On the existence of an eigenvalue below the essential spectrum

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1 Introduction

In this paper we present a new method for proving the existence of an eigenvalue below the essential spectrum for Sturm-Liouville operators with coefficients which are L^1 perturbations of periodic functions. This method combines operator theory and "standard" numerical analysis with interval arithmetic analysis. We illustrate the method by showing that the Sturm-Liouville problem (SLP) consisting of the equation

$$-y'' + \sin(x + \frac{1}{1+x^2})\, y = \lambda y, \quad x \in [0, \infty), \tag{1}$$

together with the boundary condition $y(0) = 0$, generates an operator in $L^2(0, \infty)$ that has an eigenvalue below its essential spectrum. We also establish a rigorous upper bound for this eigenvalue and a rigorous lower bound for the starting point of the essential spectrum. It is well known [1] that the operator realisation in $L^2[0, \infty)$ of the equation

$$-y'' + \sin(x)\, y = \lambda y, \quad x \in [0, \infty),$$

with the same condition $y(0) = 0$, has no eigenvalues.

2 Mathematical formulation of the method

Consider the equation

$$-y'' + qy = \lambda y \quad on \quad [0, \infty), \quad q \in L^\infty((0, \infty), \mathbb{R}), \quad \lambda \in \mathbb{R}. \tag{2}$$

It is well known that this equation is limit-point at infinity and thus together with the boundary condition $y(0) = 0$, determines a unique self-adjoint operator S in $L^2[0,\infty)$ with domain

$$\{y \in L^2(0,\infty) : y, y^{'} \in \mathrm{AC}_{\mathrm{loc}},\ -y^{''} + qy \in L^2(0,\infty),\ y(0) = 0\}.$$

Let $\sigma(q)$ and $\sigma_e(q)$ denote the spectrum and essential spectrum of S, respectively, and let

$$\sigma_0(q) = \inf \sigma_e(q).$$

Let

$$p(x) = \sin(x + \frac{1}{1+x^2}), \quad x \in [0,\infty).$$

The essential spectrum is invariant under relatively compact perturbations [2, Theorem 9.9, page 277], and p is an $L^1(0,\infty)$ perturbation of sin and consequently is a relatively compact perturbation of sin [3]. Since $\sigma_e(\sin) = \sigma_e(\cos)$ we may conclude that

$$\sigma_0(p) = \sigma_0(\sin) = \sigma_0(\cos).$$

To describe our approach to proving the existence of an eigenvalue λ_0 below the essential spectrum we introduce the notation $\lambda_0^N([a,b],q), \lambda_0^D([a,b],q)$ and $\lambda_0^P([a,b],q)$ for the lowest Neumann, Dirichlet and periodic eigenvalues, respectively, of the regular Sturm-Liouville problems (SLP) consisting of the equation

$$-y'' + qy = \lambda y \quad on \quad [a,b], \quad -\infty < a < b < \infty$$

together with either Neumann, $y^{'}(a) = 0 = y^{'}(b)$, Dirichlet, $y(a) = 0 = y(b)$ or periodic $y(a) = y(b)$, $y'(a) = y'(b)$ boundary conditions. The well-known inequality

$$\lambda_0^N([a,b],q) \le \lambda_0^P([a,b],q) < \lambda_0^D([a,b],q) \tag{3}$$

may be found in [4, Theorem 13.10, pages 209-212].

In Floquet theory, see [1], it is shown that

$$\sigma_0(\cos) = \lambda_0^P([0,2\pi],\cos). \tag{4}$$

It is well known that, for fixed q, $\lambda_0^D([0,b],q)$ is a decreasing function of b. Let

$$\lambda_0 = \lim_{b\to\infty} \lambda_0^D([0,b],\cos).$$

Our method consists of finding a provably correct lower bound for $\lambda_0^N([0,2\pi],\cos)$, λ_l say, and a provably correct upper bound for $\lambda_0^D([0,b],p)$,

λ_u say, using interval arithmetic techniques; in particular, the interval initial value problem solver AWA. (See section 3 for further details of both interval arithmetic and AWA.) Then if

$$\lambda_u < \lambda_l$$

in view of the above results (3) and (4) the existence of an eigenvalue for p below the essential spectrum will follow from Theorem 6.4 of Bailey, Everitt Weidmann and Zettl [5], the "exact spectral convergence below the essential spectrum" result.

Since interval arithmetic computations are much more "expensive" than standard numerical methods we use SLEIGN2 - a "standard" numerical analysis Fortran code - to estimate a suitable value of b and then use interval analytic methods to confirm it.

Specifically we show that there is an eigenvalue $\lambda_0(p) \in \sigma(p)$ and it satisfies the inequalities:

$$\lambda_0(p) < -0.3792 < -0.3785 \leq \sigma_0(p). \tag{5}$$

More details are provided in Sections 3 and 4 below.

3 Short introduction to interval arithmetic and the AWA algorithm

3.1 Interval initial value problem solver

In this sub-section we introduce briefly the concepts of interval arithmetic that we need to give a short account of Lohner's AWA algorithm. For an in-depth discussion of interval arithmetic, see [6], while Lohner's AWA algorithm is discussed in [7] and [8].

Denoting any of the four basic arithmetic operations by $\star$, we define, for real intervals $[a], [b]$,

$$[a] \star [b] = \{a \star b \mid a \in [a],\ b \in [b]\}$$

under the assumption $0 \notin [b]$ if $\star$ is division. Thus we can compute an enclosure for $[a] \star [b]$ by obtaining a computable upper, and lower bound, for $[a] \star [b]$ which is derived from the lower and upper bounds of $[a], [b]$ respectively, by some directed rounding facilities. Neglecting all but rounding errors any algorithm that is realised on a computer consists of finitely many operations $\star$ and thus an enclosure for the results of arithmetic operations which constitute the algorithm may be computed. In practice this simple approach would soon lead to an explosion of the interval width but many sophisticated techniques are available to control this phenomenon [6]. However most algorithms involve other approximation errors which must also contribute to the final enclosure.

Lohner's approach to computing an enclosure of the solution of initial value problems is based on the well known Taylor method for solving initial value problems. Suppose that a solution of the IVP

$$u' = f(x,u), \quad u(0) = u_0, \tag{6}$$

where $f : [0,\infty) \times R^n \to R^n$ is sufficiently smooth, is known at some point x_0. Then the solution at $x_0 + h$ is

$$u(x_0 + h) = u(x_0) + h\phi(x_0, h) + z_{x_0+h} \tag{7}$$

where $u(x_0) + h\phi(x_0, h)$ is the $(r-1)$–th degree Taylor polynomial of u expanded about x_0 and z_{x_0+h} is the associated local error. This method lends itself well to computation since the coefficients of the polynomial may be computed via an automatic differentiation package by differentiating the differential equation (6). However the error term is not known exactly since the standard formulae give, for some unknown τ,

$$z_{x_0+h} = u^{(r)}(\tau)h^r/r!, \quad \tau \in [x_0, x_0 + h]. \tag{8}$$

In Lohner's algorithm, (7) is used to advance an enclosure $[u(x_0)]$ for the solution u at x_0, to one for the solution u at $x_0 + h$ which we denote by $[u(x_0 + h)]$. Defining $f^{(r)}$ recursively by

$$f^{(0)} = f, \qquad f^{(k+1)} = \frac{\partial f^{(k)}}{\partial x} + \frac{\partial f^{(k)}}{\partial y}.f, \quad k \geq 0$$

we find a suitable enclosure for the error (8) is

$$[z_{x_0+h}] = f^{(r)}([x_0, x_0 + h], [u])h^r/r!$$

provided that some rough a priori enclosure $[u]$ for $\{u(x) : x_0 \leq x \leq x_0 + h\}$ can be computed. This is achieved by the following means. Choose some interval $[u^0] \supset [u(x_0)]$ and try to prove that

$$[u] = [u(x_0)] + [0,h] \cdot f([x_0, x_0 + h], [u^0]) \subset [u^0].$$

If this is true, and it will be for sufficiently small h, then Banach's fixed-point theorem implies that $[u]$ is an enclosure for $u(x)$ for all $x \in (x_0, x_0 + h)$. In order to achieve efficient performance and tight bounds, the details of the algorithm are more complex than this short overview can show. We refer the reader to [7] and [8] for a complete discussion of the method.

4 Summary of method for obtaining verified eigenvalue bounds for regular problems

The approach that we take to verifying enclosures for eigenvalues of Sturm-Liouville problems has been developed in [9] and covers both the regular and singular problem. In this article we shall only outline the main features of the method as applicable to our present situation, leaving the reader to consult [9] for an in depth discussion of the method. We commence with the fairly obvious observation that the eigenvalues of the boundary value problem

$$-y^{''} + qy = \lambda y, \;\; \text{on } [0, b] \tag{9}$$

$$a_1 y(0) = a_2 y^{'}(0), \;\; b_1 y(b) = b_2 y^{'}(b) \quad a_1, a_2, b_1, b_2 \in \mathbf{R},$$

where not both of a_1, a_2 and b_1, b_2 respectively are zero, are precisely those real values λ such that a non-trivial solution of (9) satisfies the prescribed boundary conditions at both 0, and b.

As is usual in the study of the eigenvalues of the Sturm Liouville problem we now introduce the Prüfer variables ρ and θ. These are defined by the transformation

$$y = \rho \sin\theta, \quad y^{'} = \rho\cos\theta. \tag{10}$$

This transformation together with the equation (9) yields the differential equation for θ

$$\frac{d\theta}{dx} = (\lambda - q(x))\sin^2\theta + \cos^2\theta \tag{11}$$

with initial condition $\theta(0;\lambda) = \theta(0) = \alpha \in [0, \pi)$ where $\tan\alpha = a_2/a_1$, $(a_1 \neq 0)$ and $\alpha = \pi/2$ otherwise. Standard results in the spectral theory of SLP allow us to classify the $n-th$ eigenvalue of (9), starting counting at $n = 0$, with the stated separated boundary conditions as that unique λ which is such that the equation (11) has a solution θ with

$$\theta(0, \lambda) = \alpha, \quad \theta(b, \lambda) = \beta + n\pi, \quad \beta \in (0, \pi],$$

where $\tan\beta = b_2/b_1$, $(b_1 \neq 0)$ and $\beta = \pi/2$ otherwise. This is the principal tool that we use to compute bounds for $\lambda_0^D([0, b], p)$ and $\lambda_0^N([0, 2\pi], \cos)$. However it is numerically more convenient to work with a pair of initial value problems for θ_L and θ_R defined by (11). The solutions $\theta_L(x, \lambda)$ and $\theta_R(x, \lambda)$ satisfy the initial conditions (12) and (13) respectively:

$$\theta_L(0) = \alpha, \tag{12}$$

$$\theta_R(b) = \beta + n\pi. \tag{13}$$

We next choose a point c, with $0 < c < b$ and define the miss-match distance

$$D(\lambda) \equiv \theta_L(c, \lambda) - \theta_R(c, \lambda). \tag{14}$$

The $n-th$ eigenvalue is the unique value λ_n with $D(\lambda_n) = 0$. By continuity we have for $\mu_1 < \mu_2$ that if $\text{sign}D(\mu_1) = -\text{sign}D(\mu_2)$ then $\lambda_n \in [\mu_1, \mu_2]$. By sign $D(\mu)$ we mean the sign of any member of the interval $D(\mu)$. This is defined only when all members of $D(\mu)$ have the same sign.

Our algorithm for enclosing the n^{th} eigenvalue λ_n proceeds as follows:

Algorithm 4.1 *1. Obtain an estimate, $\hat{\lambda}_n$, for λ_n with a numerical Sturm-Liouville solver together with an error estimate ϵ, (we have used SLEIGN2 for this);*

2. Form the quantities

$$\mu_1 = \hat{\lambda}_n - \epsilon, \quad \mu_2 = \hat{\lambda}_n + \epsilon.$$

3. Use the AWA algorithm to compute

$$D(\mu_1), \quad D(\mu_2).$$

4. If (the interval) $\text{sign}D(\mu_1) = -\text{sign}D(\mu_2)$ then $\lambda_n \in [\mu_1, \mu_2]$ otherwise increase ϵ and recompute $D(\mu_j)$, $j = 1, 2$.

5 Results

In this section we draw together the theory described in the previous sections and verify (5).

We first obtain a lower bound for $\lambda_0^N([0, 2\pi], \cos)$ and hence a lower bound for $\sigma_0(\cos)$. To do this we use SLEIGN2 to compute an estimate for

$$\lambda_0^N([0, 2\pi], \cos) \approx -0.378488$$

together with an error estimate 10^{-4}. The method of section 4 is then used with the AWA code to show

$$\lambda_0^N([0, 2\pi], \cos) \in [-0.3785, -0.378489].$$

We note that a similar procedure yields

$$\lambda_0^D([0, 2\pi], \cos) \in [-0.34767, -0.34766]$$

giving an indication of the quality of the lower bound $\lambda_0^N([0, 2\pi], \cos)$ for $\lambda_0^P([0, 2\pi], \cos)$.

Remark 5.1 *As we mentioned in Section 1, the lower bound for* $\sigma_0(\cos)$ *is obtained by computing a bound for* $\lambda_0^N([0, 2\pi], \cos)$*. This calculation involves solving the EVP, with potential* cos*, over the interval* $[0, 2\pi]$*. This would not be possible directly using the AWA code as* 2π *is not a machine representable number, and so we first transform the independent variable to the interval* $[0, 2]$*. The calculation is then performed with step size* 0.03125 *and a Taylor polynomial of degree* 15*.*

We next find upper and lower bounds for $\lambda_0(p)$. This is done by first using SLEIGN2 to provide an estimate of $\lambda_0^D([0, 20], p) \approx -0.3792153$ and error estimate 3.7×10^{-9}. Then the method of Section 4 is used to obtain an enclosure $[-0.3792154028, -0.3792154004]$ for $\lambda_0^D([0, 2], p)$. Thus (5) follows.

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