [55].

We note that the coefficients of the Sturm-Liouville differential equation above satisfy the conditions

1. $q, w \in C[0,1]$ and $w(x) > 0$ for all $x \in (0,1)$

2. $p^{-1} \in L^1_{\text{loc}}(0,1), p(x) > 0$ for all $x \in (0,1)$
3. $p^{-1} \notin L^1(0,1/2]$ and $p^{-1} \notin L^1[1/2,1)$.

Thus both endpoints 0 and 1 are singular for the differential equation. Analysis shows that the endpoint classification for this equation is

Endpoint	Parameter	Classification
0	$c \in [1,2)$	LCNO
0	$c \in [2,+\infty)$	LP
1	$d \in [1,2)$	LCNO
1	$d \in [2,+\infty)$	LP

For the endpoint 0 and for LCNO cases the boundary condition functions u, v are determined by:

Parameter	u	v
$c = 1$	1	$\ln(x)$
$1 < c < 2$	1	x^{1-c}

For the endpoint 1 and for LCNO cases the boundary condition functions u, v are determined by:

Parameter	u	v
$d = 1$	1	$\ln(1-x)$
$1 < d < 2$	1	$(1-x)^{1-d}$

Further it may be shown that the spectrum of any self-adjoint problem on $(0,1)$, with the parameters a, b, c, d, e and s satisfying the above conditions, and considered in the space $L^2((0,1);w)$ with either separated or coupled boundary conditions, is bounded below and discrete. For the analytic properties, and proofs of the spectral properties of this Heun differential equation, see the paper [7].

30. Whittaker equation

The general form of the Whittaker differential equation is

$$-y''(x) + \left(\frac{1}{4} + \frac{k^2-1}{x^2}\right) y(x) = \lambda \frac{1}{x} y(x) \text{ for all } x \in (0,+\infty)$$

where the parameter $k \in [1,+\infty)$.

Endpoint classification in the space $L^2(0,+\infty;x^{-1})$, for all $k \in [1,+\infty)$:

Endpoint	Classification
0	LP
$+\infty$	LP

This equation is studied in [49, Part II, Section 10], where there it is shown that the LP case holds at $+\infty$ and also at 0 for $k \geq 1$; the general properties of

Whittaker functions are given in [1, Chapter 13, Section 13.1.31]. The spectrum of the boundary value problem on $(0,\infty)$ is discrete and is given explicitly by:

$$\lambda_n = n + (k+1)/2, \quad n = 0, 1, 2, 3, \ldots .$$

31. Lamé equation

This differential equation has many forms; there is an extensive literature devoted to the definition, theory and properties of this equation and the associated Lamé functions; see [80, Chapter XXIII, Section 23.4] and [27, Chapter XV, Section 15.2]. The Lamé equation is a special case of the Heun equation; see [27, Chapter XV, Section 15.3] and Section 29 above.

Here we consider two cases of the Lamé equation involving the Weierstrass double periodic elliptic function $\wp$, considered for the special case when the fundamental periods $2\omega_1$ and $2\omega_2$ of $\wp$ satisfy

$$\omega_1 \in (0,\infty) \text{ and } \omega_2 = i\chi \text{ where } \chi \in (0,\infty).$$

We note that the lattice of double poles for $\wp$ is rectangular with points $[2m\omega_1 + 2n\omega_2 : m, n \in \mathbb{Z}\}$ of $\mathbb{C}$.

For the general theory of the Weierstrass elliptic function $\wp$ see [22, Chapter XIII] and [80, Chapter XX].

1. Consider the Sturm-Liouville differential equation

$$-y''(x) + k\wp(x)y(x) = \lambda y(x) \text{ for all } x \in (0, 2\omega_1)$$

where k is a real parameter, $k \in \mathbb{R}$. We note that $\wp(\cdot) : (0, 2\omega_1) \to \mathbb{R}$ and that $\wp(\cdot) \in L^1_{\text{loc}}(0, 2\omega_1)$; from [80, Chapter XX, Section 20.2] and [22, Chapter XIII, Section 13.4] it follows that

$$\wp(x) = x^{-2} + O(x^2) \text{ as } x \to 0^+ \text{ and}$$
$$\wp(x) = (2\omega_1 - x)^{-2} + O((2\omega_1 - x)^2) \text{ as } x \to 2\omega_1^-.$$

These order results for the coefficient $\wp$, at the endpoints of the interval $(0, 2\omega_1)$, taken together with the parameter k, allow of a comparison between this form of the Lamé equation and (a) the Liouville Bessel equation of Section 12 when $k \in [-1/4, +\infty)$, and (b) the Krall equation of Section 44 when $k \in (-\infty, -1/4)$. This comparison leads to the following endpoint classification for this Lamé equation in the space $L^2(0, 2\omega_1)$:

Endpoint	Parameter k	Classification
0	For $k \in (-\infty, -1/4)$	LCO
0	For $k \in [-1/4, 0)$	LCNO
0	$k = 0$	R
0	For $k \in (0, 3/4)$	LCNO
0	For $k \in [3/4, \infty)$	LP

and

Endpoint	Parameter k	Classification
$2\omega_1$	For $k \in (-\infty, -1/4)$	LCO
$2\omega_1$	For $k \in [-1/4, 0)$	LCNO
$2\omega_1$	$k = 0$	R
$2\omega_1$	For $k \in (0, 3/4)$	LCNO
$2\omega_1$	For $k \in [3/4, \infty)$	LP

The boundary condition functions u, v for the LCO and LCNO classifications at the endpoint zero can be copied from the corresponding cases for the Liouville Bessel equation in Section 12, and for the Krall equation Section 44; similarly for the endpoint $2\omega_1$.

2. Consider the Sturm-Liouville differential equation

$$-y''(x) + k\wp(x + \omega_2)y(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty)$$

where k is a real parameter, $k \in \mathbb{R}$.

From the information about the fundamental periods $2\omega_1$ and $2\omega_2$ given in case 1 above, it follows that $\wp(\cdot + \omega_2)$ is real-valued, periodic with period $2\omega_1$, and real-analytic on $\mathbb{R}$, see [1, Chapter 18, Section 18.1]; see also the corresponding case for the algebro-geometric form 3 differential equation, in Subsection 40.3.

Endpoint classification in $L^2(-\infty, +\infty)$ for all $k \in (-\infty, 0) \cup (0, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

This differential equation is of the Mathieu type, see Section 32 below, and the general properties of Sturm-Liouville differential equations with periodic coefficients given in [26, Chapter 2].

32. Mathieu equation

The general form of the Mathieu differential equation is

$$-y''(x) + 2k\cos(2x)y(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty),$$

where that parameter $k \in (-\infty, 0) \cup (0, +\infty)$.

Endpoint classification in $L^2(-\infty, +\infty)$, for all $k \in (-\infty, 0) \cup (0, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

The classical Mathieu equation has a celebrated history and voluminous literature; for the general properties of the Mathieu functions see [1, Chapter 20], [26, Chapter 2, Section 2.5], [27, Volume **III**, Chapter XVI, Section 16.2] and [80, Chapter XIX, Sections 19.1 and 19.2]. For the general properties of Sturm-Liouville differential equations with periodic coefficients see the text [26].

There are no eigenvalues for this problem on $(-\infty,+\infty)$. There may be one negative eigenvalue of the problem on $[0,\infty)$ depending on the boundary condition at the endpoint 0. The continuous (essential) spectrum is the same for the whole line or half-line problems and consists of an infinite number of disjoint closed intervals. The endpoints of these - and thus the spectrum of the problem - can be characterized in terms of periodic and semi-periodic eigenvalues of Sturm-Liouville problems on the compact interval $[0,2\pi]$; these can be computed with SLEIGN2.

The above remarks also apply to the general Sturm-Liouville equation with periodic coefficients of the same period; the so-called Hill's equation.

Of special interest is the starting point of the continuous spectrum - this is also the oscillation number of the equation. For the Mathieu equation ($p=1, q=\cos(x), w=1$) on both the whole line and the half line it is approximately -0.378; this result may be obtained by computing the first eigenvalue λ_0 of the periodic problem on the interval $[0,2\pi]$.

For extensions of this theory to Sturm-Liouville differential equations with almost periodic coefficients see the paper [60].

33. Bailey equation

The general form of the Bailey differential equation, see [11, Data base file xamples.tex; example 7], is

$$-(xy'(x))' - x^{-1}y(x) = \lambda y(x) \text{ for all } x \in (-\infty,0)\cup(0,+\infty).$$

Endpoint classification in $L^2(-\infty,0)\cup L^2(0,+\infty)$:

Endpoint	Classification
$-\infty$	LP
$0-$	LCO
$0+$	LCO
$+\infty$	LP

For both endpoints $0-$ and $0+$:

$$u(x) = \cos(\ln(|x|)) \qquad v(x) = \sin(\ln(|x|)) \text{ for all } x \in (-\infty,0)\cup(0,+\infty).$$

This example is based on the earlier studied Sears-Titchmarsh equation; see Section 58 below.

For numerical results see [11, Data base file xamples.tex; example 7].

34. Behnke-Goerisch equation

The general form of the Behnke-Goerisch differential equation, see [11, Data base file xamples.tex; example 28], is

$$-y''(x) + k\cos^2(x)y(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty)$$

where the parameter $k \in (-\infty, +\infty)$,

Endpoint classification in the space $L^2(-\infty, +\infty)$, for all $k \in (-\infty, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

This is a form of the Mathieu equation. In [15] these authors computed a number of Neumann eigenvalues of this problem using interval arithmetic with rigorous bounds.

35. Boyd equation

The general form of the Boyd equation is, see [11, Data base file xamples.tex; example 4],

$$-y''(x) - x^{-1}y(x) = \lambda y(x) \text{ for all } x \in (-\infty, 0) \cup (0, +\infty).$$

Endpoint classification in $L^2(-\infty, 0) \cup L^2(0, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$0-$	LCNO
$0+$	LCNO
$+\infty$	LP

For both endpoints $0-$ and $0+$

$$u(x) = x \qquad v(x) = x\ln(|x|) \text{ for all } x \in (-\infty, 0) \cup (0, +\infty).$$

This equation arises in a model studying eddies in the atmosphere; see [18]. There is no explicit formula for the eigenvalues of any particular boundary condition; eigenfunctions can be given in terms of Whittaker functions; see [8, Example 3].

36. Boyd equation: regularized

The form of this regularized Boyd equation is

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \text{ for all } x \in (-\infty, 0) \cup (0, +\infty)$$

where

$$p(x) = r(x)^2 \quad q(x) = -r(x)^2 \left(\ln(|x|\right)^2 \quad w(x) = r(x)^2$$

with

$$r(x) = \exp\left(-(x\ln(|x|) - x)\right) \text{ for all } x \in (-\infty, 0) \cup (0, +\infty).$$

Endpoint classification in $L^2(-\infty,0)\cup L^2(0,+\infty)$:

Endpoint	Classification
$-\infty$	LP
$0-$	R
$0+$	R
$+\infty$	LP

This is a regularized R form of the Boyd equation in Section 35; the LCNO singularity at zero has been made R but requiring the introduction of quasi-derivatives. There is a close relationship between the examples these two forms of the Boyd equation; in particular they have the same eigenvalues - see [4]. For a general discussion of regularization using non-principal solutions see [66]. For numerical results see [8, Example 3].

37. Dunford-Schwartz equation

This differential equation is considered in detail in [25, Chapter VIII, Pages 1515-20];

$$-\left((1-x^2)y'(x)\right)' + \left(\frac{2\alpha^2}{(1+x)} + \frac{2\beta^2}{(1-x)}\right) y(x) = \lambda y(x) \text{ for all } x \in (-1,+1)$$

where the independent parameters $\alpha, \beta \in [0,+\infty)$.

Boundary value problems for this differential equation are discussed in [25, Chapter XIII, Section 8].

Endpoint classification in the space $L^2(-1,+1)$ for -1:

Parameter	Classification
$0 \le \alpha < 1/2$	LCNO
$1/2 \le \alpha$	LP

Endpoint classification in the space $L^2(-1,+1)$ for $+1$:

Parameter	Classification
$0 \le \beta < 1/2$	LCNO
$1/2 \le \beta$	LP

For the LCNO cases the boundary condition functions u, v are given by

Endpoint	Parameter	u	v
-1	$\alpha = 0$	1	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$
-1	$0 < \alpha < 1/2$	$(1+x)^{\alpha}$	$(1+x)^{-\alpha}$
$+1$	$\beta = 0$	1	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$
$+1$	$0 < \beta < 1/2$	$(1-x)^{\beta}$	$(1-x)^{-\beta}$

Note that these u and v are not solutions of the differential equation but maximal domain functions.

In the case when $\alpha \in [0,1/2)$ and $\beta \in [0,1/2)$ it is shown in [25, Chapter XIII, Section 8, Page 1519] that the boundary value problem determined by the boundary conditions

$$[y,u](-1) = 0 = [y,u](1)$$

has a discrete spectrum with eigenvalues given by the explicit formula

$$\lambda_n = (n+\alpha+\beta+1)(n+\alpha+\beta) \text{ for } n = 0,1,2,\ldots;$$

the eigenfunctions are determined in terms of the hypergeometric function ${}_2F_1$.

38. Dunford-Schwartz equation: modified

This modification of the Dunford-Schwartz equation replaces one of the LCNO singularities by a LCO singularity;

$$-\left((1-x^2)y'(x)\right)' + \left(\frac{-2\gamma^2}{(1+x)} + \frac{2\beta^2}{(1-x)}\right)y(x) = \lambda y(x) \text{ for all } x \in (-1,+1)$$

where the independent parameters $\gamma, \beta \in [0,+\infty)$.

Endpoint classification in the space $L^2(-1,+1)$ for -1:

Parameter	Classification
$\gamma = 0$	LCNO
$0 < \gamma$	LCO

Endpoint classification in the space $L^2(-1,+1)$ for $+1$:

Parameter	Classification
$0 \le \beta < 1/2$	LCNO
$1/2 \le \beta$	LP

For these LCNO/LCO cases the boundary condition functions u, v are given by

Endpoint	Parameter	u	v
-1	$\gamma = 0$	1	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$
-1	$0 < \gamma$	$\cos(\gamma\ln(1+x))$	$\sin(\gamma\ln(1+x))$
$+1$	$\beta = 0$	1	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$
$+1$	$0 < \beta < 1/2$	$(1-x)^{\beta}$	$(1-x)^{-\beta}$

This is a modification of the Dunford-Schwartz equation above, see Section 37, which illustrates an LCNO/LCO mix obtained by replacing α with $i\gamma$; this changes the singularity at -1 from LCNO to LCO.

Again these u and v are not solutions of the differential equation but maximal domain functions.

39. Hydrogen atom equation

It is convenient to take this equation in two forms:

(1) $$-y''(x) + (kx^{-1} + hx^{-2})y(x) = \lambda y(x) \text{ for all } x \in (0, +\infty)$$

where the two independent parameters $h \in [-1/4, +\infty)$ and $k \in \mathbb{R}$, and

(2) $$-y''(x) + (kx^{-1} + hx^{-2} + 1)y(x) = \lambda y(x) \text{ for all } x \in (0, +\infty)$$

where the two independent parameters $h \in (-\infty, -1/4)$ and $k \in \mathbb{R}$.

Note that form (2) is introduced as a device to aid the numerical computations in the difficult LCO case; it forces the boundary value problem to have a non-negative eigenvalue.

Endpoint classification, for both forms (1) and (2), in $L^2(0, +\infty)$:

Endpoint	Form	Parameters	Classification
0	1	$h = k = 0$	R
0	1	$h = 0, k \in \mathbb{R} \setminus \{0\}$	LCNO
0	1	$-1/4 \leq h < 3/4, h \neq 0, k \in \mathbb{R}$	LCNO
0	1	$h \geq 3/4, k \in \mathbb{R}$	LP
0	2	$h < -1/4, k \in \mathbb{R}$	LCO
$+\infty$	1 and 2	$h, k \in \mathbb{R}$	LP

This is the two parameter version of the classical one-dimensional equation for quantum modelling of the hydrogen atom; see [49, Section 10].

For form (1) and all h, k there are no positive eigenvalues; form (2) is best considered in the single LCO case when some eigenvalues are positive; in form (1) there is a continuous spectrum on $[0, \infty)$; in form (2) there is a continuous spectrum on $[1, \infty)$.

If $k = 0$ the equation reduces to Bessel, see Example 2 above with $h = \nu^2 - 1/4$.

39.1. Results for form 1

In all cases below ρ is defined by

$$\rho := (h + 1/4)^{1/2} \text{ for all } h \geq -1/4.$$

(*a*) For $h \geq 3/4$ and $k \geq 0$ no boundary conditions are required; there is at most one negative eigenvalue and $\lambda = 0$ may be an eigenvalue; for $h \geq 3/4$ and $k < 0$ there are infinitely many negative eigenvalues given by

$$\lambda_n = \frac{-k^2}{(2n + 2\rho + 1)^2},\ \rho = (h + 1/4)^{1/2} > 0,\ n = 0, 1, 2, 3, \ldots$$

and $\lambda = 0$ is not an eigenvalue.

(*b*) For $h = 0$ and $k \in \mathbb{R} \setminus \{0\}$ a boundary condition is required at 0 for which

$$u(x) = x \qquad v(x) = 1 + k\, x \ln(x).$$

For some computed eigenvalues see [8] and [49, Section 10].

(*c*) For $-1/4 < h < 3/4$, *i.e.* $0 < \rho < 1$, and $h \neq 0$, *i.e.* $\rho \neq 1/2$, then a boundary condition is required at 0 for which, for all $x \in (0, +\infty)$,

$$u(x) = x^{\frac{1}{2}+\rho} \qquad v(x) = x^{\frac{1}{2}-\rho} + \frac{k}{1-2\rho} x^{\frac{3}{2}-\rho};$$

the following results hold for the non-Friedrichs boundary condition $[y, v](0) = 0$, *i.e.* $A1 = 0, A2 = 1$:

1. $k > 0,\ 0 < \rho < 1/2$ there are no negative eigenvalues
2. $k > 0,\ 1/2 < \rho < 1$ there is exactly one negative eigenvalue given by

$$\lambda_0 = \frac{-k^2}{(2\rho - 1)^2}$$

3. if $k < 0,\ 0 < \rho < 1/2$ there are infinitely many negative eigenvalues given by

$$\lambda_n = \frac{-k^2}{(2n - 2\rho + 1)^2},\ n = 0, 1, 2, 3, \ldots$$

4. if $k < 0,\ 1/2 < \rho < 1$ there are infinitely many negative eigenvalues given by

$$\lambda_n = \frac{-k^2}{(2n - 2\rho + 3)^2},\ n = 0, 1, 2, 3, \ldots$$

5. for $k = 0$ and $(A1)(A2) < 0$ there is exactly one negative eigenvalue given by:

$$\lambda_0 = \frac{4\,(A1)\,\Gamma(1+\rho)}{(A2)\,\Gamma(1-\rho)^{1/\rho}}.$$

(*d*) For $h = -1/4,\ k \in R$, the LCNO classification at 0 prevails and a boundary condition is required for which, for all $x \in (0, +\infty)$,

$$u(x) = x^{1/2} + kx^{3/2} \qquad v(x) = 2x^{1/2} + \left(x^{1/2} + kx^{3/2}\right) \ln(x).$$

For $k = 0$ and $(A1)(A2) < 0$ there is exactly one negative eigenvalue given by:

$$\lambda_0 = -c \exp(2(A1)/A2), \quad c = 4\exp(4 - 2\gamma)$$

where γ is Euler's constant: $\gamma = 0.5772156649\ldots$.

39.2. Results for form 2

For $h < -1/4,\ k \in R$, the equation is LCO at 0 (recall that we added 1 to the coefficient $q(\cdot)$ for this case, thus moving the start of the continuous spectrum from 0 to 1) for which, defining

$$\sigma := (-h - 1/4)^{1/2},$$

then, for all $x \in (0, +\infty)$,

$$u(x) = x^{1/2}\left[(1-(4h)^{-1}kx)\cos(\sigma\ln(x)) + k\sigma x\sin(\sigma\ln(x))/2\right]$$
$$v(x) = x^{1/2}\left[(1-(4h)^{-1}kx)\sin(\sigma\ln(x)) + k\sigma x\cos(\sigma\ln(x))/2\right];$$

(i) when $k = 0$ this equation reduces to the Krall equation Example 20 below (but note that the notation is different)

(ii) when $k \neq 0$ explicit formulas for the eigenvalues are not available; however we report here on the qualitative properties of the spectrum for any boundary condition at 0:

(α) for all $k \in R$ there are infinitely many negative eigenvalues tending exponentially to $-\infty$

(β) for $k > 0$ there are only a finite number of eigenvalues in any bounded interval, in particular they do not accumulate at 1

(γ) for $k \leq 0$ the eigenvalues accumulate also at 1.

(δ) for $k = 0$ and $(A1)(A2) < 0$ there is exactly one negative eigenvalue given by:

$$\lambda_0 = \frac{4\,(A1)\,\Gamma(1+\rho)}{(A2)\,\Gamma(1-\rho)^{1/\rho}}.$$

Most of these results are due to Jörgens, see [49, Section 10]; a few new results were established by the authors of [11, Data base file xamples.tex; example 13].

40. Algebro-geometric equations

A potential q of the one-dimensional Schrödinger equation

$$L[y](x) := -y''(x) + q(x)y(x) = \lambda y(x) \text{ for all } x \in I \subseteq \mathbb{R}$$

is called an algebro-geometric potential if there exists a linear ordinary differential expression P of odd-order and leading coefficient 1, which commutes with L. There are deep relationships between algebro-geometric equations and the Korteweg-de Vries hierarchy of non-linear differential equations. An overview of these properties and results can be found in the survey article [44] which contains a substantial list of references.

The main structure and properties of the algebro-geometric equations can only be observed when the differential equations are considered in the complex plane, which would take the contents of this catalogue outside the environment of the Sturm-Liouville symmetric differential equations as given in Section 3 above.

However, three forms of algebro-geometric differential equations are given here; all three examples are Sturm-Liouville equations; two cases are related to other examples in this catalogue. However, all of these three examples have to be seen within the structure of algebro-geometric potentials and the relationships to non-linear differential equations.

40.1. Algebro-geometric form 1

Let $l \in \mathbb{N}_0$; then the differential equation is

$$-y''(x) + l(l+1)x^{-2}y(x) = \lambda y(x) \text{ for all } x \in (0,\infty).$$

this is a special case of:

(i) The hydrogen atom equation of Section 39 above, which gives the endpoint classification on $(0,\infty)$ for this example.

(ii) The Liouville form of the Bessel differential equation, see Section 12 above, when the parameter $\nu = l + 1/2$; these cases of Bessel functions are named as the “spherical” Bessel functions; see [1, Chapter 10, Section 10.1] and [79, Chapter III, Section 3.41].

Endpoint classification in $L^2(0,+\infty)$:

Endpoint	Parameter	Classification
0	$l = 0$	R
0	$l \in \mathbb{N}$	LP
$+\infty$	$l \in \mathbb{N}_0$	LP

It is shown in [44] that this differential equation has two solutions of the form

$$y(x,\lambda) = \exp(ixs)\left(s^l + \sum_{j=0}^{l} a_j \frac{s^{l-j}}{x^j}\right) \text{ for all } x \in (0,\infty) \text{ and all } \lambda \in \mathbb{C},$$

where:

(i) $s^2 := \lambda$

(ii) the coefficients $\{a_j : j \in \mathbb{N}_0\}$ are determined by

$$a_0 = 1 \text{ and } a_{n+1} = i\frac{l(l+1) - n(n+1)}{2(n+1)} \text{ for all } n \in \mathbb{N}.$$

40.2. Algebro-geometric form 2

Let $g \in \mathbb{N}_0$; then the differential equation is

$$-y''(x) - \frac{g(g+1)}{\cosh(x)^2}y(x) = \lambda y(x) \text{ for all } x \in (-\infty,\infty);$$

this equation is a special case of:

(i) The hypergeometric differential equation, see Section 9 above but in particular [78, Chapter IV, Section 4.19].

(ii) The Liouville form of the Jacobi function differential equation, see Section 26 above, with the special case of $\alpha = -1/2$ and $\beta = g + 1/2$; here, the interval $(0,\infty)$ for the equation can be extended to $(-\infty,\infty)$ since the origin 0 is no longer a singular point of the equation when $\alpha = 1/2$.

Endpoint classification in $L^2(-\infty,+\infty)$:

Endpoint	Parameter	Classification
$-\infty$	$g \in \mathbb{N}_0$	LP
$+\infty$	$g \in \mathbb{N}_0$	LP

It is shown in [44] that this differential equation has two solutions of the form

$$y(x,\lambda) = \exp(ixs)\left(\sum_{n=0}^{g} a_n(s)\tanh(x)^n\right) \text{ for all } x \in (-\infty,\infty) \text{ and all } \lambda \in \mathbb{C},$$

where:

(*i*) $s^2 := \lambda$

(*ii*) the coefficients $\{a_n : n = 0,\ldots,g\}$ are determined by a five-term recurrence relation.

40.3. Algebro-geometric form 3

Let $g \in \mathbb{R}$; then this differential equation is a special case of Lamé's equation, see Section 31 above, and is given by

$$-y''(x) + g(g+1)\wp(x+\omega')y(x) = \lambda y(x) \text{ for all } x \in (-\infty,\infty),$$

where $\wp$ is the Weierstrass elliptic function with fundamental periods 2ω and $2\omega'$, where ω is real and ω' is purely imaginary.

In this situation $\wp(\cdot+\omega')$ is real-valued, periodic with period 2ω, and real-analytic on $\mathbb{R}$, see [1, Chapter 18, Section 18.1].

Endpoint classification in the space $L^2(-\infty,\infty)$:

Endpoint	Parameter	Classification
$-\infty$	$g \in \mathbb{R}$	LP
$+\infty$	$g \in \mathbb{R}$	LP

When $g \in \mathbb{N}_0$, it is shown in [44] that this differential equation (which is an example of the general Lamé differential equation) has solutions of the form

$$y(\mathbf{a},x) = \sigma(x+\omega')^{-g}\prod_{j=1}^{g}\sigma(x+\omega'-a_j)\exp\left(x\sum_{j=1}^{g}\zeta(a_j)\right) \text{ for all } x \in (-\infty,\infty),$$

where the vector $\mathbf{a} = (a_1,a_2,\ldots,a_g)$ has to satisfy the conditions

$$\sum_{\substack{j=1\\ j\neq k}}^{g}(\zeta(a_j-a_k)-\zeta(a_j)+\zeta(a_k)) = 0 \text{ for } k = 1,2,\cdots,g$$

and the spectral parameter λ is then given by

$$\lambda = (1-2g)\sum_{j=1}^{g}\wp(a_j).$$

Here σ and ζ are the Weierstrass-σ and Weierstrass-ζ functions respectively, see [1, Chapter 18, Section 18.1].

The spectrum of the unique self-adjoint operator, in the Hilbert function space $L^2(-\infty,\infty)$, generated by this example of the Lamé differential equation, consists of $g+1$ disjoint intervals, one of which is a semi-axis; these are the spectral bands of this differential operator.

Note that $\mathbf{a}$ satisfies the constraints mentioned if and only if $-\mathbf{a}$ satisfies the same constraint, since ζ is an odd function; as $\wp$ is an even function these properties lead to the same value of λ. Both the functions $y(\mathbf{a},\cdot)$ and $y(-\mathbf{a},\cdot)$ do then satisfy the same differential equation; they are linearly independent except when λ is one of the $2g+1$ band edges.

For these results and additional examples of algebro-geometric differential equations see the survey paper [44].

40.4. Algebro-geometric form 4

This form is named as the N-soliton potential.

We introduce the $N \times N$ matrix, for $1 \le j,k \le N$ and all $x \in (-\infty,\infty)$,

$$C_N(x) = \big(c_j c_k (\kappa_j + \kappa_k)^{-1} \exp(-(\kappa_j + \kappa_k)x\big)$$

with

$$c_j > 0, \kappa_j > 0, \kappa_j \neq \kappa_k \text{ for all } 1 \le j,k \le N \text{ with } j \neq k;$$

the N-soliton potential $q_N : (-\infty,\infty) \to \mathbb{R}$ is then defined by

$$q_N(x) := -2\frac{d^2}{dx^2} \ln(\det(I_N + C_N(x))) \text{ for all } x \in (-\infty,\infty)$$

(with I_N the identity matrix in $\mathbb{C}^N$). The corresponding Sturm-Liouville differential equation then reads

$$-y''(x) + q_N(x)y(x) = \lambda y(x) \text{ for all } x \in (-\infty,\infty) \text{ and } \lambda \in \mathbb{C}.$$

Since

$$q_N \in C^\infty(-\infty,\infty), \quad q_N(x) = O(\exp(-2\kappa_{j_0}|x|)) \text{ for } |x| \to \infty,$$

where $\kappa_{j_0} = \min_{1\le j\le N}(\kappa_j)$, where $\hat{\kappa} := \max_{1\le j\le N}(\kappa_j)$ for all $x \in \mathbb{R}$.the endpoint classification in $L^2(-\infty,\infty)$ is

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

Defining

$$c_{N,j,+} := c_j, \quad c_{N,j,-} := c_j^{-1} \times \begin{cases} 2\kappa_1, & j = N = 1, \\ 2\kappa_j \displaystyle\prod_{k=1}^{N} \frac{\kappa_j + \kappa_k}{\kappa_j - \kappa_k} \text{ with } k \neq j, & 1 \le j \le N,\ N \ge 2 \end{cases}$$

two independent solutions of the differential equation, associated with q_N, are then given by

$$f_{N,\pm}(x,\lambda) := \left[1 - i\sum_{j=1}^{N}(\sqrt{\lambda}+i\kappa_j)^{-1}c_{N,j,\pm}\psi_{N,j}(x)\exp\left(\mp\kappa_j x\right)\right]\exp(\pm i\sqrt{\lambda}x),$$

for all $\lambda \in \mathbb{C}$ with $\mathrm{Im}(\sqrt{\lambda}) \geq 0$, and all $x \in (-\infty,\infty)$. Here $\{\psi_{N,j}(\cdot) : j = 1,2,\ldots,N\}$ are given as follows; define the column vector

$$\Psi_N^0(x) := (c_1\exp(\kappa_1 x),\ldots,c_N\exp(-\kappa_N x))^\top,$$

and then $\Psi_N(\cdot)$ by

$$\Psi_N(x) := [I_N + C_N(x)]^{-1}\psi_N^0(x) \text{ for all } x \in (-\infty,\infty),$$

both for all $x \in (-\infty,\infty)$. Writing now

$$\Psi_N(x) = (\psi_{N,1}(x),\ldots,\psi_{N,N}(x))^\top$$

this defines the components $\{\psi_{N,j}(\cdot) : j = 1,2,\ldots,N\}$ and completes the definition of the two solutions $f_{N,\pm}$.

Now let H_N denote the (maximally defined) self-adjoint Schrödinger operator with potential q_N in $L^2(-\infty,\infty)$. Then $\psi_{N,j} \in C^\infty(-\infty,\infty)$ are exponentially decaying eigenfunctions of H_N as $|x| \to \infty$, corresponding to the negative eigenvalues $-\kappa_j^2$; thus

$$H_N\psi_{N,j} = -\kappa_j^2\psi_{N,j}, \quad 1 \leq j \leq N.$$

Moreover, H_N has spectrum

$$\{-\kappa_j^2 : 1 \leq j \leq N\} \cup [0,\infty)$$

and q_N satisfies

$$q_N(x) = -4\sum_{j=1}^{N}\kappa_j\psi_{N,j}(x)^2 < 0, \quad 0 < -q_N(x) \leq 2\hat{\kappa}^2,$$

where $\hat{\kappa} := \max_{1\leq j\leq N}(\kappa_j)$ for all $x \in \mathbb{R}$.

The potentials q_N are reflectionless since the corresponding 2×2 scattering matrix $S_N(\lambda)$ is of the form

$$S_N(\lambda) = \begin{pmatrix} T_N(\lambda) & R_N^r(\lambda) \\ R_N^\ell(\lambda) & T_N(\lambda)\end{pmatrix} \text{ for all } \lambda \geq 0,$$

with transmission coefficients given by

$$T_N(\lambda) = \prod_{j=1}^{N}\frac{\sqrt{\lambda}+i\kappa_j}{\sqrt{\lambda}-i\kappa_j} \text{ for all } \lambda \geq 0$$

and vanishing reflection coefficients from the right and left incidence

$$R_N^r(\lambda) = R_N^\ell(\lambda) = 0 \text{ for all } \lambda \geq 0.$$

Thus, the N-soliton potentials q_N can be thought of a particular construction of reflectionless potentials that adds N negative eigenvalues $-\kappa_j^2$, $1 \leq j \leq N$, to the spectrum of H_0, where $H_0 = -d^2/dx^2$ is the Schrödinger operator in $L^2(-\infty, \infty)$ associated with the trivial potential $q_0(x) = 0$ for all $x \in \mathbb{R}$, and spectrum $[0, \infty)$.

It can be shown that q_N satisfies a particular Nth stationary KdV equation, see [40, Section 1.3]. In addition, introducing an appropriate time-dependence in c_j leads to KdV N-soliton potentials, see [40, Section 1.4].

We also note that $q_g(x) = -g(g+1)[\cosh(x)]^{-2}$, treated in Subsection 40.2 above, is a special case of q_N for $N = g$ and a particular choice of κ_j and c_j, $1 \leq j \leq N$.

Reflectionless potentials q_N were first derived by Kay and Moses [51] (see also [23], [24], [39], and [41] for detailed discussions).

41. Bargmann potentials

Let $\varphi_0(x, \lambda) = s^{-1}\sin(sx)$ for $\lambda = s^2 \in \mathbb{C}$ and $x \in [0, \infty)$; then for $N \in \mathbb{N}$ introduce the $N \times N$ matrix

$$B_N(x) = (B_{N,j,k}(x)) \text{ for } 1 \leq j,k \leq N \text{ and all } x \in [0, \infty)$$

given by

$$\begin{aligned} B_{N,j,k}(x) &= \int_0^x C_j \varphi_0(t, -\gamma_j^2)\varphi_0(t, -\gamma_k^2)\,dt \\ &= C_j(2\gamma_j\gamma_k)^{-1} \begin{cases} (2\gamma_j)^{-1}\sinh(2\gamma_j x) - x, \\ \text{for } j = k, \\ (\gamma_j + \gamma_k)^{-1}\sinh((\gamma_j + \gamma_k)x) - (\gamma_j - \gamma_k)^{-1}\sinh((\gamma_j - \gamma_k)x), \\ \text{for } j \neq k, \end{cases} \\ &\quad C_j > 0,\ \gamma_j > 0 \text{ for } 1 \leq j,k \leq N. \end{aligned}$$

Bargmann potentials q_N are then defined by

$$q_N(x) = -2\frac{d^2}{dx^2}\ln(\det(I_N + B_N(x))) \text{ for all } x \in [0, \infty)$$

(I_N the identity matrix in $\mathbb{C}^N$), and the associated Sturm-Liouville differential equation reads

$$-y''(x) + q_N(x)y(x) = \lambda y(x) \text{ for all } x \in [0, \infty) \text{ and } \lambda \in \mathbb{C}.$$

It can show that

$$\int_0^\infty (1 + x)|q_N(x)|\,dx < \infty.$$

Actually, much more detailed information can be obtained to give

$$
\begin{aligned}
q_N(x) &\underset{x\downarrow 0}{=} -4\left(\sum_{j=1}^{N} C_j x\right) + o(x) \\
q_N(x) &\underset{x\uparrow\infty}{=} -2C_{j_0}^{-1}(2\gamma_{j_0})^5 \exp(-2\gamma_0 x)[1+o(1)],
\end{aligned}
$$

where $\gamma_{j_0} = \min_{1\le j\le N}(\gamma_j)$ and C_{j_0} is the corresponding normalization constant.

Hence the endpoint classification of this equation in $L^2(0,+\infty)$ is

Endpoint	Classification
0	R
$+\infty$	LP

The regular solution $\varphi_N(\cdot,\lambda)$ associated with q_N is then given by

$$
\varphi(x,\lambda) = \frac{\det\begin{vmatrix} I_N + B_N(x) & \psi(x) \\ \beta(x,\lambda) & \varphi_0(x,\lambda)\end{vmatrix}}{\det(I_N + B_N(x))}, \quad x \in [0,\infty),\ \lambda \in \mathbb{C},
$$

where the matrix in the numerator is obtained by adding to $I_N + B_N(x)$ the column ψ, the row β, and the last diagonal element φ_0. Here ψ and χ are vectors with components $C_j\varphi_0(x,-\gamma_j^2)$ and $\varphi_0(x,-\gamma_j^2)$, respectively, and

$$
\beta(x,\lambda) = \int_0^x \chi(t)^\top \varphi_0(t,\lambda)\,dt.
$$

Similarly, the Jost solution $f(x,\lambda)$ corresponding to q_N can be computed, but we omit the lengthy expression; the Jost function $F(s)$ associated with q_N finally reads

$$
F(s) = f(0,\lambda) = \prod_{j=1}^{N} \frac{s - i\gamma_j}{s + i\gamma_j} \text{ with } \lambda = s^2.
$$

This shows that the Schrödinger operator H_N in $L^2(0,+\infty)$ associated with q_N and a Dirichlet boundary condition at $x=0$ has spectrum

$$
\{-\gamma_j^2 : 1 \le j \le N\} \cup [0,\infty).
$$

Next we denote by q_0 the trivial potential $q_0(x) = 0$ for all $x \in [0,\infty)$, and by $H_0 = -d^2/dx^2$ the corresponding Schrödinger operator in $L^2(0,+\infty)$ with a Dirichlet boundary condition at $x=0$; the operator H_0 then has spectrum $[0,\infty)$.

In comparison with the trivial potential $q_0(x) = 0$, the Bargmann potential $q_N(x)$ is constructed such that the corresponding operator H_N has N additional strictly negative eigenvalues at $-\gamma_j^2$ for $1 \le j \le N$. Put differently, N negative eigenvalues $-\gamma_j^2$ have been added to the spectrum of H_0.

However, since $|F(s)| = 1$ for all $s \ge 0$, the spectral densities of H_N and H_0 coincide for $\lambda \ge 0$ (recall $\lambda = s^2$). Explicitly, the spectral function $\rho_N(\lambda)$, for all

$\lambda \in (-\infty, +\infty)$, of H_N is of the form

$$\rho_N(\lambda) = \begin{cases} (2/3)\pi^{-1}\lambda^{3/2}, & \lambda \geq 0, \\ \sum_{j=1}^{N} C_j \theta(\lambda + \gamma_j^2), & \lambda < 0 \end{cases}$$

(here $\theta(t) = 1$ for $t > 0$, $\theta(t) = 0$ for $t < 0$), which should be compared with the spectral function $\rho_0(\lambda)$ of H_0,

$$\rho_0(\lambda) = \begin{cases} (2/3)\pi^{-1}\lambda^{3/2}, & \lambda \geq 0, \\ 0, & \lambda < 0. \end{cases}$$

For Bargmann's original work we refer to [13], [14]; more details on Bargmann potentials can be found in [21, Sections III.2, IV.1 and IV.3], and the references therein (see also [42, Section 11]).

42. Halvorsen equation

The Halvorsen differential equation exhibits the difficulties created at R endpoints, both analytically and numerically, in certain circumstances;

$$-y''(x) = \lambda x^{-4} \exp(-2/x) y(x) \text{ for all } x \in (0, +\infty)$$

The endpoint classification in the weighted space $L^2((0, +\infty; x^{-4}\exp(-2/x))$:

Endpoint	Classification
0	R
$+\infty$	LCNO

For the endpoints 0 and $+\infty$ in the R and LCNO classification the boundary condition functions u, v are determined by

Endpoint	u	v
0	x	1
$+\infty$	1	x

in this example the LC boundary condition form can be used at the R endpoint 0, with u and v as shown.

Since this equation is R at 0 and LCNO at $+\infty$ the spectrum is discrete and bounded below for all boundary conditions. However, this example illustrates that even a R endpoint can cause difficulties for computation; details of the computation of eigenvalues are given in [11, Data base file xamples.tex; example 3]

At 0, the principal boundary condition entry is $A1 = 1,\ A2 = 0$; at ∞ with $u(x) = 1,\ v(x) = x$ the principal boundary condition entry is also $A1 = 1,\ A2 = 0$, but note the interchange of the definitions of u and v at these two endpoints.

43. Jörgens equation

We have this example due to Jörgens [49];

$$-y''(x) + (\exp(2x)/4 - k\exp(x))y(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty)$$

where the parameter $k \in (-\infty, +\infty)$.

Endpoint classification in the space $L^2(-\infty, +\infty)$, for all $k \in (-\infty, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

This is a remarkable example from Jörgens; numerical results are given in [11, Data base file xamples.tex; example 27]. Details of this problem are given in [49, Part II, Section 10]. For all $k \in (-\infty, +\infty)$ the boundary value problem on the interval $(-\infty, +\infty)$ has a continuous spectrum on $[0, +\infty)$; for $k \leq 1/2$ there are no eigenvalues; for $h = 0, 1, 2, 3, \ldots$ and then k chosen by $h < k - 1/2 \leq h + 1$, there are exactly $h+1$ eigenvalues and these are all below the continuous spectrum; these eigenvalues are given explicitly by

$$\lambda_n = -(k - 1/2 - n)^2, \quad n = 0, 1, 2, 3, \ldots, h.$$

44. Rellich equation

The Rellich differential equation is, where the parameter $K \in \mathbb{R}$,

$$-y''(x) + Kx^{-2}y(x) = \lambda y(x) \text{ for all } x \in (0, +\infty);$$

this equation has a long and interesting history as indicated in the references, see [72], [65], [54], [20] and [43].

Here, we consider the equation in the form, where the parameter $k \in (0, +\infty)$,

$$-y''(x) + (1 - (k^2 + 1/4)x^{-2})y(x) = \lambda y(x) \text{ for all } x \in (0, +\infty)$$

as discussed in the paper [54], and in view of the connection with the computer program SLEIGN2, see [8] and [10].

Endpoint classification, for all $k \in (0, +\infty)$, in the space $L^2(0, +\infty)$:

Endpoint	Classification
0	LCO
$+\infty$	LP

This example should be seen as a special case of the Bessel Example 2 above; solutions can be obtained in terms of the modified Bessel functions.

To help with the computations for this example the spectrum is translated by a term $+1$; this simple device is used for numerical convenience.

For problems with separated boundary conditions at endpoints 0 and ∞ there is a continuous spectrum on $[1, \infty)$ with a discrete (and simple) spectrum on $(-\infty, 1)$. This discrete spectrum has cluster points at both $-\infty$ and 1.

For the LCO endpoint at 0 the boundary condition functions are given by

$$u(x) = x^{1/2}\cos(k\ln(x)) \qquad v(x) = x^{1/2}\sin(k\ln(x)).$$

For the boundary value problem on $[0,\infty)$ with boundary condition $[y,u](0) = 0$ let the following conditions and notations apply:

(i) suppose $\Gamma(1+i) = \alpha + i\beta$ and $\mu > 0$ satisfies $\tan\left(\ln(\frac{1}{2}\mu)\right) = -\alpha/\beta$
(ii) $\theta = \mathrm{Im}(\log(\Gamma(1+i)))$
(iii) $\ln(\frac{1}{2}\mu) = \frac{1}{2}\pi + \theta + s\pi$ for $s = 0, \pm1, \pm2, ...$
(iv) $\mu_s^2 = \left(2\exp(\theta + \frac{1}{2}\pi)\right)^2 \exp(2s\pi)$ $s = 0, \pm1, \pm2, ...$

then the eigenvalues are given explicitly by $\lambda_n = -\mu^2_{-(n+1)} + 1$ $(n = 0, \pm1, \pm2, ...)$.

This problem creates major computational difficulties; see [11, Data base file xamples.tex; example 20]. The program SLEIGN2 can compute only six of these eigenvalues in a normal UNIX server, even in double precision, specifically λ_{-3} to λ_2; other eigenvalues are, numerically, too close to 1 or too close to $-\infty$. Here we list these SLEIGN2 computed eigenvalues in double precision in a normal UNIX server and compare them with the same eigenvalues computed from the transcendental equation given above; for the problem on $(0,\infty)$ with $k = 1$ and $A1 = 1.0$, $A2 = 0.0$, the results are:

Eigenvalue	eig from SLEIGN2	eig from trans. equ.
-3	$-276,562.5$	$-14,519,130$
-2	$-27,114.48$	$-27,114.67$
-1	-49.62697	-49.63318
0	0.9054452	0.9054454
1	0.9998234	0.9998234
2	0.9999997	0.9999997

45. Laplace tidal wave equation

This differential equation is given by:

$$-(x^{-1}y'(x))' + \left(kx^{-2} + k^2x^{-1}\right)y(x) = \lambda y(x) \text{ for all } x \in (0,+\infty),$$

where the parameter $k \in (-\infty,0) \cup (0,+\infty)$. This equation has been studied by many authors, in particular by Homer in his doctoral thesis [47] where a detailed list of references is to be found:

Endpoint classification in $L^2(0,\infty)$:

Endpoint	Classification
0	LCNO
$+\infty$	LP

For the endpoint 0:

$$u(x) = x^2 \qquad v(x) = x - k^{-1} \text{ for all } x \in (0,+\infty).$$

This equation is a particular case of the more general equation with this name; for details and references see [47].

There are no representations for solutions of this differential equation in terms of the well-known special functions. Thus to determine boundary conditions at the LCNO endpoint 0 use has to be made of maximal domain functions; see the $u,\ v$ functions given above. Numerical results for some boundary value problems and certain values of the parameter k, are given in [11, Data base file xamples.tex; example 8].

46. Latzko equation

This differential equation is given by

$$-((1-x^7)y'(x))' = \lambda x^7 y(x) \text{ for all } x \in (0,1].$$

Endpoint classification in $L^2(0,1]$:

Endpoint	Classification
0	R
1	LCNO

For the endpoint 1:

$$u(x) = 1 \qquad v(x) = -\ln(1-x) \text{ for all } (0,1).$$

This differential equation has a long and celebrated history; in particular it has been studied by Fichera, see [36, Pages 43 to 45]. There is a LCNO singularity at the endpoint 1 which requires the use of maximal domain functions; see the $u,\ v$ functions given above. The endpoint 0 is R but due to the fact that, in general, when a weight has the property $w(0) = 0$ then boundary value problems create difficulties for numerical computations.

This example is similar in some respects to the Legendre equation of Section 19 above.

For numerical results see [11, Data base file xamples.tex; example 7].

47. Littlewood-McLeod equation

This important example gives a Sturm-Liouville boundary value problem that has a discrete spectrum that is unbounded above and below; the differential is

$$-y''(x) + x\sin(x)y(x) = \lambda y(x) \text{ for all } x \in [0,+\infty).$$

Endpoint classification in the space $L^2(0,+\infty)$:

Endpoint	Classification
0	R
$+\infty$	LP

This differential equation is an example of the LPO endpoint classification introduced in [69, Chapter 7].

The spectral analysis of this differential equation is considered in [57] and [62]; the equation is R at 0 and LP at $+\infty$. All self-adjoint operators in $L^2[0,\infty)$

have a simple, discrete spectrum $\{\lambda_n : n = 0, \pm 1, \pm 2, \ldots\}$ that is unbounded both above and below, *i.e.*

$$\lim_{n\to-\infty} \lambda_n = -\infty \qquad \lim_{n\to+\infty} \lambda_n = +\infty.$$

Every eigenfunction has infinitely many zeros in $(0, \infty)$.

SLEIGN2, and all other codes, fail to compute the eigenvalues for this type of LP oscillatory problem. However there is qualitative information to be obtained by considering regular problems on $[0, X]$ with, say, Dirichlet boundary conditions $y(0) = Y(X) = 0$.

48. Lohner equation

This Sturm-Liouville example was one of the first differential equations to be subjected to the Lohner code, see [58], for computing guaranteed numerical bounds for eigenvalues of boundary value problems, but see the earlier paper of Plum [71]; the differential equation is

$$-y''(x) - 1000xy(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty).$$

Endpoint classification in the space $L^2(-\infty, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

In [58] Lohner computed the Dirichlet eigenvalues of certain regular problems on compact intervals, using interval arithmetic, and obtained rigorous bounds. In double precision SLEIGN2 computed eigenvalues to give numerical values that are in good agreement with these guaranteed bounds.

49. Pryce-Marletta equation

This differential equation presents a difficult problem for computational programs; it was devised by Pryce, see [69, Appendix B, Problem 60], and studied by Marletta [61]; the differential equation is:

$$-y''(x) + \frac{3(x-31)}{4(x+1)(x+4)^2}y(x) = \lambda y(x) \text{ for all } x \in [0, +\infty).$$

Endpoint classification in $L^2(0, +\infty)$:

Endpoint	Classification
0	R
$+\infty$	LP

For this differential equation boundary value problems on the interval $[0, \infty)$ are considered. Since $q(x) \to 0$ as $x \to \infty$ the continuous spectrum consists of $[0, \infty)$ and every negative number is an eigenvalue for some boundary condition at 0.

For the boundary condition $A1 = 5, A2 = 8$ at the endpoint 0, there is a negative eigenvalue λ_0 near -1.185. However the equation with $\lambda = 0$ has a solution

$$y(x) = \frac{1 - x^2}{(1 + x/4)^{5/2}} \text{ for all } x \in [0, \infty),$$

that satisfies this boundary condition, which is not in $L^2(0, \infty)$ but is "nearly" in this space. This solution deceives most computer programs; however SLEIGN2 correctly reports that λ_0 is the only eigenvalue, and the start of the continuous spectrum at 0.

Additional details of this example are to be found in the Marletta certification report on SLEIGN [61].

50. Meissner equation

The Meissner equation has constant but discontinuous coefficients; it has remarkable distribution of simple and double eigenvalues for periodic boundary conditions on the interval $(-1/2, 1/2)$: the differential equation is

$$-y''(x) = \lambda w(x) y(x) \text{ for all } x \in (-\infty, +\infty),$$

where the weight coefficient w is defined by

$$\begin{aligned} w(x) &= 1 \text{ for all } x \in (-\infty, 0] \\ &= 9 \text{ for all } x \in (0, +\infty). \end{aligned}$$

Endpoint classification in the space $L^2(-\infty, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

This equation arose in a model of a one dimensional crystal. For this constant coefficient equation with a weight function which has a jump discontinuity the eigenvalues can be characterized as roots of a transcendental equation involving only trigonometrical and inverse trigonometrical functions. There are infinitely many simple eigenvalues and infinitely many double ones for the periodic case; they are given by:

Periodic boundary conditions on $(-1/2, +1/2)$**,** *i.e.*

$$y(-1/2) = y(+1/2) \qquad y'(-1/2) = y'(+1/2).$$

We have $\lambda_0 = 0$ and for $n = 0, 1, 2, \ldots$

$$\lambda_{4n+1} = (2m\pi + \alpha)^2; \quad \lambda_{4n+2} = (2(n+1)\pi - \alpha))^2;$$
$$\lambda_{4n+3} \;=\; \lambda_{4n+4} = (2(n+1)\pi))^2.$$

where $\alpha = \cos^{-1}(-7/8)$

Semi-periodic boundary conditions on $(-1/2, +1/2)$**,** *i.e.*

$$y(-1/2) = -y(+1/2) \qquad y'(-1/2) = -y'(+1/2).$$

With $\beta = \cos^{-1}((1+\sqrt(33))/16)$ and $\gamma = \cos^{-1}((1-\sqrt(33))/16)$ these are all simple and given by, for $n = 0, 1, 2, \ldots$

$$\lambda_{4n} = (2n\pi + \beta)^2;\ \lambda_{4n+1} = (2n\pi + \gamma)^2;$$

$$\lambda_{4n+2} = (2(n+1)\pi - \gamma)^2;\ \lambda_{4n+3} = (2(n+1)\pi - \beta)^2.$$

For the general theory of periodic differential boundary value problems see [26]; for a special case with discontinuous coefficients see [45].

51. Morse equation

This differential equation has exponentially small and large coefficients; the differential equation is

$$-y''(x) + (9\exp(-2x) - 18\exp(-x))y(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty).$$

Endpoint classification in the space $L^2(-\infty, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

This differential equation on the interval $(-\infty, \infty)$ is studied in [5, Example 6]; the spectrum has exactly three negative, simple eigenvalues, and a continuous spectrum on $[0, \infty)$; the eigenvalues are given explicitly by

$$\lambda_n = -(n - 2.5)^2 \text{ for } n = 0, 1, 2.$$

52. Morse rotation equation

This differential equation is considered in [5] and is given as

$$-y''(x) + (2x^{-2} - 2000(2e(x) - e(x)^2))y(x) = \lambda y(x) \text{ for all } x \in (0, +\infty),$$

where

$$e(x) = \exp(-1.7(x - 1.3)) \text{ for all } x \in (0, +\infty).$$

Endpoint classification in the space $L^2(0, +\infty)$

Endpoint	Classification
0	LP
$+\infty$	LP

This classical problem on the interval $(0, \infty)$ has a continuous spectrum on $[0, \infty)$ and exactly 26 negative eigenvalues; it provides an invaluable numerical test for computer programs.

53. Brusencev/Rofe-Beketov equations

53.1. Example 1

The Sturm-Liouville differential equation

$$-(x^4y'(x))' - 2x^2y(x) = \lambda y(x) \text{ for all } x \in (0,\infty)$$

is considered in the paper [19]; this example provides a LC case with special properties.

Endpoint classification in $L^2(0,+\infty)$:

Endpoint	Classification
0	LP
$+\infty$	LCNO

For the endpoint $+\infty$ in the LCNO classification the boundary condition functions u, v are determined by

Endpoint	u	v
$+\infty$	x^{-1}	x^{-2}

53.2. Example 2

The Sturm-Liouville differential equation

$$-y''(x) - \left(x^{10} + x^4\text{sign}(\sin(x))\right) y(x) = \lambda y(x) \text{ for all } x \in [0,\infty)$$

is considered in the paper [73]; this example provides a LC case with special properties.

Endpoint classification in $L^2(0,+\infty)$:

Endpoint	Classification
0	R
$+\infty$	LCO

For the endpoint $+\infty$ in the LCO classification the boundary condition functions u, v may be determined as the real and imaginary parts of the expression

$$x^{-5/2}\exp(ix^6/6)Y(x) \text{ for all } x \in [1,\infty),$$

where the function $Y(\cdot)$ is the solution of the integral equation, for $x \in [1,\infty)$,

$$Y(x) = 1 + \frac{i}{2}\int_x^\infty \left(t^{-1}\text{sign}(\sin(t)) + \frac{35}{4}t^{-7}\right)\left[\exp\left(\frac{i}{3}(t^6 - x^6)\right) - 1\right] Y(t)\ dt.$$

The solution $Y(\cdot)$ of this integral equation may be obtained by the iteration method of successive approximations; in this process it has to be noted that the integrals concerned are only conditionally convergent.

54. Slavyanov equations

In the important text [77] the authors give a systematic presentation of a unified theory of special functions based on singularities of linear ordinary differential equations in the complex plane $\mathbb{C}$. In particular, in [77, Chapter 3], there is to be found an authoritative account of the definition and properties of the Heun differential equation.

In [77, Chapter 4] there is a chapter devoted to physical applications, including the use of the Heun differential equation, resulting from the application of separation techniques to boundary value problems for linear partial differential equations. From this chapter we have selected three examples of Sturm-Liouville differential equations; each equation contains a number of symbols denoting physical constants and parameters which are given here without explanation. To allow the quoted examples to be given in Sturm-Liouville form the notation for one of these parameters has been changed to play the role of the spectral parameter $\lambda \in \mathbb{C}$.

The resulting Sturm-Liouville examples given below have not yet been considered for their endpoint classification, nor for their boundary condition functions if required for LC endpoints.

54.1. Example 1

The hydrogen-molecule ion problem, see [77, Chapter 4, Section 4.1.3], gives the two differential equations:

$$-\big((1-\eta^2)Y'(\eta)\big)' + \big(n^2(1-\eta^2)^{-1} - \mu\big)\,Y(\eta) = \lambda\eta^2 Y(\eta) \text{ for all } \eta \in (-1,1)$$

and

$$-\big((1-\xi^2)X'(\xi)\big)' + \big(\kappa\xi + n^2(1-\xi^2) - \mu\big)\,X(\xi) = \lambda\xi^2 X(\xi) \text{ for all } \xi \in (1,\infty).$$

54.2. Example 2

The Teukolsky equations in astrophysics gives the equation, see [77, Chapter 4, Section 4.2.1]:

$$-((1-u^2)X'(u))' + \big(2 + (m-2u)^2(1-u^2)^{-1} - 4a\omega u - a^2\omega^2u^2\big)\,X(u) = \lambda X(u)$$

for all $u \in (-1,1)$.

54.3. Example 3

The theory of tunneling in double-well potentials, see [77, Chapter 4, Section 4.4], gives the differential equation

$$-y''(x) + V(x)y(x) = \lambda y(x) \text{ for all } x \in (-\infty,\infty)$$

with the potential V determined by

$$V(x) = -A(\operatorname{sech}^2(x+x_0) + \operatorname{sech}^2(x-x_0)) \text{ for all } x \in (-\infty,\infty);$$

here A is a number and x_0 is a parameter.

55. Fuel cell equation

This Sturm-Liouville differential equation

$$-(xy'(x))' - x^3y(x) = \lambda xy(x) \text{ for all } x \in (0, b]$$

plays an important role in a fuel cell problem as discussed in the paper [6].

Endpoint classification in the space $L^2((0,b);x)$:

Endpoint	Classification
0	LCNO
b	R

For the LCNO endpoint at 0 the u, v boundary condition functions can be taken as, see [6, Section 8]:

$$u(x) = 1 \text{ and } v(x) = \ln(x) \text{ for all } x \in (0, b].$$

Various boundary value problems are considered in [6, Section 8]; the technical requirements of the fuel cell problem require a study of the analytic properties of these boundary value problems, as the endpoint b tends to zero.

56. Shaw equation

This Sturm-Liouville differential equation is considered in the paper [76] and has the form

$$y''(x) - Q(x)y(x) = \lambda y(x) \text{ for all } x \in (0, \infty)$$

where

$$Q(x) = A - B\exp(-Cx) + Dx^{-2} \text{ for all } (0, \infty)$$

for positive real numbers A, B, C, D with $D \geq 3/4$.

Endpoint classification in $L^2(0, +\infty)$:

Endpoint	Classification
0	LP
$+\infty$	LP

In the paper [76] the following specific values for A, B, C, D are used in connection with the chemical photodissociation of methyl iodide:

$$A = 19362.8662 \quad B = 19362.8662 \times 46.4857$$

$$C = 1.3 \quad D = 2.0.$$

57. Plum equation

This Sturm-Liouville equation is one of the first to be considered for numerical computation using interval arithmetic: the equation is

$$-(y'(x))' + 100\cos^2(x)y(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty).$$

Endpoint classification in $L^2(-\infty, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

In [71] the first seven eigenvalues for periodic eigenvalues on the interval $[0, \pi]$, *i.e.*

$$y(0) = y(\pi) \qquad y'(0) = y'(\pi),$$

are computed using a numerical homotopy method together with interval arithmetic; rigorous bounds for these seven eigenvalues are obtained.

58. Sears-Titchmarsh equation

This differential equation is considered in detail in [78, Chapter IV, Section 4.14] and [75]; the equation is

$$-y''(x) - \exp(2x)y(x) = \lambda y(x) \text{ for all } x \in (-\infty, \infty)$$

and has solutions of the form, using the Bessel function J_ν and writing $\sqrt{\lambda} = s = \sigma + it$,

$$y(x, \lambda) = J_{is}(\exp(x)) \text{ for all } x \in (-\infty, \infty);$$

in the space $L^2(-\infty, \infty)$ this equation is LP at $-\infty$ and is LCO at $+\infty$. This differential equation is then another example of equations derived from the original Bessel differential equation.

This Sears-Titchmarsh differential equation is the Liouville form, see Section 7 above, of the Sturm-Liouville equation

$$-(xy'(x))' - xy(x) = \lambda x^{-1}y(x) \text{ for all } x \in (0, +\infty).$$

In the space $L^2((0, \infty); x^{-1})$ this differential equation is LP at 0 and is LCO at $+\infty$.

Endpoint classification in $L^2((0, \infty); x^{-1})$:

Endpoint	Classification
0	LP
$+\infty$	LCO

For the LCO endpoint $+\infty$ the boundary condition functions can be chosen as, for all $x \in (0, +\infty)$,

$$u(x) = x^{-1/2}\left(\cos(x) + \sin(x)\right) \quad v(x) = x^{-1/2}\left(\cos(x) - \sin(x)\right).$$

For details of boundary value problems for this Sturm-Liouville equation, on $[1,\infty)$ see [8, Example 4]. For problems on $[1,\infty)$ the spectrum is simple and discrete but unbounded both above and below, since the endpoint $+\infty$ is LCO.

Numerical results are given in [11, Data base file xamples.tex; example 6].

59. Zettl equation

This differential equation is closely linked to the classical Fourier equation 8;

$$-(x^{1/2}y'(x))' = \lambda x^{-1/2}y(x) \text{ for all } x \in (0,+\infty).$$

Endpoint classification in $L^2(0,+\infty)$

Endpoint	Classification
0	R
$+\infty$	LP

This is a devised example to illustrate the computational difficulties of regular problems which have mild (integrable) singularities, in this example at the endpoint 0 of $(0,\infty)$.

The differential equation gives $p(0) = 0$ and $w(0) = \infty$ but nevertheless 0 is a regular endpoint in the Lebesgue integral sense; however this endpoint 0 does give difficulties in the in the computational sense.

The Liouville normal form of this equation is the Fourier equation, see Section 8 above; thus numerical results for this problem can be checked against numerical results from (i) a R problem, (ii) the roots of trigonometrical equations, and (iii) as a LCNO problem (see below).

There are explicit solutions of this equation given by

$$\cos(2x^{1/2}\surd\lambda) \ \ ; \ \sin(2x^{1/2}\surd\lambda)/\surd\lambda.$$

If 0 is treated as a LCNO endpoint then $u,\ v$ boundary condition functions are

$$u(x) = 2x^{1/2} \qquad v(x) = 1.$$

The regular Dirichlet condition $y(0) = 0$ is equivalent to the singular condition $[y,u](0) = 0$. Similarly the regular Neumann condition $(py')(0) = 0$ is equivalent to the singular condition $[y,\ v](0) = 0$.

The following indicated boundary value problems have the given explicit formulae for the eigenvalues:

$$\begin{gathered} y(0) = 0 \text{ or } [y,\ u](0) = 0, \text{ and } y(1) = 0 \text{ gives} \\ \lambda_n = ((n+1)\pi)^2/4 \ (n = 0,1,...) \end{gathered}$$

$$\begin{gathered} (py')(0) = 0 \text{ or } [y,v](0) = 0, \text{ and } (py')(1) = 0 \text{ gives} \\ \lambda_n = \left((n+\tfrac{1}{2})\pi\right)^2/4 \ (n = 0,1,...). \end{gathered}$$

60. Remarks

1. The author has made use of an earlier collection of examples of Sturm-Liouville differential equations drawn up by Bailey, Everitt and Zettl, in connection with the development and testing of the computer program SLEIGN2; see [8] and [10].
2. The author has made use of major collections of Sturm-Liouville differential equations from Pryce [69] and [70], and from Fulton, Pruess and Xie [38] and [68].
3. This catalogue will continue to be developed; the author welcomes corrections to the present form, and information about additional examples to extend the scope, of the catalogue.

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62. The future

As mentioned above it is hoped to continue this catalogue as a database for Sturm-Liouville differential equations.

The main contributor to the assessment and extension of successive drafts of this catalogue is Fritz Gesztesy, who has agreed to join with the author in continuing to update the content of this database.

Together we ask that all proposals for enhancing and extending the catalogue be sent to both of us, if possible by e-mail and LaTeX file. The author's affiliation data is to be found at the end of this paper; the corresponding data for Fritz Gesztesy is:

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