Inequalities among eigenvalues of Sturm-Liouville problems

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Abstract

There are well-known inequalities among the eigenvalues of Sturm-Liouville problems with periodic, semi-periodic, Dirichlet and Neumann boundary conditions. In this paper, for an arbitrary coupled self-adjoint boundary condition, we identify two separated boundary conditions corresponding to the Dirichlet and Neumann conditions in the classical case, and establish analogous inequalities. It is also well-known that the lowest periodic eigenvalue is simple; here we prove a similar result for the general case. Moreover, we show that the algebraic and geometric multiplicities of the eigenvalues of self-adjoint regular Sturm-Liouville problems with coupled boundary conditions are the same. An important step in our approach is to obtain a representation of the fundamental solutions for sufficiently negative values of the spectral parameter. Our approach yields the existence and boundedness from below of the eigenvalues of arbitrary self-adjoint regular Sturm-Liouville problems without using operator theory.

Key words: Sturm-Liouville problems, eigenvalue inequalities, eigenvalue multiplicities, asymptotic form of solutions.
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1 Introduction

Let

$$\{\lambda_n(e^{i\theta}K); \quad n \in \mathbb{N}_0 = \{0, 1, 2, ...\}\}$$

denote the eigenvalues, listed in non-decreasing order, of the Sturm-Liouville problem (SLP) consisting of the equation

$$-(py')' + qy = \lambda wy \text{ on } J = (a, b), \tag{1.1}$$

and the coupled self-adjoint boundary condition (BC)

$$Y(b) = e^{i\theta} K Y(a), \tag{1.2}$$

where

$$i = \sqrt{-1}, \quad -\pi < \theta \leq \pi, \quad Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \quad -\infty < a < b < \infty,$$

$$K \in \mathrm{SL}(2,\mathbb{R}) =: \left\{ K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}; \, k_{ij} \in \mathbb{R}, \det K = 1 \right\}, \tag{1.3}$$

and

$$1/p, \, q, \, w \in L^1(J,\mathbb{R}), \quad p > 0, \, w > 0 \text{ a.e. on } J. \tag{1.4}$$

Here $\mathbb{R}$ denotes the set of real numbers, $L^1(J,\mathbb{R})$ the space of real valued Lebesgue integrable functions on J.

According to a well-known classical result (see [3] and [2] for the case of smooth coefficients and [7] for the general case), we have the following inequalities for $K = I$, the identity matrix:

$$\begin{aligned} \lambda_0^N & \leq \lambda_0(I) < \lambda_0(e^{i\theta} I) < \lambda_0(-I) \leq \{\lambda_0^D, \lambda_1^N\} \\ & \leq \lambda_1(-I) < \lambda_1(e^{i\theta} I) < \lambda_1(I) \leq \{\lambda_1^D, \lambda_2^N\} \\ & \leq \lambda_2(I) < \lambda_2(e^{i\theta} I) < \lambda_2(-I) \leq \{\lambda_2^D, \lambda_3^N\} \\ & \leq \lambda_3(-I) < \lambda_3(e^{i\theta} I) < \lambda_3(I) \leq \{\lambda_3^D, \lambda_4^N\} \leq \cdots, \end{aligned} \tag{1.5}$$

where $\theta \in (-\pi, \pi)$ and $\theta \neq 0$. In (1.5), λ_n^D and λ_n^N denote the n-th Dirichlet and Neumann eigenvalues, respectively, and the notation $\{\lambda_0^D, \lambda_1^N\}$ means either of λ_0^D and λ_1^N.

Inequality (1.5) has been extended in [7] from $K = I$ to

$$K = \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix}, \quad c > 0$$

and in [1] to a larger class of K, including

$$K = \begin{pmatrix} c & h \\ 0 & 1/c \end{pmatrix}, \quad c > 0, \, h \leq 0.$$

Here we find an analogue of (1.5) for an arbitrary $K \in \mathrm{SL}(2,\mathbb{R})$. A key feature of this result is the identification of two separated BC's whose eigenvalues play the role of λ_n^D and λ_n^N in (1.5). This feature plays an important role in the forthcoming update of the FORTRAN code SLEIGN2 where it is used not only to bracket the coupled eigenvalues but also to determine their indices.

As a consequence of these general inequalities we show that for *any* $K \in \mathrm{SL}(2,\mathbb{R})$ either $\lambda_0(K)$ or $\lambda_0(-K)$ is simple; thus extending the classical result that the lowest periodic eigenvalue is simple, to the general case of arbitrary coupled self-adjoint BC's. Here simple refers to both the algebraic and geometric multiplicities, since we will show below that the two multiplicities are equal for coupled boundary conditions.

Our proof of the general inequalities analogous to (1.5) actually yields a new proof of the existence and boundedness from below of the eigenvalues of arbitrary coupled self-adjoint regular SLP's without using the theory of self-adjoint operators in Hilbert space. For separated BC's such a proof is provided by the Prüfer transformation.

Our method of proof relies heavily on the approach in the book of Eastham [3] and on an asymptotic form of the fundamental solutions of (1.1) for *sufficiently negative* λ.

In Section 2 we introduce our notation and discuss some basic results. Our main results, the general inequalities and a representation of the fundamental solutions from which the asymptotic form is deduced, are stated in Section 3; the proofs are given in Section 5 after discussions of the characteristic function and multiplicities in Section 4.

2 Notation and prerequisite results

Throughout this paper, for the SLP consisting of (1.1) and (1.2) we assume that (1.3) and (1.4) hold. It follows from the well-known theory of regular self-adjoint SLP's that the SLP consisting of (1.1) and (1.2) has an infinite, but countable, number of only real eigenvalues which can be ordered to form a non-decreasing sequence (with only the double eigenvalues appearing twice). This sequence is bounded from below, but not from above.

In general, the n-th eigenvalue does not depend on the BC continuously, see [4]. Nevertheless, each simple eigenvalue is on a locally unique continuous branch of simple eigenvalues; while each double eigenvalue is on two locally unique continuous branches of eigenvalues. See [5] for details.

Let u and v be the fundamental solutions of (1.1) determined by the initial conditions

$$u(a,\lambda) = 0 = v^{[1]}(a,\lambda), \quad v(a,\lambda) = 1 = u^{[1]}(a,\lambda), \qquad \lambda \in \mathbb{C}. \tag{2.1}$$

Here and below $y^{[1]} = py'$ for any solution y of (1.1). Note that the condition (1.4) guarantees that $y, y^{[1]}$ exist and are absolutely continuous on the compact interval $[a,b]$ for any solution y of (1.1). Thus the BC (1.2) is well defined.

For any fixed $K \in \mathrm{SL}(2,\mathbb{R})$ and all $\lambda \in \mathbb{C}$ we define

$$\begin{aligned}
D(\lambda) &= k_{11}u^{[1]}(b,\lambda) - k_{21}u(b,\lambda) + k_{22}v(b,\lambda) - k_{12}v^{[1]}(b,\lambda), && (2.2)\\
A(\lambda) &= k_{11}v^{[1]}(b,\lambda) - k_{21}v(b,\lambda), && (2.3)\\
B(\lambda) &= k_{11}u^{[1]}(b,\lambda) + k_{12}v^{[1]}(b,\lambda) - k_{22}v(b,\lambda) - k_{21}u(b,\lambda), && (2.4)\\
B_1(\lambda) &= k_{11}u^{[1]}(b,\lambda) - k_{21}u(b,\lambda), && (2.5)\\
B_2(\lambda) &= k_{22}v(b,\lambda) - k_{12}v^{[1]}(b,\lambda), && (2.6)\\
C(\lambda) &= k_{22}u(b,\lambda) - k_{12}u^{[1]}(b,\lambda). && (2.7)
\end{aligned}$$

Note that

$$D(\lambda) = B_1(\lambda) + B_2(\lambda), \quad B(\lambda) = B_1(\lambda) - B_2(\lambda). \tag{2.8}$$

Let

$$\Phi(t,\lambda) = \begin{pmatrix} v(t,\lambda) & u(t,\lambda) \\ v^{[1]}(t,\lambda) & u^{[1]}(t,\lambda) \end{pmatrix}. \tag{2.9}$$

Then $\Phi(t,\lambda)$ is the fundamental matrix solution of

$$Y'(t) = [P(t) - \lambda W(t)]Y(t), \quad Y(a) = I, \tag{2.10}$$

where

$$P = \begin{pmatrix} 0 & 1/p \\ q & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}.$$

Also note that

$$K^{-1}\Phi(b,\lambda) = \begin{pmatrix} B_2(\lambda) & C(\lambda) \\ A(\lambda) & B_1(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}. \tag{2.11}$$

The function usually used to characterize the eigenvalues is

$$\det(e^{i\theta}K - \Phi(b,\lambda)) = 2\cos\theta - D(\lambda),$$

see [8], Lemma 4.5, p. 48. This identity implies the following result.

Lemma 2.1 *A number λ is an eigenvalue of the Sturm-Liouville problem consisting of (1.1) and (1.2) if and only if*

$$D(\lambda) = 2\cos\theta. \tag{2.12}$$

Moreover, we have the following lemmas.

Lemma 2.2 *A number λ is an eigenvalue of the Sturm-Liouville problem consisting of (1.1) and (1.2) of geometric multiplicity two if and only if*

$$e^{i\theta}K = \Phi(b,\lambda). \tag{2.13}$$

In this case, $\theta = 0$ or $\theta = \pi$.

Proof: See [6], Theorem 3.1. ∎

Lemma 2.3 *For the Sturm-Liouville problem consisting of (1.1) and (1.2) we have*

$$\lambda_n(e^{i\theta}K) = \lambda_n(e^{-i\theta}K) \tag{2.14}$$

for any $\theta \in (-\pi,\pi]$ and $n \in \mathbb{N}_0$. Furthermore, if z is an eigenfunction of $\lambda_n(e^{i\theta}K)$, then its complex conjugate $\bar{z}$ is an eigenfunction of $\lambda_n(e^{-i\theta}K)$.

Proof: Note that (2.14) follows from Lemma 2.1. The furthermore statement can be verified by a direct substitution. ∎

3 Main results

In this section we state our main results: general inequalities among eigenvalues of SLP's and a representation of the fundamental solutions for sufficiently negative λ. We believe that the latter result, which will be stated first and is needed for the proof of the former, is of independent interest. In this section we make no distinction between algebraic and geometric multiplicities because these are shown to be equal in Theorem 4.2.

Theorem 3.1 *There exists $\lambda_0 \in \mathbb{R}$, $k > 0$ and a continuous function*

$$\alpha : [a,b] \times (-\infty, \lambda_0] \to [0, \infty)$$

such that $\alpha(t,\lambda)$ is decreasing in λ for each $t \in (a,b]$, $\alpha_t(t,\lambda)$ exists a.e. on $[a,b]$ for each $\lambda \in (-\infty, \lambda_0]$, $(p\alpha')(t,\lambda) = p(t)\alpha_t(t,\lambda)$ is continuous on $[a,b]$ for each $\lambda \in (-\infty,\lambda_0]$, and

$$\lim_{\lambda\to-\infty} \alpha(t,\lambda) = \infty, \qquad \lim_{\lambda\to-\infty} p(t)\alpha_t(t,\lambda) = \infty, \tag{3.1}$$

for each $t \in (a,b]$. Moreover, for the fundamental solutions u and v of (1.1) we have

$$\begin{aligned}
v(t,\lambda) &= k\cosh(\alpha(t,\lambda)), & (3.2)\\
v^{[1]}(t,\lambda) &= k\,p(t)\,\alpha_t(t,\lambda)\,\sinh(\alpha(t,\lambda)), & (3.3)\\
u(t,\lambda) &= \frac{1}{k^2}\,v(t,\lambda)\int_a^t \frac{\mathrm{sech}^2(\alpha(\mathrm{s},\lambda))}{\mathrm{p(s)}}ds, & (3.4)\\
u^{[1]}(t,\lambda) &= \frac{1}{k^2}\,v^{[1]}(t,\lambda)\int_a^t \frac{\mathrm{sech}^2(\alpha(\mathrm{s},\lambda))}{\mathrm{p(s)}}ds + \frac{1}{k}\,\mathrm{sech}(\alpha(\mathrm{t},\lambda)) & (3.5)
\end{aligned}$$

on $[a,b] \times (-\infty,\lambda_0]$, where the values of the right hand sides at a are defined by continuity.

For $K \in \mathrm{SL}(2,\mathbb{R})$ we consider the separated BC's

$$y(a) = 0, \qquad k_{22}y(b) - k_{12}y^{[1]}(b) = 0, \tag{3.6}$$

and

$$y^{[1]}(a) = 0, \qquad k_{21}y(b) - k_{11}y^{[1]}(b) = 0. \tag{3.7}$$

Note that $(k_{22}, k_{12}) \neq (0,0) \neq (k_{21}, k_{11})$ since $\det K = 1$. Therefore, there exist an infinite number of eigenvalues for each of the BC's (3.6) and (3.7). Let $\{\mu_n,\ n \in \mathbb{N}_0\}$ denote the eigenvalues of (3.6) and denote by $\{\nu_n,\ n \in \mathbb{N}_0\}$ the eigenvalues of (3.7). How are μ_n and ν_n related to $\lambda_n(K)$, $\lambda_n(-K)$ and $\lambda_n(e^{i\theta}K)$? This question is answered by the following result.

Theorem 3.2 *Let $K \in \mathrm{SL}(2,\mathbb{R})$.*

(a) If $k_{11} > 0$ and $k_{12} \le 0$, then $\lambda_0(K)$ is simple, and for any $\theta \in (-\pi,\pi)$, $\theta \neq 0$, we have

$$\begin{aligned}
\nu_0 &\le \lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) \le \{\mu_0,\nu_1\}\\
&\le \lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) \le \{\mu_1,\nu_2\}\\
&\le \lambda_2(K) < \lambda_2(e^{i\theta}K) < \lambda_2(-K) \le \{\mu_2,\nu_3\}\\
&\le \lambda_3(-K) < \lambda_3(e^{i\theta}K) < \lambda_3(K) \le \{\mu_3,\nu_4\} \le \cdots.
\end{aligned} \tag{3.8}$$

(b) If $k_{11} \leq 0$ and $k_{12} < 0$, then $\lambda_0(K)$ is simple, and for any $\theta \in (-\pi, \pi)$, $\theta \neq 0$, we have

$$\begin{array}{l} \lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) \leq \{\mu_0, \nu_0\} \leq \\ \lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) \leq \{\mu_1, \nu_1\} \leq \\ \lambda_2(K) < \lambda_2(e^{i\theta}K) < \lambda_2(-K) \leq \{\mu_2, \nu_2\} \leq \\ \lambda_3(-K) < \lambda_3(e^{i\theta}K) < \lambda_3(K) \leq \{\mu_3, \nu_3\} \leq \cdots . \end{array} \tag{3.9}$$

(c) If neither case (a) nor case (b) applies to K, then either case (a) or case (b) applies to $-K$.

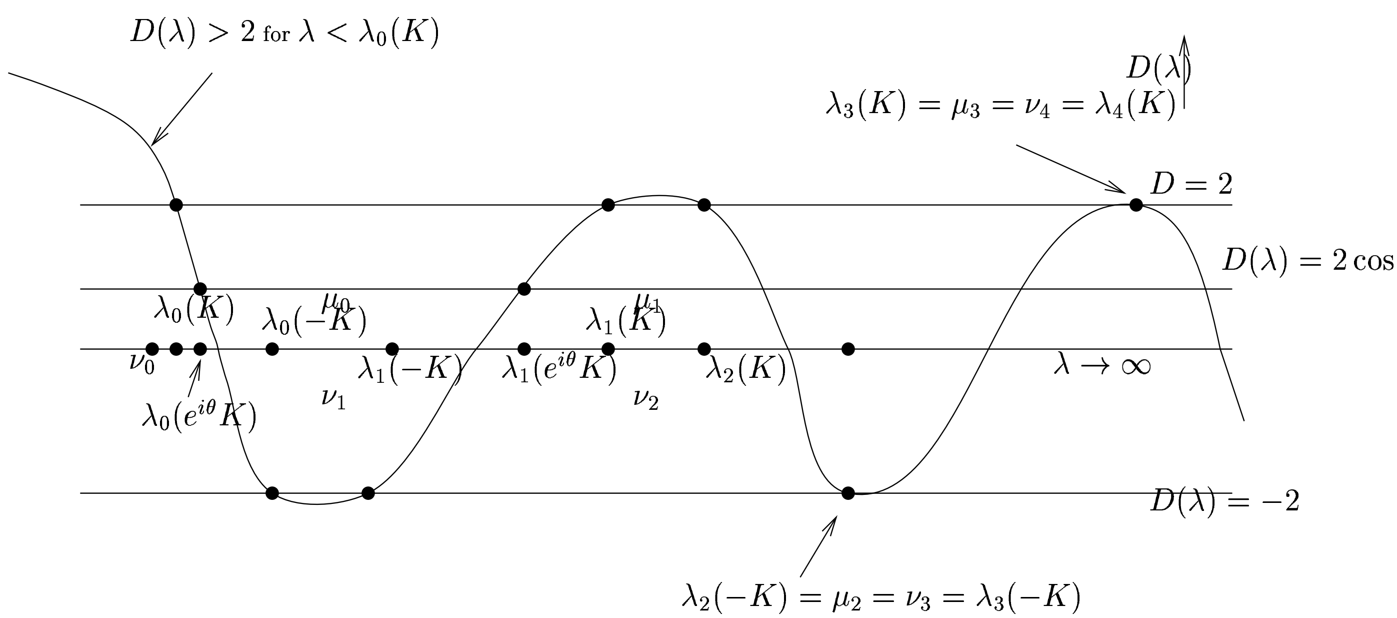
Corollary 3.1 *Let λ_n denote the n-th eigenvalue of an arbitrary self-adjoint, separated or coupled, real or non-real, boundary condition, and let λ_n^D denote the n-th Dirichlet eigenvalue. Then*

$$\lambda_n \leq \lambda_n^D, \quad n \in \mathbb{N}_0. \tag{3.10}$$

Proof: This is well known in the case of separated boundary conditions and the coupled case follows from Theorem 3.2. ▮

Corollary 3.2 *Let $K \in \mathrm{SL}(2, \mathbb{R})$ with either $k_{11} > 0$ and $k_{12} \leq 0$ or $k_{11} \leq 0$ and $k_{12} < 0$. If $\lambda_{2n+1}(K)$ is simple, where $n \in \mathbb{N}_0$, then so is $\lambda_{2n+2}(K)$. In particular, if K has a double eigenvalue, then the first double eigenvalue of K is preceded by an odd number of simple eigenvalues.*

By Theorem 3.2, for any $K \in \mathrm{SL}(2, \mathbb{R})$, either $\lambda_0(K)$ or $\lambda_0(-K)$ is simple. The inequalities in (3.8) are illustrated with the following graph:



The graph illustrating the inequalities in (3.9) is the same except that ν_n moves to the position of ν_{n+1} for all $n \in \mathbb{N}_0$.

4 The characteristic function $D(\lambda)$

In this section, for the function $D(\lambda)$ defined by (2.2), we compute $D'(\lambda)$, establish some identities involving $D(\lambda)$ and $D'(\lambda)$ and show that the geometric and algebraic multiplicities of the eigenvalues of regular self-adjoint SLP's are the same. Some properties of μ_n and ν_n are obtained thereafter.

Lemma 4.1 *For any $\lambda \in \mathbb{C}$ we have*

$$
\begin{aligned}
D'(\lambda) &= \int_a^b [A(\lambda)u^2(s,\lambda) - B(\lambda)u(s,\lambda)v(s,\lambda) - C(\lambda)v^2(s,\lambda)]w(s)ds, \qquad (4.1)\\
4C(\lambda)D'(\lambda) &= -\int_a^b [2C(\lambda)v(s,\lambda) + B(\lambda)u(s,\lambda)]^2 w(s)ds\\
&\quad -[4 - D^2(\lambda)]\int_a^b u^2(s,\lambda)w(s)ds, \qquad (4.2)\\
4A(\lambda)D'(\lambda) &= \int_a^b [2A(\lambda)u(s,\lambda) - B(\lambda)v(s,\lambda)]^2 w(s)ds\\
&\quad +[4 - D^2(\lambda)]\int_a^b v^2(s,\lambda)w(s)ds. \qquad (4.3)
\end{aligned}
$$

Proof: Let $\Phi_\lambda(t,\lambda) = \dfrac{d}{d\lambda}\Phi(t,\lambda)$. From (2.9) it follows that

$$\Phi'_\lambda = (P - \lambda W)\Phi_\lambda - W\Phi, \quad \Phi_\lambda(a,\lambda) = 0.$$

This and the variation of parameters formula imply that

$$\Phi_\lambda(t,\lambda) = -\int_a^t \Phi(t,\lambda)\Phi^{-1}(s,\lambda)W(s)\Phi(s,\lambda)ds.$$

Below we abbreviate the notation by omitting λ and, in some cases, b, as well as the integration argument s. Since

$$D(\lambda) = B_1(\lambda) + B_2(\lambda) = \text{trace}\, K^{-1}\Phi(b,\lambda)$$

by (2.11), we have that

$$
\begin{aligned}
D'(\lambda) &= \text{trace}\, K^{-1}\Phi_\lambda(b,\lambda)\\
&= -\text{trace}\, \textstyle\int_a^b K^{-1}\Phi(b,\lambda)\Phi^{-1}(s,\lambda)W(s)\Phi(s,\lambda)ds\\
&= -\text{trace}\int_a^b \begin{pmatrix} B_2 & C \\ A & B_1 \end{pmatrix}\begin{pmatrix} u^{[1]} & -u \\ -v^{[1]} & v \end{pmatrix}\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}\begin{pmatrix} v & u \\ v^{[1]} & u^{[1]} \end{pmatrix}(s)ds\\
&= -\text{trace}\int_a^b \begin{pmatrix} -B_2uv + Cv^2 & * \\ * & -Au^2 + B_1uv \end{pmatrix} w(s)ds\\
&= \int_a^b [Au^2 + (B_2 - B_1)uv - Cv^2]w(s)ds,
\end{aligned}
$$

which together with (2.8) confirm (4.1).

To establish (4.2) and (4.3), from (2.11) and (2.8) we obtain

$$B_1B_2 - AC = \det(K^{-1}\Phi_\lambda) = 1,$$

and

$$\begin{aligned} 4 - D^2 &= 4 - (B_1 + B_2)^2 = 4 - (B_1 - B_2)^2 - 4B_1B_2 \\ &= 4(1 - B_1B_2) - B^2 = -(4AC + B^2). \end{aligned}$$

Thus,

$$\begin{aligned} 4CD' &= \int_a^b [4ACu^2 - 4BCuv - 4C^2v^2]wds \\ &= \int_a^b [-(2Cv + Bu)^2 + (4AC + B^2)u^2]wds \\ &= -\int_a^b (2Cv + Bu)^2 wds - (4 - D^2)\int_a^b u^2 wds, \end{aligned}$$

i.e., (4.2) holds. The identity (4.3) can be verified similarly. ∎

Corollary 4.1 *If $\lambda \in \mathbb{R}$ satisfies $|D(\lambda)| < 2$, then $A(\lambda) \neq 0$, $C(\lambda) \neq 0$ and $D'(\lambda) \neq 0$.*

Proof: These are direct consequences of (4.2) and (4.3). ∎

Lemma 4.2 *Let μ_n and ν_n be defined as in front of Theorem 3.2. Then*
(i) $C(\lambda) = 0$ if and only if $\lambda = \mu_n$ for some $n \in \mathbb{N}_0$. For $n \in \mathbb{N}_0$, $u(\cdot, \mu_n)$ is an eigenfunction of μ_n, and this eigenfunction is unique up to constant multiples.
(ii) $A(\lambda) = 0$ if and only if $\lambda = \nu_n$ for some $n \in \mathbb{N}_0$. For $n \in \mathbb{N}_0$, $v(\cdot, \nu_n)$ is an eigenfunction of ν_n, and this eigenfunction is unique up to constant multiples.
(iii) For $\lambda = \mu_n$ and for $\lambda = \nu_n$, $n \in \mathbb{N}_0$, we have

(a) $D^2(\lambda) \geq 4$,

(b) $B_i(\lambda) \neq 0,\ i = 1, 2,$ and $B_1(\lambda) = 1/B_2(\lambda)$. Moreover,

$$u(b, \mu_n)B_2(\mu_n) = k_{12},\ n \in \mathbb{N}_0; \tag{4.4}$$

$$v(b, \nu_n)B_1(\nu_n) = k_{11},\ n \in \mathbb{N}_0. \tag{4.5}$$

Proof: Parts (i) and (ii) follow from the definitions of $C(\lambda)$ and $A(\lambda)$, respectively; Part (iii a) is a consequence of (4.2) and (4.3).

To prove (iii b), note that $A(\lambda) = 0$ for $\lambda = \nu_n$ and $C(\lambda) = 0$ for $\lambda = \mu_n$, $n \in \mathbb{N}_0$. Thus we get from (2.11) for such λ that

$$1 = \det(K^{-1}\Phi(b, \lambda)) = B_1(\lambda)B_2(\lambda).$$

To complete the proof we note that

$$v^{[1]}(b,\mu_n)u(b,\mu_n) - v(b,\mu_n)u^{[1]}(b,\mu_n) = -1.$$

This together with

$$k_{22}u(b,\mu_n) - k_{12}u^{[1]}(b,\mu_n) = C(\mu_n) = 0$$

imply that

$$u(b,\mu_n) = \frac{-1}{B_2(\mu_n)} \begin{vmatrix} 0 & -k_{12} \\ -1 & -v(b,\mu_n) \end{vmatrix} = \frac{k_{12}}{B_2(\mu_n)},$$

i.e., (4.4) holds. In the same way we can establish (4.5). ∎

Remark The existence and basic properties of the eigenvalues for separated self-adjoint BC's can be established using the Prüfer transformation. Lemmas 2.1 and 4.2, together with the continuity of $D(\lambda)$, provide a new and elementary proof of the existence of the eigenvalues for the coupled self-adjoint BC's. The boundedness from below of the eigenvalues for a fixed coupled self-adjoint BC follows from Theorem 3.1. Therefore, the existence and boundedness from below of the eigenvalues for an arbitrary self-adjoint regular SLP can be established without reference to operator theory.

Theorem 4.1 *Let $\theta = 0$ or $\theta = \pi$.*

(a) Let λ be an eigenvalue of the Sturm-Liouville problem consisting of (1.1) and (1.2) of geometric multiplicity two. Then $D'(\lambda) = 0$.

(b) Let λ be an eigenvalue of the Sturm-Liouville problem consisting of (1.1) and (1.2), and assume $D'(\lambda) = 0$. Then λ is an eigenvalue of geometric multiplicity two.

Proof: Assume that $\theta = 0$. By Lemma 2.2 and (2.11), λ is a double eigenvalue of K if and only if $A(\lambda) = C(\lambda) = 0$ and $B_1(\lambda) = B_2(\lambda) = 1$.

(a) Suppose λ is a double eigenvalue of K. Then $A(\lambda) = C(\lambda) = 0$ and $B(\lambda) = B_1(\lambda) - B_2(\lambda) = 0$. Hence, $D'(\lambda) = 0$ by (4.1).

(b) Suppose λ is an eigenvalue of K and $D'(\lambda) = 0$. Then, by Lemma 2.1, $D(\lambda) = 2$. From (4.2) and (4.3) we get

$$A(\lambda) = B(\lambda) = C(\lambda) = 0.$$

Since

$$B_1(\lambda) - B_2(\lambda) = B(\lambda) = 0 \quad \text{and} \quad B_1(\lambda) + B_2(\lambda) = D(\lambda) = 2,$$

it follows that $B_1(\lambda) = B_2(\lambda) = 1$. Thus, λ is a double eigenvalue of K.

The case $\theta = \pi$ can be established by replacing K by $-K$ in the above argument. ∎

Theorem 4.2 *The algebraic and geometric multiplicities of the eigenvalues of regular self-adjoint Sturm-Liouville problems with coupled boundary conditions are the same.*

Proof: If the coupled BC is non-real, i.e., $\theta \neq 0, \pi$, then the eigenvalues have geometric multiplicity 1 by Lemma 2.2. On the other hand, they have algebraic multiplicity 1 according to Corollary 4.1. Thus, the two multiplicities are equal in this case.

If the coupled BC is real, then Theorem 4.1 implies that an eigenvalue has geometric multiplicity 1 if and only if its algebraic multiplicity is 1. Thus, it suffices to show that if λ_* is an eigenvalue of geometric multiplicity two, then its algebraic multiplicity does not exceed two, i.e. $D''(\lambda_*) \neq 0$. By Lemma 2.2, we may assume that $\theta = 0$ and $K = \Phi(b, \lambda_*)$. Using the notation of Lemma 4.1 and noting that

$$\Phi^{-1} W \Phi = \begin{pmatrix} -uvw & -u^2 w \\ v^2 w & uvw \end{pmatrix},$$

we get

$$D'(\lambda) = \text{trace}\left[\Phi(b,\lambda_*)^{-1}\Phi(b,\lambda)\int_a^b \begin{pmatrix} uvw & u^2 w \\ -v^2 w & -uvw \end{pmatrix}(s,\lambda)\, ds\right],$$

and

$$\begin{aligned}
D''(\lambda) &= \text{trace}\left[\Phi(b,\lambda_*)^{-1}\Phi_\lambda(b,\lambda)\int_a^b \begin{pmatrix} uvw & u^2 w \\ -v^2 w & -uvw \end{pmatrix}(s,\lambda)\, ds\right] \\
&\quad + \text{trace}\left[\Phi(b,\lambda_*)^{-1}\Phi(b,\lambda)\, \partial_\lambda \begin{pmatrix} \int_a^b uvw & \int_a^b u^2 w \\ -\int_a^b v^2 w & -\int_a^b uvw \end{pmatrix}\right] \\
&= \text{trace}\left[\Phi(b,\lambda_*)^{-1}\Phi(b,\lambda) \begin{pmatrix} \int_a^b uvw & \int_a^b u^2 w \\ -\int_a^b v^2 w & -\int_a^b uvw \end{pmatrix}^2\right] \\
&\quad + \text{trace}\left[\Phi(b,\lambda_*)^{-1}\Phi(b,\lambda)\, \partial_\lambda \begin{pmatrix} \int_a^b uvw & \int_a^b u^2 w \\ -\int_a^b v^2 w & -\int_a^b uvw \end{pmatrix}\right], \\
D''(\lambda_*) &= \text{trace}\left[\begin{pmatrix} \int_a^b uvw & \int_a^b u^2 w \\ -\int_a^b v^2 w & -\int_a^b uvw \end{pmatrix}^2\right]_{\lambda=\lambda_*} \\
&= 2\left(\int_a^b u(s,\lambda_*) v(s,\lambda_*) w(s)\, ds\right)^2 \\
&\quad -2\int_a^b u(s,\lambda_*)^2 w(s)\, ds \int_a^b v(s,\lambda_*)^2 w(s)\, ds \\
&\leq 0
\end{aligned}$$

by the Cauchy-Schwarz inequality. Therefore, $D''(\lambda_*) < 0$ since u and v are linearly independent. ∎

Theorem 4.3 *Suppose $\theta = 0$ or π. Then an eigenvalue λ of the Sturm-Liouville problem consisting of (1.1) and (1.2) is double if and only if there exist $n, m \in \mathbb{N}_0$ such that $\lambda = \mu_n = \nu_m$, where μ_n and ν_m are defined in front of Theorem 3.2.*

Proof: Assume λ is a double eigenvalue of the SLP consisting of (1.1) and (1.2). From the proof of Theorem 4.1 we see that $A(\lambda) = C(\lambda) = 0$. From (2.3) and (2.7) we have

$$k_{21} v(b,\lambda) - k_{11} v^{[1]}(b,\lambda) = 0 \tag{4.6}$$

and

$$k_{22}u(b,\lambda) - k_{12}u^{[1]}(b,\lambda) = 0. \tag{4.7}$$

This means that $\lambda = \nu_m$ for some $m \in \mathbb{N}_0$ and $\lambda = \mu_n$ for some $n \in \mathbb{N}_0$.

Assume $\lambda = \mu_n = \nu_m$ for some $n, m \in \mathbb{N}_0$. From $\lambda = \mu_n$ we get that $u(t,\lambda) = u(t,\mu_n)$ is an eigenfunction of μ_n and hence satisfies (4.7). Therefore, $C(\lambda) = 0$. In the same way, $A(\lambda) = 0$ follows from $\lambda = \nu_m$. Noting that $\theta = 0$ or π implies that $D^2(\lambda) = 4$ by Lemma 2.1, and from (4.2) we have that $B(\lambda) = 0$. Then $D'(\lambda) = 0$ by (4.1), and λ is a double eigenvalue of the SLP consisting of (1.1) and (1.2) by Theorem 4.1. ∎

Corollary 4.2 *Suppose $\theta = 0$ or π. Let λ_k and λ_{k+1} be eigenvalues of the Sturm-Liouville problem consisting of (1.1) and (1.2). Then there exists $n, m \in \mathbb{N}_0$ such that*

$$\lambda_k \leq \{\mu_n, \nu_m\} \leq \lambda_{k+1}$$

where μ_n and ν_m are defined in front of Theorem 3.2.

Proof: If one of λ_k and λ_{k+1} is double, then the conclusion follows from Theorem 4.3. If both λ_k and λ_{k+1} are simple, noting that

$$D^2(\lambda_k) = D^2(\lambda_{k+1}) = 4 \quad \text{and} \quad D'(\lambda_k)\, D'(\lambda_{k+1}) < 0,$$

from (4.2) and (4.3) we have that

$$C(\lambda_k)\, C(\lambda_{k+1}) \leq 0 \quad \text{and} \quad A(\lambda_k)\, A(\lambda_{k+1}) \leq 0.$$

By the continuity of the functions $C(\lambda)$ and $A(\lambda)$, each of them has a zero on $[\lambda_k, \lambda_{k+1}]$. Then the conclusion follows from Lemma 4.2. ∎

It is seen from the definitions of μ_n and ν_n that they are functions of K. The next lemma is about the continuous dependence of $\mu_n(K)$ and $\nu_n(K)$ on K for $K \in \mathrm{SL}(2,\mathbb{R})$. See [4] for its proof.

Lemma 4.3 *For each fixed $n \in \mathbb{N}_0$, $\mu_n(K)$ and $\nu_n(K)$ are continuous functions on the sets*

$$\mathcal{K}_\infty = \{\mathcal{K} \in \mathrm{SL}(\in, \mathbb{R});\ \|_{\infty\infty} > \prime, \|_{\infty\in} \leq \prime\}$$

and

$$\mathcal{K}_\in = \{\mathcal{K} \in \mathrm{SL}(\in, \mathbb{R});\ \|_{\infty\infty} \leq \prime, \|_{\infty\in} < \prime\}.$$

Moreover, when K changes continuously from $\mathcal{K}_\infty$ to $\mathcal{K}_\in$, $\nu_0(K)$ disappears at $-\infty$ when $k_{11} = 0$ and $\nu_n(K)$ changes to $\nu_{n-1}(K)$ along a continuous eigenvalue branch, $n = 1, 2, \cdots$. On the other hand $\mu_n(K)$ is a continuous eigenvalue branch for fixed n, $n \in \mathbb{N}_0$.

5 Proofs of the main results

Proof of Theorem 3.1: Consider the boundary value problem consisting of (1.2) and the BC

$$(py')(a) = (py')(b) = 0.$$

Let λ_0 be the smallest eigenvalue of this Neumann problem. Then $v(\cdot, \lambda_0)$ is an eigenfunction for λ_0. Hence $v(\cdot, \lambda_0)$ has no zero on $[a, b]$. So, there exists $k > 0$ such that $v(t, \lambda_0) \geq k$ for $t \in [a, b]$. Denote by $\phi(t, \lambda) \in (0, \pi)$ the Prüfer angle for $v(t, \lambda)$, i.e.,

$$\phi(t, \lambda) = \operatorname{arccot} \frac{v^{[1]}(t, \lambda)}{v(t, \lambda)}. \tag{5.1}$$

Then for $t \in (a, b]$ and $\lambda \leq \lambda_0$, the Sturm Comparison Theorem implies that $\phi(t, \lambda) \leq \phi(t, \lambda_0)$ since $\phi(a, \lambda) = \phi(a, \lambda_0) = \pi/2$. Hence we have

$$\frac{v'(t, \lambda)}{v(t, \lambda)} \geq \frac{v'(t, \lambda_0)}{v(t, \lambda_0)}, \quad \text{for } t \in (a, b] \ a.e., \ \lambda \leq \lambda_0, \tag{5.2}$$

as long as $v(t, \lambda) > 0$, which implies that $v(t, \lambda) \geq v(t, \lambda_0) \geq k$ for $t \in (a, b]$ and $\lambda \leq \lambda_0$. In the same way we see that $v(t, \lambda)$ is strictly decreasing in λ on $(-\infty, \lambda_0]$ for each fixed $t \in (a, b]$.

There is a unique $\alpha : [a, b] \times (-\infty, \lambda_0] \to [0, \infty)$ determined by (3.2) which is continuous. Moreover, $\alpha(t, \lambda)$ is decreasing in λ on $(-\infty, \lambda_0]$ for each $t \in (a, b]$, $\alpha_t(t, \lambda)$ exists, and $p(t)\alpha_t(t, \lambda)$ is continuous. By the reduction of order formula we see that u satisfies (3.4) and $v^{[1]}$, $u^{[1]}$ satisfy (3.3), (3.5), respectively. We only need to show that (3.1) holds for $t \in (a, b]$.

From the theory of the Prüfer transformation (see [7]), we know that for ϕ defined by (5.1), $\lim_{\lambda \to -\infty} \phi(t, \lambda) = 0$ for $t \in (a, b]$. So,

$$\lim_{\lambda \to -\infty} \frac{v^{[1]}(t, \lambda)}{v(t, \lambda)} = \infty, \quad \text{for } t \in (a, b],$$

i.e.,

$$\lim_{\lambda \to -\infty} p(t)\alpha_t(t, \lambda) \tanh(\alpha(t, \lambda)) = \infty, \quad \text{for } t \in (a, b].$$

Note that $0 \leq \tanh(\alpha(t, \lambda)) \leq 1$, we get that

$$\lim_{\lambda \to -\infty} p(t)\alpha_t(t, \lambda) = \infty, \quad \text{for } t \in (a, b]. \tag{5.3}$$

Now, we show that

$$\lim_{\lambda \to -\infty} \alpha(t, \lambda) = \infty, \quad \text{for } t \in (a, b]. \tag{5.4}$$

Assume the contrary. Without loss of generality, let $\lim_{\lambda \to -\infty} \alpha(b, \lambda) = r < \infty$. Then

$$\alpha(b, \lambda) \leq r \qquad \text{on} (-\infty, \lambda_0]. \tag{5.5}$$

From (5.3) we conclude that there is $L \le \lambda_0$ such that $p(b)\alpha_t(b, L) > 0$. By the continuity of $p(\cdot)\alpha_t(\cdot, L)$ we have that $p(t)\alpha_t(t, L) > 0$ on $[c, b]$ for some $c \in (a, b)$. In view of (5.2) with λ_0 replaced by L we see that for each $\lambda \le L$, $p(t)\alpha_t(t, \lambda) > 0$ on $[c, b]$. This together with (5.5) imply that $0 \le \alpha(t, \lambda) \le r$ for all $t \in (c, b]$ and $\lambda \le L$. Hence $k \le v(t, \lambda) \le k \cosh r := r_1$ for $t \in [c, b]$ and $\lambda \le \lambda_0$, and $v^{[1]}(c, \lambda) \ge 0$. However, for $\lambda \le L$, from (1.1) we have

$$\begin{aligned} v(b, \lambda) &= v(c, \lambda) + v^{[1]}(c, \lambda) \int_c^b \frac{dt}{p(t)} \\ &\quad + \int_c^b \frac{1}{p(t)} \int_c^t (q(s) - \lambda w(s)) v(s, \lambda)\, ds\, dt \\ &> \int_c^b \frac{1}{p(t)} \int_c^t q(s) v(s, \lambda)\, ds\, dt \\ &\quad - \lambda \int_c^b \frac{1}{p(t)} \int_c^t w(s) v(s, \lambda)\, ds\, dt \\ &\to \infty \text{ as } \lambda \to -\infty. \end{aligned}$$

We reach a contradiction and, therefore, complete the proof. ∎

Proof of Theorem 3.2: (a) Fix a $K \in \mathcal{K}_\infty$. Then there exists a continuous function $K(\cdot) : [0, 1] \to \mathcal{K}_\infty$ such that $K(0) = I$ and $K(1) = K$. From Lemma 4.3, for each fixed $n \in \mathbb{N}_0$, both $\mu_n(K(s))$ and $\nu_n(K(s))$ are continuous on $[0, 1]$. Moreover, for $n \in \mathbb{N}_0$,

$$\mu_n(K(0)) = \mu_n(I) = \lambda_n^D, \quad \nu_n(K(0)) = \nu_n(I) = \lambda_n^N,$$

$$\mu_n(K(1)) = \mu_n, \quad \nu_n(K(1)) = \nu_n.$$

When necessary we will indicate the dependence of $D(\lambda)$ on K by introducing the notation $D_K(\lambda)$. It is known ([7], Section 13) that for each $n \in \mathbb{N}_0$,

$$(-1)^n D_I(\lambda_n^N) \ge 2 \quad \text{and} \quad (-1)^{n+1} D_I(\lambda_n^D) \ge 2,$$

and from (1.5)

$$\lambda_0^N < \{\lambda_0^D, \lambda_1^N\} < \{\lambda_1^D, \lambda_2^N\} < \cdots.$$

Thus, the continuity of $K(s)$, $\mu_n(K(s))$ and $\nu_n(K(s))$ together with Part (iii a) of Lemma 4.2 yield that for every $n \in \mathbb{N}_0$

$$(-1)^n D(\nu_n) \ge 2 \quad \text{and} \quad (-1)^{n+1} D(\mu_n) \ge 2 \tag{5.6}$$

and

$$\nu_0 < \{\mu_0, \nu_1\} < \{\mu_1, \nu_2\} < \cdots. \tag{5.7}$$

In particular, $\nu_0 < \mu_0$, $D(\nu_0) \ge 2$, and $D(\mu_0) \le -2$. Thus there exists $\lambda_{n_0}(K) \in [\nu_0, \mu_0)$ for some $n_0 \in \mathbb{N}_0$. This implies that $\lambda_0(K) \le \lambda_{n_0}(K) < \mu_0$. Hence, by Theorem 4.3, $\lambda_0(K)$ is simple.

Now, we show that the inequalities in (3.8) hold. From Theorem 3.1 one deduces that as $\lambda \to -\infty$, $v(b,\lambda)$ and $v^{[1]}(b,\lambda)$ approach infinity. By the Bounded Convergence Theorem and the decreasing property of α in λ,

$$\lim_{\lambda\to-\infty}\int_a^b \frac{\operatorname{sech}^2(\alpha(\mathrm{s},\lambda))}{\mathrm{p(s)}}ds = 0.$$

Then we see that among the four functions $u(b,\lambda), v(b,\lambda), u^{[1]}(b,\lambda)$ and $v^{[1]}(b,\lambda)$, $v^{[1]}(b,\lambda)$ grows the fastest and $u(b,\lambda)$ the slowest, as $\lambda \to -\infty$. Also note that if $k_{12} = 0$, then $k_{11}k_{22} = 1$, and hence $k_{22} > 0$. From (2.2) we see that

$$\lim_{\lambda\to-\infty} D(\lambda) = \infty.$$

To show (3.8), or equivalently, to justify the graph in Section 3, we only need to prove that
(i) $\nu_0 \le \lambda_0(K)$, and hence $D(\lambda) > 2$ for $\lambda < \nu_0$;
(ii) the only eigenvalue for K in $[\nu_0, \nu_1)$ is $\lambda_0(K)$;
(iii) for each $n \in \mathbb{N}_0$, there are exactly two eigenvalues for $-K$ in (ν_{2n}, ν_{2n+2}), i.e., $\lambda_{2n}(-K)$ and $\lambda_{2n+1}(-K)$;
(iv) for each $n \in \mathbb{N}_0$, there are exactly two eigenvalues for K in (ν_{2n+1}, ν_{2n+3}), i.e., $\lambda_{2n+1}(K)$ and $\lambda_{2n+2}(K)$.

Assume (i) does not hold. Since $\lim_{\lambda\to-\infty} D(\lambda) = \infty$, there exist $\lambda_k(K)$ and $\lambda_{k+1}(K)$ such that $\lambda_k(K) \le \lambda_{k+1}(K) < \nu_0$. By Corollary 4.2, there exists an $\nu_n \in [\lambda_k(K), \lambda_{k+1}(K)]$ for some $n \in \mathbb{N}_0$, which contradicts the fact that ν_0 is the first eigenvalue for the BC (3.7). Thus, we have verified Part (i).

Assume (ii) does not hold. Then

$$\nu_0 \le \lambda_0(K) < \lambda_1(K) < \nu_1$$

since $\lambda_0(K)$ is simple. Noting that $D(\lambda_0(K)) = D(\lambda_1(K)) = 2$ and $D'(\lambda_0(K)) < 0$, we have

$$\min_{\lambda\in[\lambda_0(K),\lambda_1(K)]} D(\lambda) < 2 \quad \text{and} \quad D(\lambda_\#) = \min_{\lambda\in[\lambda_0(K),\lambda_1(K)]} D(\lambda) \text{ for some } \lambda_\# \in (\lambda_0(K), \lambda_1(K)).$$

If $D(\lambda_\#) \in (-2, 2)$, then $D'(\lambda_\#) = 0$, contradicting Corollary 4.1. If $D(\lambda_\#) \le -2$, then

$$\lambda_0(K) < \lambda_0(-K) \le \lambda_1(-K) < \lambda_1(K),$$

and by Corollary 4.2 for $\theta = \pi$, there exists

$$\nu_n \in [\lambda_0(-K), \lambda_1(-K)] \subset (\lambda_0(K), \lambda_1(K)) \subset (\nu_0, \nu_1)$$

for some $n \in \mathbb{N}_0$, which is impossible. This establishes Part (ii). The proofs of Parts (iii) and (iv) are similar.

(b) Given K with $k_{11} \le 0$ and $k_{12} < 0$, the path

$$s \mapsto K(s) =: \begin{pmatrix} 1 + s(k_{11} - 1) & k_{12} \\ k_{21}(s) & k_{22} \end{pmatrix}, \quad s \in [0, 1],$$

connects $K(0) \in \mathcal{K}_\infty$ to $K = K(1)$, where

$$k_{21}(s) = \frac{[1 + s(k_{11} - 1)]k_{22} - 1}{k_{12}}.$$

Now, we also have

$$\lim_{\lambda \to -\infty} D_{K(s)}(\lambda) = \infty.$$

Then, as above, we can show that (3.9) for this K follows from (3.8) for $K(0)$, the continuity of $D_{K(s)}(\lambda)$ and Lemma 4.3.

(c) This is clear. ∎

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