Sturm-Liouville problems with an infinite number of interior singularities

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Abstract

The generalized Sturm-Liouville problems in this paper stem from the ideas of Christ Shubin and Stolz, based on introducing singularities at a countable number of regular points on the real line.

This idea is generalized to the introduction of a countable number of regular or limit-circle singular points. These results are shown to link with the work of Everitt and Zettl concerned with operator theory generated by a countable number of symmetric differential expressions defined on intervals of the real line.

The results show the the Titchmarsh-Weyl dichotomy for integrable-square solutions can be extended and the corresponding m-coefficient introduced. All associated self-adjoint operators can be characterised.

There are many applications of these results to one-dimensional Schrödinger equations thus extending earlier work of Gesztesy and Kirsch.

1 Introduction

This paper[1] is concerned with a generalized form of the classical Sturm-Liouville theory, *i.e.* those results arising from the study of the differential equation

$$M[y] \equiv -(py')' + qy = \lambda wy \quad \text{on} \quad (a, b). \tag{1.1}$$

This generalization arises from earlier results of Everitt and Zettl [9] and [10], Shubin Christ and Stolz [4], and the recent study of coupled boundary conditions for the equation (1.1) given by Bailey, Everitt and Zettl in [2].

Essentially the generalization is to consider Sturm-Liouville differential equations (1.1) on more than one interval and then linking the solutions across regular or limit-circle singular end-points by means of interface boundary conditions, in such a form to sustain the holomorphic properties of the solutions with respect to the complex parameter λ. These interface boundary conditions are introduced in Section 10 below and lead to the holomorphic properties of the generalized solutions.

This development requires some discussion of 'single interval' initial and boundary value problems; see Sections 3 to 6, and in particular Theorem 2 in Section 5 on a generalized initial value problem.

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The interface boundary conditions, here introduced in the canonical form given in [2], lead to a generalized Green's formula, see Section 11, which then allows an extension of the Titchmarsh-Weyl theory to Sturm-Liouville problems with an infinite number of interior singularities.

In an appropriate Hilbert function space the maximal and minimal operators T_1 and T_0, with $T_0^* = T_1$ and $T_1^* = T_0$, are introduced in Section 12, the Weyl limit-point/limit-circle classifications extended in Section 13, the Titchmarsh-Weyl m-coefficient defined in Section 14, and the self-adjoint extensions of T_0 are classified in Section15.

Section 16 is devoted to the consideration of the equivalence of the resulting self-adjoint operators to the self-adjoint differential operators on multiple intervals, defined by the extension of the Glazman-Krein-Naimark theory, as given in [9] and [10]. This equivalence throws light on the concept of 'maximality' of the boundary conditions introduced in [10, Section 3]; in particular examples of such boundary conditions can now be exhibited which show that the removal of one independent condition leads to the loss of maximality.

Finally in Section 17 examples are given to illustrate the sensitivity of the limit-point and limit-circle classification to the choice of the infinite set of intervals and to the choice of interface boundary conditions.

2 Notations

$\mathbb{R}$ and $\mathbb{C}$ denote the real and complex fields; $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the positive and non-negative real numbers; $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ denote the positive, non-negative and all integers.

Let (a,b) and $[\alpha,\beta]$ denote open and compact intervals of $\mathbb{R}$.

L and AC denote Lebesgue integration and absolute continuity; $L^p_{\text{loc}}(a,b)$, with $p \geq 1$, and $AC_{\text{loc}}(a,b)$ denote the spaces of complex-valued functions that are p-integrable and absolutely continuous on all compact intervals of (a,b).

If $w : (a,b) \to \mathbb{R}_0^+$ and is Lebesgue measurable with w positive almost everywhere on (a,b), then $L^2((a,b) : w)$ denotes the Hilbert function space of all functions $f : (a,b) \to \mathbb{C}$ such that

$$\int_a^b w(x)\,|f(x)|^2\,dx < +\infty$$

with inner product

$$(f,g)_w := \int_a^b w(x)f(x)\bar{g}(x)\,dx. \tag{2.1}$$

If U is an arbitrary open set of $\mathbb{C}$ then $\mathbf{H}(U)$ denotes the set of all functions $\varphi : U \to \mathbb{C}$ that are holomorphic in U.

The symbol '($x \in K$)' is to read as 'for all elements x of the set K'.

3 The Sturm-Liouville differential equation

The minimal conditions on the coefficients p, q and w in the equation (1.1) , with $\lambda = \mu + i\nu \in \mathbb{C}$, for the existence of global solutions on (a,b), with holomorphic properties in the parameter λ, that are consistent with the development of spectral properties in the Hilbert function space $L^2((a,b):w)$, see (2.1), are

$$\left.\begin{array}{ll}\text{(i)} & p,q,w:(a,b)\to\mathbb{R}\\ \text{(ii)} & p^{-1}(\equiv 1/p),q,w\in L^1_{\text{loc}}(a,b)\\ \text{(iii)} & w(x)>0 \text{ for almost all } x\in(a,b).\end{array}\right\} \tag{3.1}$$

These conditions lead to

Theorem 1 *Under the conditions* (3.1), *given* $c\in(a,b)$ *and* $\xi(\cdot)$, $\eta(\cdot)\in\mathbf{H}(\mathbb{C})$ *there exists a unique mapping* $y:(a,b)\times\mathbb{C}\to\mathbb{C}$ *with the properties*

1. $y(\cdot,\lambda)$ *and* $(py')(\cdot,\lambda)\in AC_{loc}(a,b)\quad(\lambda\in\mathbb{C})$
2. $y(x,\cdot)$ *and* $(py')(x,\cdot)\in\mathbf{H}(\mathbb{C})\quad(x\in(a,b))$
3. $y(c,\lambda)=\xi(\lambda)$ *and* $(py')(c,\lambda)=\eta(\lambda)\quad(\lambda\in\mathbb{C})$
4. $y(\cdot,\lambda)$ *satisfies* (1.1) *almost everywhere on* $(a,b)\quad(\lambda\in\mathbb{C})$.

Proof. See [5, Chapter 1], [12, Chapter V] and [14, Chapter I]. ■

Remark With the minimal conditions (3.1) holding it is essential to use the quasi-derivative py' instead of the classical derivative y', in the statement of Theorem 1.

¿From Theorem 1 follows the definition of the 'Titchmarsh' solutions $\theta_\alpha,\varphi_\alpha$ of the equation 1.1.

Definition 1 *Let the pair* (M,w) *be given satisfying the minimal conditions* (3.1); *let* $c\in(a,b)$; *let the parameter* $\alpha\in[0,\pi)$; *then the solutions* $\theta_\alpha,\varphi_\alpha$ *are defined by the initial conditions, for all* $\lambda\in\mathbb{C}$,

$$\left.\begin{array}{ll}\theta_\alpha(c,\lambda)=\cos(\alpha) & (p\theta_\alpha')(c,\lambda)=-\sin(\alpha)\\ \varphi_\alpha(c,\lambda)=\sin(\alpha) & (p\varphi_\alpha')(c,\lambda)=\cos(\alpha)\end{array}\right\} \tag{3.2}$$

Remarks

1. From Theorem 1 it follows that $\theta_\alpha(x,\cdot)$ and $(p\theta_\alpha')(x,\cdot)\in\mathbf{H}(\mathbb{C})$ for all $x\in(a,b)$; similarly for φ. Also from the real initial conditions (3.2) it follows that, for all $x\in(a,b)$ and for all $\lambda\in\mathbb{C}$,

$$\bar{\theta}_a(x,\lambda)=\theta_\alpha(x,\bar{\lambda})\quad(\overline{p\theta_\alpha'})(x,\lambda)=(p\theta_\alpha')(x,\bar{\lambda}). \tag{3.3}$$

   There is a similar result for the solution φ_α.

2. The pair $\theta_\alpha,\varphi_\alpha$ forms a basis for all solutions of the equation (1.1) on (a,b), and this for all $\lambda\in\mathbb{C}$. The generalized Wronskian

$$W(\theta_\alpha,\varphi_\alpha)(x,\lambda):=\theta_\alpha(x,\lambda)(p\varphi_\alpha')(x,\lambda)-(p\theta_\alpha')(x,\lambda)\varphi_\alpha(x,\lambda) \tag{3.4}$$

   satisfies

$$W(\theta_\alpha,\varphi_\alpha)(x,\lambda)=1\quad(x\in(a,b)\text{ and }\lambda\in\mathbb{C}). \tag{3.5}$$

4 Separated boundary conditions

¿From the differential equation (1.1), and hence given the coefficients p and q, define the differential expression $M : D(M) \times (a,b) \to \mathbb{C}$ by

$$D(M) := \{\, f : (a,b) \to \mathbb{C} : f,\ pf' \in AC_{\text{loc}}(a,b) \,\} \tag{4.1}$$

and

$$M[f](x) := -(p(x)f'(x))' + q(x)f(x) \quad (\, x \in (a,b) \text{ and } f \in D(M)\,). \tag{4.2}$$

The Green's formula for M is, given any $[\alpha,\beta] \subset (a,b)$,

$$\int_\alpha^\beta \{\overline{g}(x)M[f](x) - f(x)\overline{M[g]}(x)\}\,dx = [f,g](x)|_\alpha^\beta \quad (f,g \in D(M)\,) \tag{4.3}$$

where $[f,g](\cdot) : D(M) \times D(M) \times (a,b) \to \mathbb{C}$, defined by

$$[f,g](x) := f(x)(p\overline{g}')(x) - (pf')(x)\overline{g}(x), \tag{4.4}$$

is the skew-symmetric bilinear form of M.

Given also the weight w and then the space $L^2((a,b):w)$ define the maximal domain Δ of the pair (M,w) by

$$\Delta := \{f \in D(M) : f,\ w^{-1}M[f] \in L^2((a,b):w)\,\}. \tag{4.5}$$

This domain Δ is dense in $L^2((a,b):w)$; see[12, Chapter V].

¿From (4.3) it follows that the limits

$$\lim_{x \to a+(b-)} [f,g](x) \tag{4.6}$$

exist and are finite for all $f,g \in \Delta$.

The following notation is now introduced; for all $f,g \in \Delta$

$$[f,g](a^+) := \lim_{x \to a+} [f,g](x) \text{ and } [f,g](b^-) := \lim_{x \to b-} [f,g](x). \tag{4.7}$$

Given the pair (M,w) on the interval (a,b) the end-points, now denoted by a^+ and b^-, are classified at a^+ (b^-) as

1. either *regular*
2. or *limit-circle* in $L^2((a,b):w)$
3. or *limit-point* in $L^2((a,b):w)$.

These mutually exclusive classifications are now classical; for details see [5, Chapter 9], [12, Chapter V] or [14, Chapter II].

Lemma 1 *If both the end-points a^+ and b^- are, independently, either regular or limit-circle in $L^2((a,b):w)$ then*

1.

$$\theta_\alpha(\cdot,\lambda) \text{ and } \varphi_\alpha(\cdot,\lambda) \in \Delta \quad (\,\lambda \in \mathbb{C}\,) \tag{4.8}$$

2. *for all* $f \in \Delta$ *the functions defined by, for all* $\lambda \in \mathbb{C}$,

$$\lambda \longmapsto [\theta_\alpha(\cdot,\lambda), f]\left(a^+(b^-)\right) \quad \text{and} \quad \lambda \longmapsto [\varphi_\alpha(\cdot,\lambda), f]\left(a^+(b^-)\right) \tag{4.9}$$

belong to the class $\mathbf{H}(\mathbb{C})$.

Proof. For 1 see [5, Chapter 9], [12, Section 17.5] and [14, Chapter II].

For 2 see the results in [7] and [8]. ■

In this paper it is essential to restrict both a^+ and b^-, but independently, to be either regular or limit-circle in $L^2((a,b):w)$; the limit-point case at both end-points has to be excluded; the reason for this exclusion is discussed in the next Section.

Lemma 2 *Let the pair* (M,w) *be given on* (a,b) *satisfying the minimal conditions* (3.1); *then*

1. *If* a^+ (b^-) *is limit-point in* $L^2((a,b):w)$

$$[f,g]\left(a^+\,(b^-)\right) = 0 \quad (\, f,g \in \Delta\,) \tag{4.10}$$

2. *If* a^+ (b^-) *is regular or limit-circle in* $L^2((a,b):w)$ *there exist pairs* $\gamma,\delta \in \Delta$ *such that*

$$\left.\begin{array}{cc} (i) & \gamma,\delta : (a,b) \to \mathbb{R} \\ (ii) & [\gamma,\delta]\,(a^+\,(b^-)) = 1 \end{array}\right\} \tag{4.11}$$

Proof. These results are known; see [7] and [12]. ■

To generate self-adjoint operators in $L^2((a,b):w)$ from the pair (M,w) homogeneous boundary conditions at a^+ and/or b^- may be required. In the limit-point case no boundary condition is required but if a^+ or b^- is regular or limit-circle then separated or coupled boundary conditions are necessary. (For coupled conditions see the named section below.)

If a^+ or b^- is regular or limit-circle then, given the pair $\gamma,\delta \in \Delta$ as in (4.11), all separated boundary conditions can be determined in the form,with $f \in \Delta$,

$$[f, A\gamma + B\delta]\left(a^+\,(b^-)\right) = 0 \tag{4.12}$$

where $A,B \in \mathbb{R}$ with $A^2 + B^2 > 0$. If a^+ is regular then the separated condition 4.12 reduces to the classical condition

$$f(a^+)\cos(\alpha) - (pf')(a^+)\sin(\alpha) = 0$$

for some $\alpha \in [0,\pi)$. Similarly if b^- is regular.

5 Generalized initial value problems

Separated boundary conditions lead to an extension of the initial value problem given in Theorem 1.

Theorem 2 *Let the pair (M, w) be given on the interval (a, b), satisfying the minimal conditions* (3.1)*; let the end-point classification hold*

$$\left.\begin{array}{ll} (i) & a^{+} \text{ is either regular or limit-circle in } L^2((a,b):w), \\ & \text{and independently,} \\ (ii) & b^{-} \text{ is either regular or limit-circle in } L^2((a,b):w); \end{array}\right\} \quad (5.1)$$

let the pair $\gamma, \delta \in \Delta$ be given satisfying the conditions (4.11)*; let $\xi(.)$, $\eta(\cdot) \in \mathbf{H}(\mathbb{C})$ be given; then there exists a unique mapping $\psi : (a, b) \times \mathbb{C}$ with the properties:*

1. $\psi(\cdot, \lambda)$ *and* $(p\psi')(\cdot, \lambda) \in AC_{loc}(a, b) \quad (\lambda \in \mathbb{C})$
2. $\psi(x, \cdot)$ *and* $(p\psi')(x, \cdot) \in \mathbf{H}(\mathbb{C}) \quad (x \in (a, b))$
3. $[\psi(\cdot, \lambda), \gamma(\cdot)]\,(a^+\,(b^-)) = \xi(\lambda)$ *and* $[\psi(\cdot, \lambda), \delta(\cdot)]\,(a^+\,(b^-)) = \eta(\lambda) \quad (\lambda \in \mathbb{C})$
4. $\psi(\cdot, \lambda)$ *satisfies the differential equation* (1.1) *almost everywhere on* (a, b)
5. $\psi(\cdot, \lambda) \in \Delta \quad (\lambda \in \mathbb{C})$.

Remarks 1. If $a^+\,(b^-)$ is regular then this result is equivalent to the existence result given in Theorem 1.

2. This theorem fails to hold if the end-point, a or b, is in the limit-point classification.

Proof. The proof depends upon the use of the Plücker identity, see [7, Section 2] and [8, Section 4]; let $f_r, g_r \in D(M)$ for $r = 1, 2, 3$; then

$$\det_{r,s=1,2,3} [[f_r, g_s]]\,(x) = 0 \quad (x \in (a, b)). \tag{5.2}$$

To determine the solution ψ with the required properties for the end-point a^+ write, using the basis solutions $\theta_\alpha, \varphi_\alpha$ of (3.2) for some choice of $\alpha \in [0, \pi)$,

$$\psi(x, \lambda) = A(\lambda)\theta_\alpha(x, \lambda) + B(\lambda)\varphi_\alpha(x, \lambda) \quad (x \in (a, b) \quad \lambda \in \mathbb{C})$$

with the coefficients $A(\cdot)$ and $B(\cdot)$ to be determined.

¿From the generalized initial conditions 3 of the Theorem and using (4.8) we obtain the pair of linear equations

$$A(\lambda)[\theta_\alpha(\cdot, \lambda), \gamma(\cdot)](a^+) + B(\lambda)[\varphi_\alpha(\cdot, \lambda), \gamma(\cdot)](a^+) = \xi(\lambda) \quad (\lambda \in \mathbb{C})$$

$$A(\lambda)[\theta_\alpha(\cdot, \lambda), \delta(\cdot)](a^+) + B(\lambda)[\varphi_\alpha(\cdot, \lambda), \delta(\cdot)](a^+) = \eta(\lambda) \quad (\lambda \in \mathbb{C}).$$

These equations have a unique solution for A and B, for each $\lambda \in \mathbb{C}$, if the determinant $d(\cdot)$ defined by

$$d(\lambda) := [\theta_\alpha(\cdot, \lambda), \gamma(\cdot)](a^+)[\varphi_\alpha(\cdot, \lambda), \delta(\cdot)](a^+) - [\varphi_\alpha(\cdot, \lambda), \gamma(\cdot)](a^+)[\theta_\alpha(\cdot, \lambda), \delta(\cdot)](a^+)$$

satisfies $d(\lambda) \neq 0 \ (\lambda \in \mathbb{C})$.

To evaluate $d(\cdot)$ make the following substitution into the Plücker identity (5.2)

$$f_1, f_2, f_3 : \theta_\alpha(\cdot, \lambda), \varphi_\alpha(\cdot, \lambda), \gamma(\cdot) \qquad g_1, g_2, g_3 : \theta_\alpha(\cdot, \bar{\lambda}), \varphi_\alpha(\cdot, \bar{\lambda}), \delta(\cdot).$$

Recalling that such terms as $[\theta_\alpha(\cdot,\lambda),\varphi_\alpha(\cdot,\bar{\lambda})](x)$ are, from the Green's formula (4.3), independent of the value of x, and using the properties (3.2), (3.3) and (4.11), the identity (5.2) shows that

$$d(\lambda) = 1 \quad (\,\lambda \in \mathbb{C}\,).$$

The linear equations given above now yield the following solutions

$$A(\lambda) = [\varphi_\alpha(\cdot,\lambda),\delta(\cdot)](a^+)\xi(\lambda) - [\varphi_\alpha(\cdot,\lambda),\gamma(\cdot)](a^+)\eta(\lambda) \quad (\,\lambda \in \mathbb{C}\,)$$

$$B(\lambda) = -[\theta_\alpha(\cdot,\lambda),\delta(\cdot)](a^+)\xi(\lambda) + [\theta_\alpha(\cdot,\lambda),\gamma(\cdot)](a^+)\eta(\lambda) \quad (\,\lambda \in \mathbb{C}\,).$$

All the required results for the solution ψ are now seen to be satisfied; in particular property 2 follows from (4.9) of Lemma 1 above. ■

6 Coupled boundary conditions

The canonical form of symmetric (self-adjoint) coupled boundary conditions introduced in [2] leads to the appropriate definition of interface boundary conditions required for the generalized Sturm-Liouville boundary value problem. This requires the end-point classification for the pair (M,w) on (a,b) given in (5.1), *i.e.* both end-points a^+ and b^- are to be in the regular or limit-circle case in $L^2((a,b):w)$. This is the only environment in which coupled boundary can be considered for the differential equation (1.1).

We first choose $\gamma,\delta \in \Delta$ with the properties

$$\left.\begin{array}{ll} (i) & \gamma,\delta : (a,b) \to \mathbb{R} \\ (ii) & [\gamma,\delta](a^+) = [\gamma,\delta](b^-) = 1. \end{array}\right\} \tag{6.1}$$

Such a choice is possible in many ways; for some details see [2, Section 2]. Note that γ,δ must be members of the set Δ but they may or may not be solutions of the differential equation (1.1); however one construction for γ,δ is to take, for some $\mu \in \mathbb{R}$ and $\alpha \in [0,\pi)$

$$\gamma(x) = \theta_\alpha(x,\mu) \quad \delta(x) = \varphi_\alpha(x,\mu) \quad (\,x \in (a,b)) \tag{6.2}$$

and to use (3.5). Other constructions are possible but then may require the 'patching lemma' of Naimark, see [12, Section 17.3, Lemma 2].

Now define the 2×1 column matrix $\mathbb{F} : \Delta \times (a,b) \to \mathbb{C}\times\mathbb{C}$ by

$$\mathbb{F}(f,x) \equiv \mathbb{F}(x) := \begin{bmatrix} [f,\gamma](x) \\ [f,\delta](x) \end{bmatrix} \quad (\,f \in \Delta \text{ and } x \in (a,b)\,), \tag{6.3}$$

then the canonical form of the coupled boundary conditions is

$$\mathbb{F}(b^-) = \exp(i\tau)\mathbf{K}\mathbb{F}(a^+) \tag{6.4}$$

where

$$\left.\begin{array}{r} \tau \in (-\pi,\pi] \\ \mathbf{K} \in SL_2(\mathbb{R}). \end{array}\right\} \tag{6.5}$$

Here $SL_2(\mathbb{R})$ is the group of all 2×2 matrices $\mathbf{K} = [k_{rs}]$ with $k_{rs} \in \mathbb{R}\,(\,r,s = 1,2\,)$ and $\det \mathbf{K} = 1$.

Since, with the given end-point classification, all solutions of the differential equation are in the domain Δ the symmetric, coupled boundary condition (6.3) can also be applied to such solutions to determine a self-adjoint boundary value problem in $L^2((a,b):w)$. For details of this form of coupled boundary conditions see [2, Section 2].

The coupled boundary condition (6.4) can be used to define self-adjoint operators from the pair (M,w) in the space $L^2((a,b):w)$; the spectral multiplicity of these operators may be, but cannot exceed, order 2.

In this paper the canonical form (6.4) is used to create interface boundary conditions; see Section 10 below.

7 Intervals of $\mathbb{R}$

We suppose given a countable set of open intervals $\{\,(a_n,b_n) : n\in\mathbb{Z}\,\}$ such that for all $n\in\mathbb{Z}$

$$(i)\ a_n,b_n\in\mathbb{R} \quad (ii)\ a_n<b_n \quad (iii)\ a_{n+1}=b_n. \tag{7.1}$$

¿From these conditions it is clear that the sequences $\{\,a_n : n\in\mathbb{Z}\,\}$ and $\{\,b_n : n\in\mathbb{Z}\,\}$ are monotonic decreasing and increasing respectively; define then

$$\mathbf{a} := \lim_{n\to-\infty} a_n \qquad \mathbf{b} := \lim_{n\to\infty} b_n \tag{7.2}$$

so that

$$-\infty\le\mathbf{a}<\mathbf{b}\le+\infty.$$

The open intervals $\{\,(a_n,b_n) : n\in\mathbb{Z}\,\}$ are all disjoint but "abut" at the end-points. The open interval $(\mathbf{a},\mathbf{b})\subseteq\mathbb{R}$ and may be the whole real line. Define the set of points of $\mathbb{R}$

$$\langle\mathbf{a},\mathbf{b}\rangle := \bigcup_{n\in\mathbb{Z}}(a_n,b_n). \tag{7.3}$$

There are a number of earlier contributions to the study of Sturm-Liouville problems on sets of intervals. In [9] only a finite number of intervals is involved but otherwise satisfy the conditions (7.1). In ([4]) the main interest is when the intervals are taken to be $\{(n,n+1) : n\in\mathbb{Z}\,\}$ and $(\mathbf{a},\mathbf{b})=\mathbb{R}$, with a more general situation considered in the Appendix. In [10] the open intervals $\{(a_n,b_n) : n\in\mathbb{Z}\}$ are arbitrary and may overlap; also for some $n_-\in\mathbb{Z}$ and $n_+\in\mathbb{Z}$ it may be that $a_{n_-}=-\infty$ and/or $a_{n_+}=+\infty$.

The analysis in this paper adapts to all these possible choices of the intervals $\{\,(a_n,b_n) : n\in\mathbb{Z}\,\}$ but the results are stated in terms of the conditions (7.1) given above.

8 Differential equations and operators

Given the sequence of intervals determined by (7.1) let the sequence $\{p_n,q_n,w_n : n\in\mathbb{Z}\}$ of coefficients be defined so that each triple (p_n,q_n,w_n) satisfies the basic conditions (3.1) on (a_n,b_n), for all $n\in\mathbb{Z}$.

Define the sequence of differential expressions $\{\ M_n : n\in\mathbb{Z}\,\}$ by, see (4.1) and (4.2),

$$D(M_n) := \{\, f:(a_n,b_n)\to\mathbb{C} : f,p_nf'\in AC_{\mathrm{loc}}(a_n,b_n)\,\} \tag{8.1}$$

and, for all $f \in D(M_n)$,

$$M_n[f] := -(p_n f')' + q_n f \quad \text{on} \quad (a_n, b_n). \tag{8.2}$$

The Green's formula for M_n is similar to (4.3) with the bilinear form defined by, for all $f, g \in D(M_n)$,

$$[f, g]_n(x) := f(x)(p_n \bar{g}')(x) - (p_n f')(x)\bar{g}(x) \quad (\, x \in (a_n, b_n)\,). \tag{8.3}$$

The corresponding differential equation is

$$M_n[y] \equiv -(p_n y')' + q_n y = \lambda w_n y \quad \text{on} \quad (a_n, b_n) \tag{8.4}$$

with solution properties as described in Section 3 above.

Let L^2_n denote the Hilbert function space $L^2((a_n, b_n) : w_n)$ with inner-product $(\cdot,\cdot)_n$, as in (2.1).

Define the maximal domain $\Delta_n \subset L^2_n$ by, for all $n \in \mathbb{Z}$,

$$\Delta_n := \{\, f \in D(M_n) : f, w_n^{-1} M_n[f] \in L^2_n \,\}. \tag{8.5}$$

¿From Green's formula it follows that, for all $f, g \in \Delta_n$,

$$[f, g]_n(\cdot) : [a_n^+, b_n^-] \to \mathbb{C}$$

where $[f, g]_n(a_n^+)$ and $[f, g]_n(b_n^-)$ are determined by taking limits in $\mathbb{C}$.

The maximal operator $T_{1,n} : D(T_{1,n}) \subset L^2_n \to L^2_n$ is defined by , for all $n \in \mathbb{Z}$,

$$D(T_{1,n}) := \Delta_n \quad \text{and} \quad T_{1,n} f := w_n^{-1} M_n[f] \quad (\, f \in D(T_{1,n})\,). \tag{8.6}$$

The end-point restriction on a_n^+ and b_n^-, to which reference is made in Section 5 above, is now applied to all the intervals of the set $\langle \mathbf{a}, \mathbf{b} \rangle$. This restriction requires that the coefficients (p_n, q_n, w_n) be chosen so that, for all $n \in \mathbb{Z}$,

$$\left.\begin{array}{ll} (i) & a_n^+ \text{ is either regular or limit-circle in } L^2((a_n, b_n) : w_n), \\ & \text{and independently,} \\ (ii) & b_n^- \text{ is either regular or limit-circle in } L^2((a_n, b_n) : w_n). \end{array}\right\} \tag{8.7}$$

This requirement eliminates the possibility that any of the end-points of the set $\langle \mathbf{a}, \mathbf{b} \rangle$ are in the limit-point classification in the corresponding L^2-space.

For each $n \in \mathbb{Z}$ choose a point $c_n \in (a_n, b_n)$ and a number $\sigma_n \in [0, \pi)$. A solution basis $\{u_n, v_n\}$ for the differential equation (8.4) on (a_n, b_n) is then determined by, and this for all $\lambda \in \mathbb{C}$,

$$\left.\begin{array}{ll} u_n(c_n, \lambda) = \cos(\sigma_n) & (p_n u_n')(c_n, \lambda) = -\sin(\sigma_n) \\ v_n(c_n, \lambda) = \sin(\sigma_n) & (p_n v_n')(c_n, \lambda) = \cos(\sigma_n). \end{array}\right\} \tag{8.8}$$

The sequence $\{c_n : n \in \mathbb{Z}\}$ is important in subsequent sections; the points of $\{c_n\}$ can be chosen freely from the open intervals $\{(a_n, b_n)\}$; a prescription is

$$c_n = \frac{1}{2}(a_n + b_n) \quad (\, n \in \mathbb{Z}\,). \tag{8.9}$$

Note that, from (7.2),

$$\lim_{n \to -\infty} c_n = \mathbf{a} \quad \lim_{n \to \infty} c_n = \mathbf{b}. \tag{8.10}$$

It follows that the generalized Wronskian $W(u_n, v_n)$ satisfies

$$W(u_n, v_n)(x, \lambda) := u_n(x, \lambda)(p_n u_n')(x, \lambda) - (p_n u_n')(x, \lambda)v_n(x, \lambda) = 1 \tag{8.11}$$

for all $x \in (a_n, b_n)$ and for all $\lambda \in \mathbb{C}$.

¿From the classification condition (8.7) it follows that, as with (4.8),

$$u_n(\cdot, \lambda) \text{ and } v_n(\cdot, \lambda) \in D(T_{1,n}) \equiv \Delta_n \quad (\, \lambda \in \mathbb{C}\,). \tag{8.12}$$

Furthermore, since the initial conditions (8.8) are real-valued, the Wronskian result (8.11) implies

$$[u_n(\cdot, \lambda), v_n(\cdot, \bar{\lambda})]_n(x) = 1 \quad (\, x \in [a_n^+, b_n^-] \text{ and } \lambda \in \mathbb{C}\,). \tag{8.13}$$

9 Differential equations on the set $\langle \mathbf{a}, \mathbf{b} \rangle$

Given the set $\langle \mathbf{a}, \mathbf{b} \rangle \subset \mathbb{R}$, as defined by (7.3), and the sequence of coefficients $\{p_n, q_n, w_n : n \in \mathbb{Z}\}$, satisfying (3.1) for all $n \in \mathbb{Z}$, consider the generalized Sturm-Liouville differential equation

$$-(\mathbf{P}\mathbf{Y}')' + \mathbf{Q}\mathbf{Y} = \lambda \mathbf{W}\mathbf{Y} \quad \text{on} \quad \langle \mathbf{a}, \mathbf{b} \rangle \tag{9.1}$$

where

1. $\mathbf{P} := \{p_n : n \in \mathbb{Z}\}$ and similarly for $\mathbf{Q}$ and $\mathbf{W}$;

2. $\mathbf{Y} := \{y_n : n \in \mathbb{Z}\}$ where each component y_n is a solution of $M_n[y] = \lambda w_n y$ on (a_n, b_n), for the same value of the parameter λ, for all $n \in \mathbb{Z}$

3. $\mathbf{Q}\mathbf{Y} := \{q_n y_n : n \in \mathbb{Z}\}$ and $\mathbf{W}\mathbf{Y} := \{w_n y_n : n \in \mathbb{Z}\}$

4. $\mathbf{P}\mathbf{Y}' := \{p_n y_n' : n \in \mathbb{Z}\}$ and $(\mathbf{P}\mathbf{Y}')' := \{\, (p_n y_n')' : n \in \mathbb{Z}\}$

For use in the following sections $\mathbf{Y}$ is taken to be a solution of (9.1) only if all the above given conditions hold.

We write (9.1) in the symbolic form

$$\mathbf{M}[\mathbf{Y}] = \lambda \mathbf{W}\mathbf{Y} \quad \text{on} \quad \langle \mathbf{a}, \mathbf{b} \rangle \tag{9.2}$$

and, in the same vein, for all $\lambda \in \mathbb{C}$,

$$\mathbf{Y}(\cdot, \lambda) \text{ and } (\mathbf{P}\mathbf{Y}')(\cdot, \lambda) : \langle \mathbf{a}, \mathbf{b} \rangle \to \mathbb{C}.$$

Note that in view of the possible limit-circle classification at the end-points $\{a_n^+, b_n^- : n \in \mathbb{Z}\}$, the solution $\mathbf{Y}(\cdot, \lambda)$ and quasi-derivative $(\mathbf{P}\mathbf{Y}')(\cdot, \lambda)$ may not be defined at these end-points.

The Hilbert function space $\mathbf{L}^2 \equiv \mathbf{L}^2(\langle \mathbf{a}, \mathbf{b} \rangle : \mathbf{W})$ is defined as the set of all function sequences $\mathbf{F} = \{f_n : n \in \mathbb{Z}\}$ with

$$(i)\ f_n \in L_n^2 \ (\, n \in \mathbb{Z}\,)$$

$$(ii) \quad \int_{\langle \mathbf{a}, \mathbf{b} \rangle} \mathbf{W}\,|\mathbf{F}|^2 = \sum_{n \in \mathbb{Z}} \int_{a_n}^{b_n} w_n\,|f_n|^2 < \infty. \tag{9.3}$$

The inner-product in $\mathbf{L}^2$ is given by

$$(\mathbf{F},\mathbf{G}) = \int_{\langle \mathbf{a},\mathbf{b}\rangle} \mathbf{W}\mathbf{F}\bar{\mathbf{G}} = \sum_{n\in\mathbb{Z}} \int_{a_n}^{b_n} w_n f_n \bar{g}_n. \tag{9.4}$$

It is not difficult to show that $\mathbf{L}^2$ is a Hilbert space and also, as in [9, Section 2], that this space is to be identified with the direct (orthogonal) sum

$$\sum_{n\in\mathbb{Z}} \oplus L_n^2.$$

In a similar form we can define the spaces $\mathbf{L}^2(\langle \mathbf{a}, c_0] : \mathbf{W})$ and $\mathbf{L}^2([c_0, \mathbf{b}\rangle : \mathbf{W})$ and their respective norms and inner-products.

In the next Section "interface" boundary conditions are introduced at the end-points $\{a_n^+, b_n^- : n \in \mathbb{Z}\}$ so that:

1. General solutions of the equation (9.1) can be determined from initial conditions at a single point of $\langle \mathbf{a}, \mathbf{b}\rangle$; this leads to a theorem for this generalized equation similar to the classical Theorem 1 for the single equation (1.1)
2. The Weyl limit-point/limit-circle classification in $\mathbf{L}^2$ can be extended for the end-points $\mathbf{a}$ and $\mathbf{b}$
3. The Titchmarsh-Weyl theory can be developed for the generalized differential equation (9.1) to obtain the existence of solutions and self-adjoint differential operators in the space $\mathbf{L}^2$
4. The Titchmarsh-Weyl m-coefficient can be introduced and shown to be a classical Nevanlinna function on the complex plane $\mathbb{C}$.

10 Interface boundary conditions

These conditions are based on the ideas in the recent paper [2] and are discussed above in Section 6.

We recall that $SL_2(\mathbb{R})$ is the group of all 2×2 matrices $\mathbf{K} = [k_{r,s}]$ with $k_{r,s} \in \mathbb{R}$ $(r, s = 1, 2)$ and $\det \mathbf{K} = 1$.

For each $n \in \mathbb{Z}$ choose two elements $\gamma_n, \delta_n \in \Delta_n$, see (8.5), such that, compare with (6.1),

$$\left.\begin{array}{ll} (i) & \gamma_n, \delta_n : (a_n, b_n) \to \mathbb{R} \\ (ii) & [\gamma_n, \delta_n](a_n^+) = [\gamma_n, \delta_n](b_n^-) = 1. \end{array}\right\} \tag{10.1}$$

Such a choice is always possible; for example for all $n \in \mathbb{Z}$ choose $\mu_n \in \mathbb{R}$ and then define

$$\gamma_n(x) := u_n(x, \mu_n) \quad \delta_n(x) := v_n(x, \mu_n) \quad (\, x \in (a_n, b_n)\,)$$

and use (8.13).

Let $\mathbf{Y}$ be any solution of the generalized equation (9.1) and for $n \in \mathbb{Z}$ define the 2×1 matrix $\mathbb{Y}_n : (a_n, b_n) \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by

$$\mathbb{Y}_n(x, \lambda) := \begin{bmatrix} [y_n(\cdot, \lambda), \gamma_n(\cdot)]_n(x) \\ [y_n(\cdot, \lambda), \delta_n(\cdot)]_n(x) \end{bmatrix} \quad (\, x \in (a_n, b_n) \text{ and } \lambda \in \mathbb{C}\,). \tag{10.2}$$

¿From the regular/limit-circle classification condition (8.7) it follows that the limits $\mathbb{Y}_n(a_n^+,\lambda)$ and $\mathbb{Y}_n(b_n^-,\lambda)$ exist in $\mathbb{C}$ and are finite for all $n \in \mathbb{Z}$ and all $\lambda \in \mathbb{C}$.

An interface solution $\mathbf{Y}$ is determined by starting the component y_0 as a classical solution of the differential equation $M_0[y] = \lambda w_0 y$ at the point c_0 of the interval (a_0, b_0). For each $n \in \mathbb{Z} \setminus \{0\}$ the solution component y_n is determined by inductive construction, both as n increases and decreases from 0, on using a given sequence of interface boundary conditions.

The most general form of interface boundary condition that ensures all the components of the solution $\mathbf{Y}$ and the first quasi-derivative $\mathbf{PY}'$ are holomorphic on $\mathbb{C}$, is obtained by choosing a sequence $\{\, \mathbf{A}_n : n \in \mathbb{Z}\,\}$ of 2×2 matrices over $\mathbb{C}$ with $\det \mathbf{A}_n \neq 0 \ (\, n \in \mathbb{Z}\,)$ and then to require that

$$\mathbb{Y}_{n+1}(a_{n+1}^+,\lambda) = \mathbf{A}_n \mathbb{Y}_n(b_n^-,\lambda) \quad (\, \lambda \in \mathbb{C} \text{ and } n \in \mathbb{Z}\,).$$

Theorem 3 given below applies equally well in this general case.

However for the extension of the Titchmarsh-Weyl theory to the generalised equation is essential to choose a special form for the matrix sequence $\{\mathbf{A}_n\}$; this choice is determined by the form of symmetric coupled boundary conditions discussed in Section 6 above. The structural reason for this choice is made more definite in Section 11 below, on the generalised Green's formula.

For each $n \in \mathbb{Z}$ choose $\tau_n \in [0, \pi)$ and $\mathbf{K}_n \in SL_2(\mathbb{R})$ and define, see (6.4),

$$\mathbf{A}_n := \exp(i\tau_n)\mathbf{K}_n \quad (\, n \in \mathbb{Z}\,);$$

then the interface condition requires that

$$\mathbb{Y}_{n+1}(a_{n+1}^+,\lambda) = \exp(i\tau_n)\mathbf{K}_n \mathbb{Y}_n(b_n^-,\lambda) \quad (\, \lambda \in \mathbb{C} \text{ and } n \in \mathbb{Z}\,). \tag{10.3}$$

It is this requirement that ensures the continuation of the solution from one interval to the two adjacent intervals in such a form that the holomorphic properties in λ are preserved, and that the extended Green's formula is available.

Note that once the sequence $\{\gamma_n, \delta_n : n \in \mathbb{Z}\}$ of interface elements is determined according to the specification (10.1), and then kept fixed, **all** interface boundary conditions can now be determined by varying the sequences $\{\tau_n : n \in \mathbb{Z}\}$ and $\{\mathbf{K}_n : n \in \mathbb{Z}\}$.

This procedure leads to:

Theorem 3 *Let the sequences* $\{\,(a_n, b_n) : n \in \mathbb{Z}\,\}$, $\{p_n, q_n, w_n : n \in \mathbb{Z}\}$, $\{\gamma_n, \delta_n : n \in \mathbb{Z}\}$ *and* $\{\tau_n, \mathbf{K}_n : n \in \mathbb{Z}\,\}$ *be determined as Sections* 7, 8, 8, 9, 10 *and* 10 *respectively. Let* $\xi(\cdot)$, $\eta(\cdot) \in \mathbf{H}(\mathbb{C})$ *be given, and for some* $r \in \mathbb{Z}$ *let* $c \in (a_r, b_r)$ *be chosen.*

Then there exists a unique generalized solution $\mathbf{Y} = \{y_n : n \in \mathbb{Z}\}$, *as defined in Section 9 above, of*

$$\mathbf{M}[\mathbf{Y}] = \lambda \mathbf{W}\mathbf{Y} \, on \, \langle \mathbf{a}, \mathbf{b} \rangle \tag{10.4}$$

with the properties:

$$y_r(c,\lambda) = \xi(\lambda) \quad (p_r y_r')(c,\lambda) = \eta(\lambda) \quad (\, \lambda \in \mathbb{C}\,) \tag{10.5}$$

$$\mathbb{Y}_{n+1}(a_{n+1}^+,\lambda) = \exp(i\tau_n)\mathbf{K}_n \mathbb{Y}_n(b_n^-,\lambda) \quad (\, \lambda \in \mathbb{C} \text{ and } n \in \mathbb{Z}\,) \tag{10.6}$$

$$y_n(x,\cdot) \text{ and } (p_n y_n')(x,\cdot) \in \mathbf{H}(\mathbb{C}) \quad (\, x \in (a_n, b_n) \text{ and } n \in \mathbb{Z}\,) \tag{10.7}$$

$$y_n(\cdot,\lambda) \in \Delta_n \quad (\, \lambda \in \mathbb{C} \text{ and } n \in \mathbb{Z}\,). \tag{10.8}$$

For any given $\lambda \in \mathbb{C}$ *the generalised holomorphic solutions of (*10.4*) form a linear manifold of two-dimensions.*

Proof. See below.

Remarks 1. The consequences of this Theorem may be stated symbolically as; given all the information for the Theorem the generalized differential equation

$$\mathbf{M}[\mathbf{Y}] = \lambda \mathbf{W}\mathbf{Y} \quad \text{on} \quad \langle \mathbf{a}, \mathbf{b} \rangle$$

has a unique solution $\mathbf{Y}$, given the initial data $\xi(\cdot)$, $\eta(\cdot) \in \mathbf{H}(\mathbb{C})$, with the properties

$$\mathbf{Y}(c, \lambda) = \xi(\lambda) \quad (\mathbf{P}\mathbf{Y}')(c, \lambda) = \eta(\lambda) \quad (\, \lambda \in \mathbb{C}\,)$$

$$\mathbf{Y}(\cdot, \lambda) \quad \text{and} \quad (\mathbf{P}\mathbf{Y}')(\cdot, \lambda) \in AC_{\text{loc}}\, \langle \mathbf{a}, \mathbf{b} \rangle \quad (\, \lambda \in \mathbb{C}\,)$$

$$\mathbf{Y}(x, \cdot) \quad \text{and} \quad (\mathbf{P}\mathbf{Y}')(x, \cdot) \in \mathbf{H}(\mathbb{C}) \quad (\, x \in \langle \mathbf{a}, \mathbf{b} \rangle\,).$$

2. The choice of the interval (a_r, b_r), and hence the point c, to initiate the solution is convenient for the notation but is arbitrary; without loss of generality we can assume that $r = 0$ and that $c \in (a_0, b_0)$.

Proof. The proof is by countable inductive construction.

¿From Theorem 1 construct the elements y_0 and $p_0 y_0' : (a_0, b_0) \times \mathbb{C} \to \mathbb{C}$ as the unique solution of $M_0[y] = \lambda w y$ on (a_0, b_0) with the initial conditions (10.5).

We show now how the component y_1 of $\mathbf{Y}$ can be defined on (a_1, b_1) with the required properties. The same method gives the definition and properties of y_{-1} on (a_{-1}, b_{-1}). The proof is then completed by induction on $\mathbb{Z}$ in both positive and negative directions.

The function $y_0(\cdot, \lambda) \in \Delta_0$ for all $\lambda \in \mathbb{C}$; hence from the definition (10.2) the limit $\mathbb{Y}_0(b_0^-, \lambda)$ exists in $\mathbb{C}$. Define the functions $\zeta_1, \zeta_2 : \mathbb{C} \to \mathbb{C}$ by

$$\zeta_1(\lambda) := [y_0(\cdot, \lambda), \gamma_0(\cdot)](b_0^-) \quad \text{and} \quad \zeta_2(\lambda) := [y_0(\cdot, \lambda), \delta_0(\cdot)](b_0^-) \quad (\, \lambda \in \mathbb{C}\,);$$

from [8, Sections 3 and 6], or see Lemma 1 in Section 4 above, it follows that $\zeta_1, \zeta_2 \in \mathbf{H}(\mathbb{C})$.

To make good the interface condition (10.3) with $n = 0$ it is necessary to determine the component y_1 so that $\mathbb{Y}_1(a_1^+, \lambda) = \exp(i\tau_0)\mathbf{K}_0 \mathbb{Y}_0(b_0^-, \lambda)$, for all $\lambda \in \mathbb{C}$. Writing $\mathbf{K}_0 = [k_{rs}]$ this vector equation is equivalent to requiring

$$[y_1(\cdot, \lambda), \gamma_1(\cdot)](a_1^+) = \exp(i\tau_0)\,(k_{11}\zeta_1(\lambda) + k_{12}\zeta_2(\lambda)) \quad (\, \lambda \in \mathbb{C}\,) \tag{10.9}$$

$$[y_1(\cdot, \lambda), \delta_1(\cdot)](a_1^+) = \exp(i\tau_0)\,(k_{21}\zeta_1(\lambda) + k_{22}\zeta_2(\lambda)) \quad (\, \lambda \in \mathbb{C}\,). \tag{10.10}$$

The right-hand sides of (10.9) and (10.10) represent functions that belong to the class $\mathbf{H}(\mathbb{C})$.

The existence of the component y_1 with all the required properties now follows from the generalized initial value result given in Theorem 2.

Clearly the initial solutions $\{y_0\}$ form a two-dimensional linear manifold and this continues for the generalized solutions $\{\mathbf{Y}\}$.

This completes the proof. ■

11 The generalized Green's formula

The Green's formula for the single interval case is given in Section 4 above.

The formula in the generalized case takes a similar form but this result is essentially due to the special canonical form of the interface boundary conditions given in (10.3).

The domain $\mathbf{D}(\mathbf{M},\mathbf{W})$ of the generalized differential expression $\mathbf{M}$ and the weight $\mathbf{W}$, see (9.1) and (9.2), may be defined in the form

$$\mathbf{D}(\mathbf{M},\mathbf{W}) := \{\mathbf{F} = \{f_n : n \in \mathbb{Z}\} : f_n \in \Delta_n \ (n \in \mathbb{Z})\}\,; \tag{11.1}$$

see also (8.1) and (8.2).

When the interface boundary conditions are introduced it is necessary to work with a sub-domain of $\mathbf{D}(\mathbf{M},\mathbf{W})$; this sub-domain depends upon the choice of the two sequences $\{\tau_n : n \in \mathbb{Z}\}$ and $\{\mathbf{K}_n : n \in \mathbb{Z}\}$ introduced in Sections 6 and 10 above. For this reason the notation for this domain is written as $\mathbf{D}(\mathbf{M},\mathbf{W} : \tau,\mathbf{K})$. To define this domain the interface boundary condition (10.3) is involved but now applied to the general vector $\mathbf{F}$ instead of the solution vector $\mathbf{Y}$ (see (6.3) for the definition of $\mathbb{F}$ but now extended to $\mathbb{F}_n$):

$$\mathbf{D}(\mathbf{M},\mathbf{W} : \tau,\mathbf{K}) := \left\{\mathbf{F} \in \mathbf{D}(\mathbf{M},\mathbf{W}) : \mathbb{F}_{n+1}(a_{n+1}^+) = \exp(i\tau_n)\mathbf{K}_n\mathbb{F}_n(b_n^-) \quad (n \in \mathbb{Z})\right\}. \tag{11.2}$$

The generalized Green's formula for $\mathbf{M}$ and the domain $\mathbf{D}(\mathbf{M},\mathbf{W} : \tau,\mathbf{K})$ is given in the form of

Lemma 3 *Given all the conditions of Theorem 3 above; let $m, n \in \mathbb{Z}$ with $n < m$; let $\alpha \in (a_n, b_n)$ and $\beta \in (a_m, b_m)$; for all $\mathbf{F},\mathbf{G} \in \mathbf{D}(\mathbf{M},\mathbf{W})$ define the integral*

$$\begin{aligned}\int_\alpha^\beta \left\{\bar{\mathbf{G}}\mathbf{M}(\mathbf{F}) - \mathbf{F}\bar{\mathbf{M}}(\mathbf{G})\right\} := &\int_\alpha^{b_n} \left\{\bar{g}_n M_n[f_n] - f_n \bar{M}_n[g_n]\right\} \\ &+ \sum_{r=n+1}^{m-1} \int_{a_r}^{b_r} \left\{\bar{g}_r M_r[f_r] - f_r \bar{M}_r[g_r]\right\} \\ &+ \int_{a_m}^{\beta} \left\{\bar{g}_m M_m[f_m] - f_m \bar{M}_m[g_m]\right\}. \end{aligned} \tag{11.3}$$

Then for all $\mathbf{F},\mathbf{G} \in \mathbf{D}(\mathbf{M},\mathbf{W} : \tau,\mathbf{K})$

$$\int_\alpha^\beta \left\{\bar{\mathbf{G}}\mathbf{M}(\mathbf{F}) - \mathbf{F}\bar{\mathbf{M}}(\mathbf{G})\right\} = [f_m, g_m]_m(\beta) - [f_n, g_n]_n(\alpha). \tag{11.4}$$

Proof. It is at this point that the interface boundary conditions applied in the definition (11.2) are essential; the result (11.4) holds on the domain $\mathbf{D}(\mathbf{M},\mathbf{W} : \tau,\mathbf{K})$ but not, in general, on $\mathbf{D}(\mathbf{M},\mathbf{W})$.

Using the Green's formula (4.3) on the separate intervals involved in (11.3) we find in the notation of Section 4, and on taking the end-points limits as required

$$\begin{aligned}\int_\alpha^\beta \left\{\bar{\mathbf{G}}\mathbf{M}(\mathbf{F}) - \mathbf{F}\bar{\mathbf{M}}(\mathbf{G})\right\} &= [f_n, g_n]_n(b_n^-) - [f_n, g_n]_n(\alpha) \\ &\quad + \sum_{r=n+1}^{m-1} \left\{[f_r, g_r]_r(b_r^-) - [f_r, g_r]_r(a_r^+)\right\} \\ &\quad + [f_m, g_m]_m(\beta) - [f_m, g_m]_m(a_m^+) \\ &= [f_m, g_m]_m(\beta) - [f_n, g_n]_n(\alpha) + \sum \text{ (say)} \end{aligned} \tag{11.5}$$

where

$$\sum = \sum_{r=n}^{m-1} \left\{[f_r, g_r]_r(b_r^-) - [f_{r+1}, g_{r+1}]_{r+1}(a_{r+1}^+)\right\}.$$

¿From the results given in [2, Lemma 2.1] it follows that the interface boundary condition, see (11.2),

$$\mathbb{F}_{r+1}(a_{r+1}^{+}) = \exp(i\tau_r)\mathbf{K}_r\mathbb{F}_r(b_r^{-}) \quad (r \in \mathbb{Z}) \tag{11.6}$$

imply that

$$[f_{r+1}, g_{r+1}]_{r+1}(a_{r+1}^{+}) = [f_r, g_r]_r(b_r^{-}) \quad (r \in \mathbb{Z}).$$

In turn this implies that $\sum$ of (11.5) is zero and the required result now follows. ■

Remarks 1. It is possible to extend (11.4) to include the end-points $\{a_n\}$ and $\{b_n\}$ by taking limits with α and β.

2. It is shown in [2] that the conditions (11.6) are the canonical form of the most general interface conditions to ensure that the Green's formula takes the form (11.4), that is to ensure that all the intermediate terms cancel out to zero.

12 Differential operators in $\mathbf{L}^2(\langle \mathbf{a}, \mathbf{b} \rangle : \mathbf{W})$

The general theory of differential operators generated by Lagrange symmetric (formally self-adjoint) quasi-differential expressions on single intervals is considered in [12, Chapters V and VI] and [15, Chapters], and for sets of intervals in [9] and [10]. The ideas used in these works are applied here to generate differential operators from the formal expression $\mathbf{M}$ in the Hilbert function space $\mathbf{L}^2(\langle \mathbf{a}, \mathbf{b} \rangle : \mathbf{W})$.

Suppose given all the data in Theorem 3 of Section 10 above. The space $\mathbf{L}^2(\langle \mathbf{a}, \mathbf{b} \rangle : \mathbf{W})$, the sequences $\mathbf{P}, \mathbf{Q}, \mathbf{W}$ and the differential expression $\mathbf{M}$ are all defined in Section 9 above; the domains $\mathbf{D}(\mathbf{M}, \mathbf{W})$ and $\mathbf{D}(\mathbf{M}, \mathbf{W} : \tau, \mathbf{K})$ in Section 11. Define the expression $\mathbf{W}^{-1}\mathbf{M}$ by

$$\mathbf{W}^{-1}\mathbf{M}[\mathbf{F}] := \{w_n^{-1}M_n[f_n] : n \in \mathbb{Z}\}.$$

Given the set $\{\mathbf{M}, \mathbf{W}, \tau, \mathbf{K}\}$ the maximal domain $\mathbf{\Delta}$ is defined by

$$\mathbf{\Delta} := \{\mathbf{F} \in \mathbf{D}(\mathbf{M}, \mathbf{W} : \tau, \mathbf{K}) : \mathbf{F}, \mathbf{W}^{-1}\mathbf{M}[\mathbf{F}] \in \mathbf{L}^2(\langle \mathbf{a}, \mathbf{b} \rangle : \mathbf{W})\} \tag{12.1}$$

and the maximal differential operator $\mathbf{T}_1 : \mathbf{D}(\mathbf{T}_1) \to \mathbf{L}^2(\langle \mathbf{a}, \mathbf{b} \rangle : \mathbf{W})$ by

$$\mathbf{D}(\mathbf{T}_1) := \mathbf{\Delta} \tag{12.2}$$

and

$$\mathbf{T}_1\mathbf{F} := \mathbf{W}^{-1}\mathbf{M}[\mathbf{F}] \quad (\mathbf{F} \in \mathbf{D}(\mathbf{T}_1)). \tag{12.3}$$

The corresponding minimal operator $\mathbf{T}_0$ is given by (for the sequence $\{c_n : n \in \mathbb{Z}\}$ see (8.8) and (8.9))

$$\mathbf{D}(\mathbf{T}_0) := \left\{ \mathbf{F} \in \mathbf{D}(\mathbf{T}_1) : \lim_{n \to \pm\infty} [f_n, g_n]_n(c_n) = 0 \; (\mathbf{G} \in \mathbf{D}(\mathbf{T}_1)) \right\} \tag{12.4}$$

The results given in the following theorem are similar to the corresponding results for the single interval case:

Theorem 4 *Let all the conditions of Theorem 3 of Section 10 hold; let the operators* $\mathbf{T}_0$ *and* $\mathbf{T}_1$ *be defined by* (12.1) *to* (12.4). *Then*

1. $\mathbf{D}(\mathbf{T}_0)$ *is a dense linear manifold of* $\mathbf{L}^2(\langle \mathbf{a}, \mathbf{b} \rangle : \mathbf{W})$

2. $\mathbf{T}_0$ *is closed and symmetric in* $\mathbf{L}^2(\langle \mathbf{a}, \mathbf{b} \rangle : \mathbf{W})$

3. $\mathbf{T}_1$ *is closed in* $\mathbf{L}^2(\langle \mathbf{a}, \mathbf{b} \rangle : \mathbf{W})$

4. $\mathbf{T}_0^* = \mathbf{T}_1$ *and* $\mathbf{T}_1^* = \mathbf{T}_0$.

Proof. The proof of these results follows the same lines as in [12, Chapter V] with the additional methods given in [10, Sections 3 and 4]. The details are omitted from this paper. ■

13 The Weyl classification and the Titchmarsh solutions in $\mathbf{L}^2(\langle \mathbf{a}, \mathbf{b} \rangle : \mathbf{W})$

The Titchmarsh solutions for the single interval case are given in Section 3 above. The corresponding solutions for the generalized case are defined as follows (as in the case of Theorem 3 these solutions are given, for convenience only, initial values at the point c_0 in the interval (a_0, b_0)):

Definition 2 *Let the pair* $(\mathbf{M}, \mathbf{W})$ *with intervals* $\langle \mathbf{a}, \mathbf{b} \rangle$ *and domain* $\mathbf{D}(\mathbf{M}, \mathbf{W} : \tau, \mathbf{K})$ *be given; let the parameter* $\alpha \in [0, \pi)$*; then* $\mathbf{\Theta}_\alpha \equiv \{\theta_{\alpha,n} : n \in \mathbb{Z}\}$ *and* $\mathbf{\Phi}_\alpha \equiv \{\varphi_{\alpha,n} : n \in \mathbb{Z}\}$ *are generalised solutions of the differential equation* $\mathbf{M}[\mathbf{Y}] = \lambda \mathbf{W} \mathbf{Y}$ *on* $\langle \mathbf{a}, \mathbf{b} \rangle$ *determined by the initial conditions at the point* c_0*, for all* $\lambda \in \mathbb{C}$,

$$\begin{aligned} \theta_{\alpha,0}(c_0, \lambda) = \cos(\alpha) \quad (p_0 \theta'_{\alpha,0})(c_0, \lambda) = -\sin(\alpha) \\ \varphi_{\alpha,0}(c_0, \lambda) = \sin(\alpha) \quad (p_0 \varphi'_{\alpha,0})(c_0, \lambda) = \cos(\alpha) \end{aligned} \tag{13.1}$$

and then extended to the set $\langle \mathbf{a}, \mathbf{b} \rangle$ *as in Theorem* 3 *of Section* 10.

The pair

$$\{\mathbf{\Theta}_\alpha, \mathbf{\Phi}_\alpha\}, \tag{13.2}$$

for each $\alpha \in [0, \pi)$, forms a basis for all solutions of the second-order generalised differential equation $\mathbf{M}[\mathbf{Y}] = \lambda \mathbf{W} \mathbf{Y}$ on $\langle \mathbf{a}, \mathbf{b} \rangle$ subject to the interface boundary conditions determined by the pair $\{\tau, \mathbf{K}\}$.

Let $\lambda = \mu \in \mathbb{R}$; it should be noted that although the initial component $\varphi_0(\cdot, \mu)$ is real-valued on (a_0, b_0) in general any other component $\varphi_n(., \mu)$, with $n \neq 0$, may well be complex-valued on (a_n, b_n) due to the factors $\{\exp(i\tau_n)\}$ in the interface boundary conditions.

The classical theory of the Weyl dichotomy to give the limit-point(LP)/limit-circle(LC) classification at the two end-points $\mathbf{a}$ and $\mathbf{b}$ now extends to the generalized differential equation (9.1), but now with the addition of the interface parameters $\{\tau_n, \mathbf{K}_n : n \in \mathbb{Z}\}$. It is necessary to recognise the rôle played by the interface parameters since the existence of solutions of (9.1), holomorphic in the spectral parameter λ, depends on the interface parameters.

For the theory of the Weyl dichotomy for the single interval case see [5, Chapter 9], [14, Chapter II] and [15]. With the availability of the generalized Green's formula (11.4) this theory extends, with suitable adjustments, to the generalised case. Thus we state without proof

Theorem 5 (Weyl) *Given the set* $\{\mathbf{M}, \mathbf{W}, \tau, \mathbf{K}\}$ *on* $\langle \mathbf{a}, \mathbf{b} \rangle$, *let all the conditions of Theorem* 3 *be satisfied; let the solutions* $\mathbf{\Theta}_\alpha$ *and* $\mathbf{\Phi}_\alpha$ *be defined by the initial conditions* (13.1)*.*

Then for all $\alpha \in [0, \pi)$ ***either***

1. $\mathbf{b}$ $(\mathbf{a})$ *is in the **limit-point** case and*

$$\mathbf{\Theta}_\alpha(\cdot,\lambda) \text{ and } \mathbf{\Phi}_\alpha(\cdot,\lambda) \notin \mathbf{L}^2([c_0,\mathbf{b}\rangle : \mathbf{W}) \quad (\,\lambda \in \mathbb{C}\setminus\mathbb{R}\,) \tag{13.3}$$

$$\big(\mathbf{\Theta}_\alpha(\cdot,\lambda) \text{ and } \mathbf{\Phi}_\alpha(\cdot,\lambda) \notin \mathbf{L}^2(\langle\mathbf{a},c_0] : \mathbf{W}) \quad (\,\lambda \in \mathbb{C}\setminus\mathbb{R}\,)\big)$$

2. ***or*** $\mathbf{b}$ $(\mathbf{a})$ *is in the **limit-circle** case and*

$$\mathbf{\Theta}_\alpha(\cdot,\lambda) \text{ and } \mathbf{\Phi}_\alpha(\cdot,\lambda) \in \mathbf{L}^2([c_0,\mathbf{b}\rangle : \mathbf{W}) \quad (\,\lambda \in \mathbb{C}\,) \tag{13.4}$$

$$\big(\mathbf{\Theta}_\alpha(\cdot,\lambda) \text{ and } \mathbf{\Phi}_\alpha(\cdot,\lambda) \in \mathbf{L}^2(\langle\mathbf{a},c_0] : \mathbf{W}) \quad (\,\lambda \in \mathbb{C}\,)\big)\,.$$

Remark As in the classical case the LP/LC classification is independent of the spectral parameter λ. However the classification does depend upon:

1. The set of intervals $\{\,(a_n,b_n) : n \in \mathbb{Z}\,\}$
2. The set of coefficients $\{\mathbf{P},\mathbf{Q},\mathbf{W}\}$
3. The set of interface parameters $\{\tau_n,\mathbf{K}_n : n \in \mathbb{Z}\,\}$

There are some results concerning the dependence of the LP/LC classification on these parameters; see Section 17 below and the results in [3] and [4].

14 The m-coefficient

The theory of the Titchmarsh-Weyl m-coefficient for the single interval case is developed in [14, Chapters II and III] and in [5, Chapter 9]; there is additional information in [7]. As in the LP/LC classification this theory of the m-coefficient extends to the generalized case and again we state the existence theorem without proof.

It is convenient to introduce the notation

$$\mathbb{C}_\pm := \{\,\lambda \in \mathbb{C} : \operatorname{Im}(\lambda) \gtrless 0\}$$

for the upper and lower half-planes of $\mathbb{C}$.

Theorem 6 (Titchmarsh) *Given the set* $\{\mathbf{M},\mathbf{W} : \tau,\mathbf{K}\}$ *on* $\langle\mathbf{a},\mathbf{b}\rangle$, *let all the conditions of Theorem 3 be satisfied; let the solutions* $\mathbf{\Theta}_\alpha$ *and* $\mathbf{\Phi}_\alpha$ *be defined by the initial conditions* (13.1). *Then for all* $\alpha \in [0,\pi)$

1. *If* $\mathbf{b}$ $(\mathbf{a})$ *is in the limit-point case then there exists a unique analytic function* $m_\alpha(\cdot) : \mathbb{C}_+ \cup \mathbb{C}_- \to \mathbb{C}$ $\;(n_\alpha : \mathbb{C}_+ \cup \mathbb{C}_- \to \mathbb{C})$ *such that*

$$m_\alpha \in \mathbf{H}(\mathbb{C}_+ \cup \mathbb{C}_-) \quad (n_\alpha \in \mathbf{H}(\mathbb{C}_+ \cup \mathbb{C}_-)) \tag{14.1}$$

and

$$\mathbf{\Theta}_\alpha(\cdot,\lambda) + m_\alpha(\lambda)\mathbf{\Phi}_\alpha(\cdot,\lambda) \in \mathbf{L}^2([c_0,\mathbf{b}\rangle : \mathbf{W}) \quad (\,\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-) \tag{14.2}$$

$$\big(\mathbf{\Theta}_\alpha(\cdot,\lambda) + n_\alpha(\lambda)\mathbf{\Phi}_\alpha(\cdot,\lambda) \in \mathbf{L}^2(\langle\mathbf{a},c_0] : \mathbf{W}) \quad (\,\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-)\big)$$

and

$$\int_{c_0}^{\mathbf{b}} \mathbf{W}(\cdot)\,|\mathbf{\Theta}_\alpha(\cdot,\lambda) + m_\alpha(\lambda)\mathbf{\Phi}_\alpha(\cdot,\lambda)|^2 = \frac{\operatorname{Im}(m_\alpha(\lambda))}{\operatorname{Im}(\lambda)} \quad (\,\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-\,) \tag{14.3}$$

$$\left(\int_{\mathbf{a}}^{\mathbf{c_0}} \mathbf{W}(\cdot)\,|\mathbf{\Theta}_\alpha(\cdot,\lambda) + n_\alpha(\lambda)\mathbf{\Phi}_\alpha(\cdot,\lambda)|^2 = -\frac{\operatorname{Im}(n_\alpha(\lambda))}{\operatorname{Im}(\lambda)} \quad (\,\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-\,)\right).$$

2. If **b** (**a**) *is in the limit-circle case then exists a family of analytic functions*

$$\{m_{\alpha,\kappa}(\cdot) : \kappa \in [0,\pi)\} \quad (\{n_{\alpha,\kappa}(\cdot) : \kappa \in [0,\pi)\})$$

with

$$m_{\alpha,\kappa}(\cdot) : \mathbb{C}_+ \cup \mathbb{C}_- \to \mathbb{C} \ (\kappa \in [0,\pi)) \quad (n_{\alpha,\kappa}(\cdot) : \mathbb{C}_+ \cup \mathbb{C}_- \to \mathbb{C} \ (\kappa \in [0,\pi))) \tag{14.4}$$

such that for all $\kappa \in [0,\pi)$

$$m_{\alpha,\kappa} \in \mathbf{H}(\mathbb{C}_+ \cup \mathbb{C}_-) \quad (n_{\alpha,\kappa} \in \mathbf{H}(\mathbb{C}_+ \cup \mathbb{C}_-)) \tag{14.5}$$

and

$$\boldsymbol{\Theta}_\alpha(\cdot,\lambda) + m_{\alpha,\kappa}(\lambda)\boldsymbol{\Phi}_\alpha(\cdot,\lambda) \in \mathbf{L}^2([c_0,\mathbf{b}\rangle : \mathbf{W}) \quad (\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-) \tag{14.6}$$

$$\big(\boldsymbol{\Theta}_\alpha(\cdot,\lambda) + n_{\alpha,\kappa}(\lambda)\boldsymbol{\Phi}_\alpha(\cdot,\lambda) \in \mathbf{L}^2(\langle\mathbf{a},c_0] : \mathbf{W}) \quad (\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-)\big)$$

and

$$\int_{c_0}^{\mathbf{b}} \mathbf{W}(\cdot)\,|\boldsymbol{\Theta}_\alpha(\cdot,\lambda) + m_{\alpha,\kappa}(\lambda)\boldsymbol{\Phi}_\alpha(\cdot,\lambda)|^2 = \frac{\operatorname{Im}(m_{\alpha,\kappa}(\lambda))}{\operatorname{Im}(\lambda)} \quad (\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-) \tag{14.7}$$

$$\left(\int_{\mathbf{a}}^{\mathbf{c_0}} \mathbf{W}(\cdot)\,|\boldsymbol{\Theta}_\alpha(\cdot,\lambda) + n_{\alpha,\kappa}(\lambda)\boldsymbol{\Phi}_\alpha(\cdot,\lambda)|^2 = -\frac{\operatorname{Im}(n_{\alpha,\kappa}(\lambda))}{\operatorname{Im}(\lambda)} \quad (\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-)\right).$$

Remarks 1. In the limit-circle case the parameter κ is connected with the Weyl circle; for details see [7, Section 10] where, however, in the notation used τ replaces κ.

2. The positive and negative signs on the right-hand sides of the integral terms involving the end-points **b** and **a** respectively, represent a sign change from the notation used in [14, Chapter II].

There is a corollary to Theorem 6 to record the result that these Titchmarsh-Weyl m-coefficients (the negative of the n-coefficients) are Nevanlinna (Herglotz, Pick, Riesz) functions; this result leads to integral representations and connections with the spectral properties of the associated self-adjoint operators in the spaces $\mathbf{L}^2([c_0,\mathbf{b}\rangle : \mathbf{W})$ ($\mathbf{L}^2(\langle\mathbf{a},c_0] : \mathbf{W})$).

Corollary 1 *¿From the results of Theorem 6 it follows that, for all* $\kappa \in [0,\pi)$ *as applicable,*

$$m_\alpha \text{ and } m_{\alpha,\kappa} : \mathbb{C}_\pm \to \mathbb{C}_\pm$$

$$-n_\alpha \text{ and } -n_{\alpha,\kappa} : \mathbb{C}_\pm \to \mathbb{C}_\pm.$$

Proof. Clear. ■

Corollary 2 *Let the set* $\{\mathbf{M},\mathbf{W} : \tau,\mathbf{K}\}$ *be either limit-point or limit-circle at* **a**; *similarly, and independently, at* **b**. *Let* $\boldsymbol{\Theta}_\alpha + n_\alpha\boldsymbol{\Phi}_\alpha$ *be **any** Titchmarsh solution at* **a**; *let* $\boldsymbol{\Theta}_\alpha + m_\alpha\boldsymbol{\Phi}_\alpha$ *be **any** Titchmarsh solution at* **b**. *Then for all* $\lambda \in \mathbb{C}\setminus\mathbb{R}$ *the pair*

$$\{\boldsymbol{\Theta}_\alpha + n_\alpha\boldsymbol{\Phi}_\alpha, \boldsymbol{\Theta}_\alpha + m_\alpha\boldsymbol{\Phi}_\alpha\}$$

is linearly independent and a solution basis for the generalized differential equation

$$\mathbf{M}[\mathbf{Y}] = \lambda\mathbf{W}\mathbf{Y} \quad \text{on} \quad \langle\mathbf{a},\mathbf{b}\rangle\,.$$

Proof. We need only prove that this pair of solutions has linearly independent initial conditions at the point $c_0 \in (a_0,b_0)$, see (13.1), since then the result follows from Theorem 2.

A calculation shows that the classical Wronskian of the pair of solutions at c_0 takes the value

$$n_\alpha - m_\alpha$$

and this term is not zero on $\mathbb{C}\setminus\mathbb{R}$ from Corollary 1 above. ■

15 Self-adjoint operators in $\mathbf{L}^2(\langle\mathbf{a},\mathbf{b}\rangle:\mathbf{W})$

For the set $\{\mathbf{M},\mathbf{W}:\tau,\mathbf{K}\}$ the minimal and maximal operators, $\mathbf{T}_0$ and $\mathbf{T}_1$ with $\mathbf{T}_0\subseteq\mathbf{T}_1$, in the space $\mathbf{L}^2(\langle\mathbf{a},\mathbf{b}\rangle:\mathbf{W})$ are defined in Section 12, with $\mathbf{T}_0^*=\mathbf{T}_1$ and $\mathbf{T}_1^*=\mathbf{T}_0$.

The deficiency indices (d^+,d^-) of the closed symmetric operator in $\mathbf{L}^2(\langle\mathbf{a},\mathbf{b}\rangle:\mathbf{W})$ are given by the dimension of the eigenspaces of the adjoint operator T_0^* at the points $\pm i\in\mathbb{C}$ respectively. From Theorem 4 and (12.3) we obtain

$$d^{\pm}:=\dim\{\mathbf{Y}\in\mathbf{D}(\mathbf{T}_1):\mathbf{M}[\mathbf{Y}]=\pm i\mathbf{W}\mathbf{Y}\quad\text{on}\quad\langle\mathbf{a},\mathbf{b}\rangle\};\tag{15.1}$$

thus from Theorem 2 we find that

$$d^{\pm}\in\{0,1,2\}.\tag{15.2}$$

In the classical case the real-valued form of the coefficients $\{p,q,w\}$ on (a,b) implies that, in addition to (15.2), we have $d^+=d^-$. In this the generalized case the fact that the interface boundary conditions are complex-valued prevents an application of the same argument. However we have

Theorem 7 *Let the set* $\{\mathbf{M},\mathbf{W}:\tau,\mathbf{K}\}$ *be given on* $\langle\mathbf{a},\mathbf{b}\rangle$ *; let the definition of LP/LC end-points be given as in Section 13 above. Let the deficiency indices* $\{d^+,d^-\}$ *of* $\mathbf{T}_0$ *be defined as in* (15.1)*. Then*

$$d^+=d^-=d\ (say)$$

and:

1. $d=0$ *if and only if LP at* $\mathbf{a}$ *and LP at* $\mathbf{b}$
2. $d=1$ *if and only if* ***either*** *LP at* $\mathbf{a}$ *and LC at* $\mathbf{b}$ ***or*** *LC at* $\mathbf{a}$ *and LP at* $\mathbf{b}$
3. $d=2$ *if and only if LC at* $\mathbf{a}$ *and LC at* $\mathbf{b}$.

Proof. The proof is numbered respectively as in the statement of the Theorem.

1. For $\lambda=\pm i$ the only $\mathbf{L}^2(\langle\mathbf{a},\mathbf{b}\rangle:\mathbf{W})$ solutions near $\mathbf{a}$ and $\mathbf{b}$ are linearly independent upon the Titchmarsh solutions given in (14.2); these two solutions are linearly independent on $\langle\mathbf{a},\mathbf{b}\rangle$ and this implies that the eigenspaces of $\mathbf{T}_0^*=\mathbf{T}_1$ at $\pm i$ are null; hence $d^+=d^-=0$.

2. Suppose LP at $\mathbf{a}$ and LC at $\mathbf{b}$ then for $\lambda=\pm i$ the only solution in $\mathbf{L}^2(\langle\mathbf{a},\mathbf{b}\rangle:\mathbf{W})$ near $\mathbf{a}$ is the Titchmarsh solution (14.2); this solution can be continued to $\mathbf{b}$ in $\mathbf{L}^2(\langle\mathbf{a},\mathbf{b}\rangle:\mathbf{W})$ since the two linearly independent Titchmarsh solutions near $\mathbf{b}$ are both in $\mathbf{L}^2(\langle\mathbf{a},\mathbf{b}\rangle:\mathbf{W})$; thus for both $\lambda=\pm i$ there is only one linearly independent solution in $\mathbf{L}^2(\langle\mathbf{a},\mathbf{b}\rangle:\mathbf{W})$ and so $d^+=d^-=1$. There is a similar argument for the case with LC at $\mathbf{a}$ and LP at $\mathbf{b}$.

3. In this case all solutions of $\mathbf{M}[\mathbf{Y}]=\lambda\mathbf{W}\mathbf{Y}$ are in $\mathbf{L}^2(\langle\mathbf{a},\mathbf{b}\rangle:\mathbf{W})$ are so $d^+=d^-=2$. ■

The determination of self-adjoint extensions of $\mathbf{T}_0$ (equivalently self-adjoint restrictions of $\mathbf{T}_1$) depends upon the application of boundary conditions at the end-points $\mathbf{a}$ and $\mathbf{b}$. We consider here only separated boundary conditions, see Section 4 above, although it is quite possible to introduce coupled boundary conditions, see Section 6 above, but only in the case that $d = 2$.

Symbolically a separated boundary condition at $\mathbf{b}$ is written as

$$[\mathbf{F}, \mathbf{\Omega_b}](\mathbf{b}) = 0 \tag{15.3}$$

where, see Section 12, $\mathbf{F} \in \mathbf{\Delta} = \mathbf{D}(\mathbf{T}_1)$ and the boundary condition function $\mathbf{\Omega_b} \equiv \{\omega_n : n \in \mathbb{Z}\}$ has to satisfy

$$\mathbf{\Omega_b} \in \mathbf{D}(\mathbf{T}_1) \setminus \mathbf{D}(\mathbf{T}_0) \tag{15.4}$$

and

$$[\mathbf{\Omega_b}, \mathbf{\Omega_b}](\mathbf{b}) = 0. \tag{15.5}$$

The first of these two conditions is to ensure that if a boundary condition is required then it is linearly independent. The second condition is to ensure that the boundary condition is symmetric.

In these formulae a term such as $[\mathbf{F}, \mathbf{G}](\mathbf{b})$, with $\mathbf{F}, \mathbf{G} \in \mathbf{\Delta}$, is defined by

$$[\mathbf{F}, \mathbf{G}](\mathbf{b}) := \lim_{n \to \infty} [f_n(\cdot), g_n(\cdot)]_n(c_n) \tag{15.6}$$

where the sequence $\{c_n\}$ is defined in Section 8 above; see in particular (8.9). The limit in (15.6) exists and is finite in $\mathbb{C}$ from an application of the generalised Green's formula (11.4).These separated boundary conditions need only be applied when $\mathbf{b}$, equally well $\mathbf{a}$, is in the LC classification.

Note that the definition (15.6) implies:

Lemma 4 *Let* $\mathbf{F}, \mathbf{G} \in \mathbf{\Delta}$; *let* $\{x_n : n \in \mathbb{Z}\}$ *be any sequence of real numbers such that* $x_n \in [a_n^+, b_n^-]$ $(n \in \mathbb{Z})$. *Then*

$$\lim_{n \to +\infty} [f_n(\cdot), g_n(\cdot)]_n(x_n) = [\mathbf{F}, \mathbf{G}](\mathbf{b}) \text{ and } \lim_{n \to -\infty} [f_n(\cdot), g_n(\cdot)]_n(x_n) = [\mathbf{F}, \mathbf{G}](\mathbf{a}).$$

Proof. This follows from the condition $\mathbf{F}, \mathbf{G} \in \mathbf{\Delta} \subset \mathbf{L}^2$ and use of the Green's formula given in Lemma 3 above. ■

The reason for these remarks remark lies in an extension of Lemma 2 in Section 4; we state this extension but without proof as:

Lemma 5 *Let the set* $\{\mathbf{M}, \mathbf{W} : \tau, \mathbf{K}\}$ *be given on* $\langle \mathbf{a}, \mathbf{b} \rangle$; *let the generalised form*

$$[\cdot, \cdot](\mathbf{b}) : \mathbf{\Delta} \times \mathbf{\Delta} \to \mathbb{C}$$

be defined by (15.6)*; then*

1. *If* $\mathbf{b}$ *is limit-point in* $\mathbf{L}^2(\langle \mathbf{a}, \mathbf{b} \rangle : \mathbf{W})$ *then*

$$[\mathbf{F}, \mathbf{G}](\mathbf{b}) = 0 \quad (\mathbf{F}, \mathbf{G} \in \mathbf{\Delta})$$

2. *If* $\mathbf{b}$ *is limit-circle in* $\mathbf{L}^2(\langle \mathbf{a}, \mathbf{b} \rangle : \mathbf{W})$ *then there exist pairs* $\mathbf{F}, \mathbf{G} \in \mathbf{\Delta}$ *such that*

$$[\mathbf{F}, \mathbf{G}](\mathbf{b}) \neq 0 \quad (\mathbf{F}, \mathbf{G} \in \mathbf{\Delta}).$$

There is a similar result for the end-point $\mathbf{a}$.

Proof. This follows the method in [6], see also [14, Chapter II]. ■

To apply a separated boundary condition at $\mathbf{b}$, when the LC classification holds, define the family of boundary condition functions $\{\mathbf{\Omega}_{\mathbf{b}}^{\kappa}\}$ by

Definition 3 *Let the LC condition hold at* $\mathbf{b}$; *let the parameter* $\kappa \in [0,\pi)$; *let* $\mu_{\mathbf{b}} \in \mathbb{R}$; *for some* $\alpha \in [0,\pi)$ *take the basis* $\{\mathbf{\Theta}_\alpha, \mathbf{\Phi}_\alpha\}$ *as given in* (13.2)*; then define*

$$\mathbf{\Omega}_{\mathbf{b}}^{\kappa}(x) := \cos(\kappa)\mathbf{\Theta}_\alpha(x,\mu_{\mathbf{b}}) + \sin(\kappa)\mathbf{\Phi}_\alpha(x,\mu_{\mathbf{b}}) \quad (\, x \in \langle \mathbf{a}, \mathbf{b} \rangle \,). \tag{15.7}$$

When $\mathbf{a}$ *is LC there is a similar definition for a boundary condition function* $\mathbf{\Omega}_{\mathbf{a}}$ *at* $\mathbf{a}$; *let* $\kappa \in [0,\pi)$ *and let* $\mu_{\mathbf{a}} \in \mathbb{R}$; *define:*

$$\mathbf{\Omega}_{\mathbf{a}}^{\kappa}(x) := \cos(\kappa)\mathbf{\Theta}_\alpha(x,\mu_{\mathbf{a}}) + \sin(\kappa)\mathbf{\Phi}_\alpha(x,\mu_{\mathbf{a}}) \quad (\, x \in \langle \mathbf{a}, \mathbf{b} \rangle \,). \tag{15.8}$$

We have then:

Lemma 6 *Let the LC condition hold at both* $\mathbf{a}$ *and* $\mathbf{b}$; *let* $\mathbf{\Omega}_{\mathbf{a}}^{\kappa}$ *and* $\mathbf{\Omega}_{\mathbf{b}}^{\kappa}$ *be defined as above; then*

$$[\mathbf{\Omega}_{\mathbf{a}}^{\kappa}, \mathbf{\Omega}_{\mathbf{a}}^{\kappa}](\mathbf{a}) = 0 \quad \textit{and} \quad [\mathbf{\Omega}_{\mathbf{b}}^{\kappa}, \mathbf{\Omega}_{\mathbf{b}}^{\kappa}](\mathbf{b}) = 0.$$

Proof. Write, for short,

$$\mathbf{\Omega}_{\mathbf{b}}^{\kappa}(x,\mu_{\boldsymbol{\beta}}) = \{\omega_n(x,\mu)\}.$$

Consider the end-point $\mathbf{b}$; let $n \in \mathbb{N}$; from the generalized Green's formula (11.4) on the interval $[c_0, c_n]$;

$$[\omega_n, \omega_n]_n(c_n) - [\omega_0, \omega_0]_0(c_0) = \sum$$

where $\sum$ represents a sum of terms of the form, here α and β represent appropriate interval end-points,

$$\int_\alpha^\beta \{\bar{\omega}_r(\cdot,\mu) M_r[\{\omega_r(\cdot,\mu)] - \omega_r(\cdot,\mu)\bar{M}_r[\omega_r(\cdot,\mu)]\}$$

$$= (\mu - \bar{\mu}) \int_\alpha^\beta w_r \left|\omega_r(\cdot,\mu)\right|^2 = 0$$

since $\mu \in \mathbb{R}$; thus $\sum = 0$. The term $[\omega_0, \omega_0]_0(c_0) = 0$ since $\omega_0(\cdot,\mu)$ is real-valued on $[c_0, b_0]$. Thus

$$[\omega_n, \omega_n]_n(c_n) = 0 \quad (\, n \in \mathbb{N}\,)$$

and the required result follows.

There is a similar proof for $\mathbf{\Omega}_{\mathbf{a}}^{\kappa}$. ■

The LC set of boundary condition functions $\{\mathbf{\Omega}_{\mathbf{b}}^{\kappa} : \kappa \in [0,\pi)\}$ are independent in the sense of

Lemma 7 *Let the LC condition hold at* $\mathbf{b}$; *let* $\kappa_1, \kappa_2 \in [0,\pi)$ *with* $\kappa_1 \neq \kappa_2$; *then*

$$[\mathbf{\Omega}_{\mathbf{b}}^{\kappa_1}, \mathbf{\Omega}_{\mathbf{b}}^{\kappa_2}](\mathbf{b}) \neq \mathbf{0}. \tag{15.9}$$

Proof. ¿From the definition (15.6) a computation shows that

$$[\mathbf{\Omega}_{\mathbf{b}}^{\kappa_1},\mathbf{\Omega}_{\mathbf{b}}^{\kappa_2}](\mathbf{b}) = \sin(\kappa_2-\kappa_1) \neq 0$$

since $-\pi < \kappa_2 - \kappa_1 < \pi$ and $\kappa_2 - \kappa_1 \neq 0$. ■

There is a similar LC result for the set $\{\mathbf{\Omega}_{\mathbf{a}}^{\kappa} : \kappa \in [0,\pi)\}$.

We can now describe the collection of all self-adjoint extensions $\{\mathbf{S}\}$ of the minimal operator $\mathbf{T}_0$. Recall the connections between the deficiency index $d \in \{0,1,2\}$ and the LP/LC classification of the end-points $\mathbf{a}$ and $\mathbf{b}$.

Case 1. $d = 0$ LP at $\mathbf{a}$ and at $\mathbf{b}$.

No boundary conditions at $\mathbf{a}$ or $\mathbf{b}$ are required; there is only a unique self-adjoint extension $\mathbf{S}$ given by $\mathbf{S} = \mathbf{T}_0 = \mathbf{T}_1$ and

$$\mathbf{D}(\mathbf{S}) := \mathbf{D}(\mathbf{T}_1)$$

$$\mathbf{SF} := \mathbf{W}^{-1}\mathbf{M}[\mathbf{F}] \quad (\mathbf{F} \in \mathbf{D}(\mathbf{S})).$$

Case 2a. $d = 1$ LC at $\mathbf{a}$ and LP at $\mathbf{b}$.

One separated boundary condition at $\mathbf{a}$ and no boundary condition at $\mathbf{b}$; there is a continuum of self-adjoint extensions $\{\mathbf{S}_\kappa\}$ described by

$$\mathbf{D}(\mathbf{S}_\kappa) := \{\mathbf{F} \in \mathbf{D}(\mathbf{T}_1) : [\mathbf{F},\mathbf{\Omega}_{\mathbf{a}}^{\kappa}](\mathbf{a}) = 0\} \quad (\kappa \in [0,\pi))$$

$$\mathbf{S}_\kappa\mathbf{F} := \mathbf{W}^{-1}\mathbf{M}[\mathbf{F}] \quad (\mathbf{F} \in \mathbf{D}(\mathbf{S}_\kappa)) \quad (\kappa \in [0,\pi))$$

Case 2b. $d = 1$ LP at $\mathbf{a}$ and LC at $\mathbf{b}$.

This case is similar to case 2a with $\mathbf{a}$ and $\mathbf{b}$ interchanged.

Case 3. $d = 2$ LC at $\mathbf{a}$ and at $\mathbf{b}$.

Two separated boundary conditions, one each at $\mathbf{a}$ and $\mathbf{b}$; there is a continuum of self-adjoint extensions $\{\mathbf{S}_{\kappa-,\kappa+}\}$ described by, for κ_-, $\kappa_+ \in [0,\pi)$,

$$\mathbf{D}(\mathbf{S}_{\kappa-,\kappa+}) = \{\mathbf{F} \in \mathbf{D}(\mathbf{T}_1) : (i)\ [\mathbf{F},\mathbf{\Omega}_{\mathbf{a}}^{\kappa-}](\mathbf{a}) = 0\ (ii)\ [\mathbf{F},\mathbf{\Omega}_{\mathbf{b}}^{\kappa+}](\mathbf{b}) = 0\}$$

$$\mathbf{S}_{\kappa-,\kappa+}\mathbf{F} := \mathbf{W}^{-1}\mathbf{M}[\mathbf{F}] \quad (\mathbf{F} \in \mathbf{D}(\mathbf{S}_{\kappa-,\kappa+})).$$

The proof of self-adjointness in all these cases follows the proof as in the single interval case; see [12, Chapter V] and [10, Sections 1 to 3].

It is also possible to generate self-adjoint operators in the two Hilbert spaces $\mathbf{L}^2(\langle \mathbf{a}, c_0] : \mathbf{W})$ and $\mathbf{L}^2([c_0, \mathbf{b}\rangle : \mathbf{W})$. Here the point c_0 is a regular point for the differential equation

$$\mathbf{M}[\mathbf{Y}] = \lambda\mathbf{WY} \text{ on } \langle \mathbf{a}, c_0]$$

or $[c_0, \mathbf{b}\rangle$ and a boundary condition is essential at this point. Such a boundary condition can be applied, for some choice of $\alpha \in [0,\pi)$, using the solution component $\varphi_{\alpha,0}(\cdot,\mu)$ of $\mathbf{\Phi}_\alpha(\cdot,\mu)$ for some $\mu \in \mathbb{R}$; the condition then appears in the form, for $\mathbf{F} \in \mathbf{D}(\mathbf{M},\mathbf{W} : \tau,\mathbf{K})$,

$$[f_0,\varphi_{\alpha,0}]_0(c_0) = 0 \text{ or equivalently } f_0(c_0)\cos(\alpha) - (p_0 f_0')(c_0)\sin(\alpha) = 0.$$

The decision to use a boundary condition at the end-point $\mathbf{a}$ or $\mathbf{b}$ depends upon the LP/LC classification of these points in the relevant $\mathbf{L}^2$ space.

As an example consider the case when $\mathbf{b}$ is LC; then the domain of a self-adjoint operator, with a choice of $\alpha \in [0, \pi)$ and $\kappa \in [0, \pi)$, is given by

$$\mathbf{D}(\mathbf{S}_{\alpha,\kappa}) := \{\mathbf{D}(\mathbf{M}, \mathbf{W} : \tau, \mathbf{K}) \cap \mathbf{L}^2([c_0, \mathbf{b}\rangle) : [f_0, \varphi_{\alpha,0}]_0(c_0) = 0 \text{ and } [\mathbf{F}, \mathbf{\Omega}^\kappa_\mathbf{b}](\mathbf{b}) = 0\}.$$

The operator is defined by

$$\mathbf{S}_{\alpha,\kappa}\mathbf{F} := \mathbf{W}^{-1}\mathbf{M}[\mathbf{F}] \quad (\mathbf{F} \in \mathbf{D}(\mathbf{S}_{\alpha,\kappa})).$$

16 Interval boundary conditions

In this section we discuss the connection between the results in [4] and in this paper, within the more general framework of quasi-differential boundary value problems on countable sets of intervals of $\mathbb{R}$, given in [10].

Restricting the general order boundary-value problems considered in [10], to second-order Sturm-Liouville problems provides an environment into which the ideas of this paper can be embedded. The boundary conditions in [10] are more general than the interface conditions applied in this paper but then the solutions do not possess the "interface connectedness" seen in the results given in the previous Sections of this paper.

The notations in this Section call for some changes to be made to the notations originally adopted in the paper [10]; in particular, since this notation appears frequently below, the generalised bi-linear form $[\mathbf{f}, \mathbf{g}]$ defined in [10, (2.8)] is here replaced with the notation $[[\mathbf{F}, \mathbf{G}]]$.

Consider the application of [10, Theorem 3.1] to the system arising from the data and conditions given in Sections 8 and 9 above. On each interval (a_n, b_n) the pair (M_n, w_n) generates a minimal closed operator in $L^2((a_n, b_n) : w_n)$ with deficiency indices $d_n^+ = d_n^- = 2$; this follows for all $n \in \mathbb{Z}$ from the end-point classification condition (8.7). From [10, Corollary 2.4] the system $\{\, (M_n, w_n) : n \in \mathbb{Z}\,\}$ generates a minimal operator $\mathbf{T}_{\min}$, see [10, (2.7)], in the Hilbert space $\mathbf{L}^2 \equiv \sum_{n\in\mathbb{Z}} \oplus L_n^2$ (as discussed in (9.4)); $\mathbf{T}_{\min}$ has deficiency indices (d^+, d^-) given by, see [10, (2.16)],

$$d^+ = d^- = \sum_{n\in\mathbb{Z}} 2 = \aleph_0.$$

Thus to generate self-adjoint operators from this system in $\mathbf{L}^2$ the extended Glazman-Krein-Naimark (GKN) boundary conditions of [10, Theorem 3.1] have to be applied to give an extension of the operator $\mathbf{T}_{\min}$; these boundary conditions are applied to the domain of the maximal operator $\mathbf{T}_{\max}$ and take the form

$$[[\mathbf{f}, \mathbf{B}_n]] \equiv \sum_{r\in\mathbb{Z}} [f_r, B_{n,r}]_r = 0 \quad (\, n \in \mathbb{Z}\,); \tag{16.1}$$

here both $\mathbf{F} = \{f_r : r \in \mathbb{Z}\}$ and $\mathbf{B}_n = \{B_{n,r} : r \in \mathbb{Z}\} \in \mathbf{L}^2$, with f_r and $B_{n,r} \in D(T_{1,r})$ for all $r \in \mathbb{Z}$ (see (8.6) for the maximal operator $T_{1,r}$). In (16.1) the term $[\cdot, \cdot]_r$ is defined by

$$[f_r, g_r]_r := [f_r, g_r](b_r^-) - [f_r, g_r](a_r^+) \quad (\, f_r, g_r \in \Delta_r\,)$$

and the infinite series is absolutely convergent, see [10, Section 2].

The boundary conditions vectors $\{\mathbf{B}_n : n \in \mathbb{Z}\}$ (16.1) have to satisfy certain conditions given in [10, (3.2)]; of these conditions the two that are important here are:

(i) The *linear independence* condition

(ii) The *symmetry* condition:

$$[[\mathbf{B}_m, \mathbf{B}_n]] = 0 \quad (\, m, n \in \mathbb{Z}\,) \tag{16.2}$$

(iii) The *maximality* condition: the only vector $\mathbf{B}$ that satisfies

$$[[\mathbf{B}, \mathbf{B}_n]] = 0 \quad (\, n \in \mathbb{Z}\,) \tag{16.3}$$

is the null vector.

The domains for self-adjoint operators $\mathbf{S}$ are then defined by requiring

$$[[\mathbf{F}, \mathbf{B}_n]] = 0 \quad (\, n \in \mathbb{Z}\,). \tag{16.4}$$

We now give some details to show the self-adjoint operators $\mathbf{S}$ defined in Section 15 through the introduction of interface boundary conditions are special cases of self-adjoint operators resulting from the analysis in [10] through sequences $\{\mathbf{B}_n : n \in \mathbb{Z}\}$ of boundary condition functions satisfying (16.2) and (16.3).

This equivalence result throws light on how to construct sequences $\{\mathbf{B}_n\}$ of boundary condition functions that are and are not, maximal in the sense of (16.3).

The choice of elements of the sequence $\{\mathbf{B}_n\}$ falls into two parts:

1. To choose some of the elements so that all the interface boundary conditions, see (10.3) or (11.2), are represented as interval conditions of the form (16.4)
2. To choose the remaining elements, at most two are necessary, so that the LC boundary conditions, if required, at $\mathbf{a}$ and/or $\mathbf{b}$ are represented as interval conditions of the form (16.4).

For 1. these elements are constructed as follows:

The "patching lemma" of Naimark, see [12, Section 17.3, Lemma 2], is required at several stages; this enables the alteration of elements of the maximal domains $\{\Delta_n : n \in \mathbb{Z}\}$ on compact sub-intervals of $\{(a_n, b_n) : n \in \mathbb{Z}\}$, in order to separate the end-point effects of any element.

Consider,for $n \in \mathbb{Z}$, the interface condition for the elements of the domain $\mathbf{D}(\mathbf{M}, \mathbf{W} : \tau, \mathbf{K})$ given in (11.2), *i.e.*

$$\mathbb{F}(a_{n+1}^+) = \exp(i\tau_n)\mathbf{K}_n\mathbb{F}(b_n^-).$$

This represents the two linear equations, writing $\mathbf{K}_n = [k_{rs}(n)]$,

$$\begin{aligned} [f_{n+1}, \gamma_{n+1}](a_{n+1}^+) &= \exp(i\tau_n)\left(k_{11}(n)[f_n, \gamma_n](b_n^-) + k_{12}(n)[f_n, \delta_n](b_n^-)\right) \\ &= [f_n, \exp(-i\tau_n)\left(k_{11}(n)\gamma_n + k_{12}(n)\delta_n\right)](b_n^-) \end{aligned} \tag{16.5}$$

$$\begin{aligned} [f_{n+1}, \delta_{n+1}](a_{n+1}^+) &= \exp(i\tau_n)\left(k_{21}(n)[f_n, \gamma_n](b_n^-) + k_{22}(n)[f_n, \delta_n](b_n^-)\right) \\ &= [f_n, \exp(-i\tau_n)\left(k_{21}(n)\gamma_n + k_{22}(n)\delta_n\right)](b_n^-). \end{aligned} \tag{16.6}$$

For this given $n \in \mathbb{Z}$ and the end-points $b_n = a_{n+1}$ we construct the interval boundary condition sequences $\mathbf{B}_{n,1}^{\pm} = \{B_{n,1,r}^{\pm} : r \in \mathbb{Z}\}$ and $\mathbf{B}_{n,2}^{\pm} = \{B_{n,2,r}^{\pm} : r \in \mathbb{Z}\}$ as follows: choose $\varepsilon_n > 0$ so that

$$a_n < a_n + \varepsilon_n < b_n - \varepsilon_n < b_n$$

and then

$$B_{n,1,n}^{-} = \exp(-i\tau_n)\,(k_{11}(n)\gamma_n + k_{12}(n)\delta_n) \text{ on } [b_n - \varepsilon_n, b_n); \quad B_{n,1,n}^{+} = \gamma_{n+1} \text{ on } (a_{n+1}, a_{n+1} + \varepsilon_n]$$

$$B_{n,2,n}^{-} = \exp(-i\tau_n)\,(k_{21}(n)\gamma_n + k_{22}(n)\delta_n) \text{ on } [b_n - \varepsilon_n, b_n); \quad B_{n,2,n}^{+} = \delta_{n+1} \text{ on } (a_{n+1}, a_{n+1} + \varepsilon_n].$$

with all other elements of $\{B_{n,1,r}^{\pm}\}$ and $\{B_{n,2,r}^{\pm}\}$ set to zero, patching to ensure that $B_{n,s,n}^{-} \in D(T_{1,n})$ and $B_{n,s,n}^{+} \in D(T_{1,n+1})$ for $s = 1, 2$. It follows that the interface boundary conditions (16.5) and (16.6) are equivalent to, in the interval boundary condition notation of [10, (2.8) and Theorem 3.1],

$$[[\mathbf{F}, \mathbf{B}_{n,s}]] = 0 \quad (s = 1, 2); \tag{16.7}$$

here it is to be noted again that the symbol $[[\cdot, \cdot]]$ represents the generalised interval bilinear form defined by [10, (2.8)]. To conform to the requirements of [10, Theorem 3.1] it is necessary to establish the results

$$[[\mathbf{B}_{n,s}, \mathbf{B}_{n,t}]] = 0 \quad (\,s, t = 1, 2). \tag{16.8}$$

We have, see again [10, (2.8)],

$$[[\mathbf{B}_{n,1}, \mathbf{B}_{n,1}]] = [\exp(-i\tau_n)\,(k_{11}(n)\gamma_n + k_{12}(n)\delta_n)\,, \exp(-i\tau_n)\,(k_{11}(n)\gamma_n + k_{12}(n)\delta_n)]_n\,(b_n^-) -$$

$$-[\gamma_{n+1}, \gamma_{n+1}]_{n+1}(a_{n+1}^+)$$

and it may be seen that both terms on the right-hand side are zero. Again

$$[[\mathbf{B}_{n,1}, \mathbf{B}_{n,2}]] = [\exp(-i\tau_n)\,(k_{11}(n)\gamma_n + k_{12}(n)\delta_n)\,, \exp(-i\tau_n)\,(k_{21}(n)\gamma_n + k_{22}(n)\delta_n)]_n -$$

$$-[\gamma_{n+1}, \delta_{n+1}]_{n+1}(a_{n+1}^+);$$

in this a calculation shows that both the two terms on the on the right-hand side have the value one, and so the left-hand side takes the value zero as required.

Thus the required conditions (16.8) are satisfied.

We may now regard the two sequences of interval boundary condition functions $\{\mathbf{B}_{n,1} : n \in \mathbb{Z}\}$ and $\{\mathbf{B}_{n,2} : n \in \mathbb{Z}\}$ as well-defined in terms of the interface boundary condition functions $\{\gamma_n, \delta_n, \tau_n, \mathbf{K}_n : n \in \mathbb{Z}\}$;and that these conditions are equivalent. Moreover it is clear from the above calculations and the definition of the sequences $\{\mathbf{B}_{n,1} : n \in \mathbb{Z}\}$ and $\{\mathbf{B}_{n,2} : n \in \mathbb{Z}\}$ that these GKN boundary condition functions are linearly independent, and that the interval symmetry conditions [10, (3.1),(3)] are satisfied, *i.e.*

$$[[\mathbf{B}_{n,s}, \mathbf{B}_{n,t}]] = 0 \quad (\,r, s = 1, 2 \text{ and } n \in \mathbb{Z}). \tag{16.9}$$

The question to be asked now is 'are the symmetric GKN boundary condition functions $\{\mathbf{B}_{n,s} : n \in \mathbb{Z} \text{ and } s = 1, 2\}$ maximal in the defined sense of [10, (3.2)]'. In general the answer is in the negative since account has to be taken of the two singular end-points $\mathbf{a}$ and $\mathbf{b}$ of the interface problem. This then leads to the second of the two numbered remarks made above.

For 2. the elements have to be constructed as follows:

(i) if $\{\mathbf{M}, \mathbf{W} : \boldsymbol{\tau}, \mathbf{K}\}$ is LP at both $\mathbf{a}$ and $\mathbf{b}$ then no additional boundary conditions are required

(ii) if $\mathbf{b}$ is LC then a boundary condition of the form, see (15.3) and (15.5), has to be added with a boundary condition function $\mathbf{B}_{\mathbf{b}}^{\kappa} = \{\beta_n^{\kappa} : n \in \mathbb{Z}\}$, for some $\kappa \in [0, \pi)$, given by, with $\varepsilon > 0$ chosen so that $a_0 < a_0 + \varepsilon < b_0 - \varepsilon < b_0$ and $\beta_0^{\kappa}(\cdot)$ patched on (a_0, b_0),

$$\begin{aligned}
\beta_n^{\kappa}(x) &= 0 \quad (\, x \in (a_n, b_n) \quad \text{and} \quad n \in \mathbb{Z} \setminus \mathbb{N}_0) \\
&= 0 \quad (\, x \in (a_0, a_0 + \varepsilon]\,) \quad \text{and} \quad n = 0 \\
&= \cos(\kappa)\theta_{\alpha,0}(x, \mu) + \sin(\kappa)\varphi_{\alpha,0}(x, \mu) \quad (\, x \in [b_0 - \varepsilon, b_0)) \quad \text{and} \quad n = 0 \\
&= \cos(\kappa)\theta_{\alpha,n}(x, \mu) + \sin(\kappa)\varphi_{\alpha,n}(x, \mu) \quad (\, x \in (a_n, b_n) \quad \text{and} \quad n \in \mathbb{N}\,)
\end{aligned}$$

and the interface boundary condition

$$[\mathbf{F}, \mathbf{B}_{\mathbf{b}}^{\kappa}]\,(\mathbf{b}) \equiv \lim_{n \to \infty} [f_n(\cdot), \beta_n^{\kappa}(\cdot)]_n(c_n) = 0,$$

or equivalently the GKN boundary condition

$$[[\mathbf{F}, \mathbf{B}_{\mathbf{b}}^{\kappa}]] = 0$$

(iii) if $\mathbf{a}$ is LC then a boundary condition of the form has to be added with a boundary condition function $\mathbf{B}_{\mathbf{a}}^{\kappa} = \{\beta_n^{\kappa} : n \in \mathbb{Z}\}$, for some $\kappa \in [0, \pi)$, given by, with $\beta_0^{\kappa}(\cdot)$ patched on (a_0, b_0),

$$\begin{aligned}
\beta_n^{\kappa}(x) &= \cos(\kappa)\theta_{\alpha,n}(x, \mu) + \sin(\kappa)\varphi_{\alpha,n}(x, \mu) \quad (\, x \in (a_n, b_n) \quad \text{and} \quad n \in \mathbb{Z} \setminus \mathbb{N}_0) \\
&= \cos(\kappa)\theta_{\alpha,0}(x, \mu) + \sin(\kappa)\varphi_{\alpha,0}(x, \mu) \quad (\, x \in (a_0, a_0 + \varepsilon]\,)) \quad \text{and} \quad n = 0 \\
&= 0 \quad (\, x \in [b_0 - \varepsilon, b_0) \quad \text{and} \quad n = 0 \\
&= 0 \quad (\, x \in (a_n, b_n) \quad \text{and} \quad n \in \mathbb{N}\,)
\end{aligned}$$

and the interval boundary condition

$$[\mathbf{F}, \mathbf{B}_{\mathbf{a}}^{\kappa}](\mathbf{a}) \equiv \lim_{n \to -\infty} [f_n(\cdot), \beta_n^{\kappa}(\cdot)]_n(c_n) = 0,$$

or equivalently the GKN boundary condition

$$[[\mathbf{F}, \mathbf{B}_{\mathbf{a}}^{\kappa}]] = 0.$$

The equivalence of the interface and GKN boundary conditions stated in (ii) and (iii) above is proved by use of the Green's formula of Lemma 3 and the result given in Lemma 4.

Consider now the amalgamated set of GKN boundary condition functions, for some choice of $\kappa_1, \kappa_2 \in [0, \pi)$,

$$\{\, \mathbf{B}_{\mathbf{a}}^{\kappa_1}, \mathbf{B}_{\mathbf{b}}^{\kappa_2}, \mathbf{B}_{n,s}^{\pm} : n \in \mathbb{Z} \quad \text{and} \quad s \in \{1, 2\}\,\}. \tag{16.10}$$

An examination based on the definitions of these GKN functions and use of Lemma 6 and (16.9) above, shows the complete set (16.10) is symmetric in the definition of [10, (3.1) (3)]. The linear independence and the maximality of the set (16.10) depends upon the LP / LC classification of the set $\{\mathbf{M}, \mathbf{W} : \tau, \mathbf{K}\}$ in $\mathbf{L}^2 \equiv \mathbf{L}^2(\langle \mathbf{a}, \mathbf{b} \rangle : \mathbf{W})$ as follows:

1. If both **a** and **b** are LP then the operator S in H defined in [10, Theorem 3.1, (3.6)] using only the set
$$\{\, \mathbf{B}^{\pm}_{n,s} : n \in \mathbb{Z} \quad \text{and} \quad s \in \{1,2\} \,\} \tag{16.11}$$
is seen to identify with the operator $\mathbf{S}$ defined in $\mathbf{L}^2$ in Case 1 of Section15. Thus S is self-adjoint in H and so the set (16.11) is maximal in the sense of [10, (3.2)]; this set is also seen to be linearly independent in the sense of [10, (3.1) (2)].

2. If **a** is LC and **b** is LP then the operator then the operator S in H defined in [10, Theorem 3.1 (3.6)], using now the set
$$\{\, \mathbf{B}^{\kappa_1}_{\mathbf{a}}, \mathbf{B}^{\pm}_{n,s} : n \in \mathbb{Z} \quad \text{and} \quad s \in \{1,2\} \,\}, \tag{16.12}$$
is seen to identify with the self-adjoint operator $\mathbf{S}_{\kappa_1}$ in $\mathbf{L}^2$ in Case 2a of Section 15. Thus S is self-adjoint in H and so the set (16.12) is maximal in the sense of [10, (3.2)]; this set is also seen to be linearly independent in the sense of [10, (3.1) (2)].

   However in this case if the single function $\mathbf{B}^{\kappa_1}_{\mathbf{a}}$ is removed then self-adjointness no longer holds and the remaining set is no longer maximal in the sense of [10, (3.2)].

   There is a similar result in the case when **a** is LP and **b** is LC, when Case 2b of Section 15 holds.

3. If both **a** and **b** are LC then the appropriate set of GKN functions is given by (16.10) to give the self-adjointness of the corresponding S in H of [10, Theorem 3.1]; this is equivalent to the situation of Case 3 of Section 15.

   Again if either one or both of $\mathbf{B}^{\kappa_1}_{\mathbf{a}}$, $\mathbf{B}^{\kappa_2}_{\mathbf{b}}$ is removed from the set (16.10) then the resulting set is no longer maximal in the sense [10, (3.2)].

17 Some comments on the limit-point/limit-circle classification

There is an extensive literature devoted to sufficient conditions for 'classical' Sturm-Liouville differential equations, as given by (1.1), to be classified, *i.e.* results studying the dependence of the LP/LC classification on the three coefficients p, q and w, and the interval (a, b). In spite of the most intensive efforts since the original LP/LC classification given by Herman Weyl in 1910, there are few results in the direction of obtaining necessary conditions. Thus an attempt to provide general LP/LC criteria dependent upon on the sequences of coefficients $\{p_n, q_n, w_n\}$ and the interface boundary parameters $\{\tau_n, \mathbf{K}_n\}$ for differential expressions with interior singularities, is bound to meet with the same, if not even greater, difficulties. Thus results for either LP or LC classifications for these generalized problems is best left to concrete situations arising from applications; see, for example, [4, Theorems 1 and 2], [3, Section 6] and a recent result in [13].

Our goal in this Section is to point out the following known results:

1. The LP/LC classification of the end-points **a** and **b** for a given set $\{\mathbf{M}, \mathbf{W} : \tau, \mathbf{K}\}$, as discussed in Section 13 above, depends upon the interface sequences $\{\tau_n, \mathbf{K}_n\}$ even although the pair $\{\mathbf{M}, \mathbf{W}\}$, including the sequence of intervals $\{(a_n, b_n)\}$, is kept fixed.

2. On the other hand it is possible to specify the pair $\{\mathbf{M}, \mathbf{W}\}$, with only relatively mild restrictions on the coefficients $\{p_n, q_n, w_n\}$ and on the intervals $\{(a_n, b_n)\}$, in such a way that the LP classification holds for all choices of the interface parameters $\{\tau_n, \mathbf{K}_n\}$.

For Result 1. above we refer to an example considered in [4, Section 2]. This case starts with the Schrödinger differential expression $-y'' + qy$ on $\mathbb{R}$ with $q \in L^1_{\text{loc}}(\mathbb{R})$, and an infinite number of additional interface boundary conditions at points $\{a_n\}$, $\{b_n\}$ with $\mathbf{a} = -\infty$ and $\mathbf{b} = +\infty$ as in Section 7 above. With the given restriction on the potential q all the interior end-points a_n^+ and b_n^- are regular for the given differential expression considered in $L^2(a_n, b_n)$, and this for all $n \in \mathbb{Z}$. For such an example we can replace the general interface boundary conditions (10.3) with the pointwise conditions

$$\begin{bmatrix} y_{n+1}(a_{n+1}^+, \lambda) \\ y'_{n+1}(a_{n+1}^+, \lambda) \end{bmatrix} = \exp(i\tau_n)\mathbf{K}_n \begin{bmatrix} y_n(b_n^-, \lambda) \\ y_n(b_n^-, \lambda) \end{bmatrix} \tag{17.1}$$

for numbers $\tau_n \in [0, \pi)$ and matrices $\mathbf{K}_n \in SL_2(\mathbb{R})$ for all $n \in \mathbb{Z}$. (Compare with the remark on coupled boundary conditions for the regular case given in Section 6 above.) In [4] the special case

$$q(x) = 0 \ (x \in \mathbb{R}), \ \tau_n = 0 \quad \text{and} \quad \mathbf{K}_n = \begin{bmatrix} 1 & 0 \\ V_n & 1 \end{bmatrix} \quad (\, n \in \mathbb{Z}\,) \tag{17.2}$$

is studied, and the maximal operator $\mathbf{T}_1$, as defined in (12.2) and (12.3), is denoted formally by

$$\mathbf{T}_1 = -\frac{d^2}{dx^2} + q(x) + \sum_{n \in \mathbb{Z}} V_n \delta(x - a_n). \tag{17.3}$$

This form is motivated by the fact that the interface boundary conditions determined by (17.1) and (17.2) correspond physically to 'δ-interactions' as discussed in greater detail in [13]. In [4, Section 2] it is shown that if

$$q(x) = 0 \ (\, x \in \mathbb{R}\,) \text{ and } a_0 = 0, \ a_{n+1} - a_n = 1/(n+1) \ (\, n \in \mathbb{Z}\,) \tag{17.4}$$

and the sequence $\{V_n : n \in \mathbb{Z}\}$ is chosen in an appropriate form, then $\mathbf{T}_1$ given by (17.3) is in the LC case at $+\infty$. On the other hand if the sequence $\{V_n\}$ is determined by $V_n = 0$ ($n \in \mathbb{Z}$) then (17.3) and (17.4) reduce $\mathbf{T}_1$ to a classical case that is clearly LP at $+\infty$.

For Result 2. we state a result [11, Theorem 2.2] that, in the notation of this paper, essentially gives

Theorem 8 (Gesztesy and Kirsch) *Let the sequence of intervals $\{\,(a_n, b_n) : n \in \mathbb{Z}\,\}$, as given in Section 7, satisfy the additional condition, for some $\varepsilon_0 > 0$,*

$$|b_n - a_n| \geq \varepsilon_0 \quad (\, n \in \mathbb{Z}\,)$$

so that $\mathbf{a} = -\infty$ and $\mathbf{b} = +\infty$;for all $n \in \mathbb{Z}$ let $q_n \in L^1_{loc}(a_n, b_n)$ and let the differential expression

$$M_n[y] := -y'' + q_n y \quad \text{on} \quad (a_n, b_n)$$

be in the limit-circle classification at a_n^+ and b_n^-.

Further assume that, for all $\varepsilon > 0$, the negative part of the potential $q^{(\varepsilon)} \in L^1_{loc}(\mathbb{R})$ defined by

$$q^{(\varepsilon)}(x) := \begin{cases} q_n(x) \ (\, x \in [a_n - \varepsilon, b_n - \varepsilon] \text{ and } n \in \mathbb{Z}\,) \\ 0 \quad \text{elsewhere} \end{cases}$$

is relatively bounded, with relative bound $a_\varepsilon < 1$, with respect to the operator H_0 defined by

$$D(H_0) := \{f \in L^2(\mathbb{R}) : f, f' \in AC_{loc}(\mathbb{R}) \text{ and } f'' \in L^2(\mathbb{R})\}$$

and

$$H_0 f := f'' \quad (\, f \in D(H_0)\,).$$

Then the maximal operator $\mathbf{T}_1$, see (12.2) and (12.3), defined from the differential expression, in the space $\mathbf{L}^2(\langle -\infty, +\infty\rangle)$,

$$-\mathbf{Y}'' + \mathbf{QY} \quad on \quad \langle -\infty, +\infty\rangle \tag{17.5}$$

*is self-adjoint for **all** interface boundary condition parameters $\{\tau_n, \mathbf{K}_n : n \in \mathbb{Z}\}$.*

Proof. See [11, Theorem 2.2]. ■

Corollary 3 *Under all the conditions of Theorem 8 the generalized differential expression* (17.5) *is in the limit-point classification at $\pm\infty$ in $\mathbf{L}^2(\mathbb{R})$, for all choices of the interface boundary condition parameters $\{\tau_n, \mathbf{K}_n : n \in \mathbb{Z}\}$.*

Proof. See Theorem 5 above and the consequences of Theorem 8. ■

Remarks 1. A sufficient condition for the relative bound of $q^{(\varepsilon)}$ to satisfy the required conditions of Theorem 8 is the negative parts $q_n^- := \min(q_n, 0)$ of the individual coefficients q_n satisfy

$$\sup_{n\in\mathbb{Z}} \sup_{x\in[a_n+\varepsilon, b_n-\varepsilon]} |q_n^-(x)| < +\infty$$

for all $\varepsilon > 0$. For details of this result see [11].

2. The interface conditions introduced in [11] are given in a form different form from the conditions stated in (10.3) above, but either form is equivalent to the other form. In particular certain limit-circle interface conditions are introduced in [11] together with a number of concrete examples. One of these examples

$$a_n = (n - \frac{1}{2})\pi \quad b_n = (n + \frac{1}{2})\pi \quad (\, n \in \mathbb{Z}\,)$$

and

$$p(x) = w(x) = 1 \quad q(x) = \frac{\nu}{\cos^2(x)} \quad (\, \langle -\infty, +\infty\rangle\,)$$

where the parameter $\nu > -\frac{1}{4}$, allows of explicit analysis; see [11, Section 4].

3. General limit-point criteria for interface boundary conditions are given in [3, Section 2].

4. For a detailed assessment of examples of one-dimensional Schrödinger differential equations in quantum mechanics that are solvable, see [1].

References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden. *Solvable models in quantum mechanics.* (Springer-Verlag, Heidelberg and New York; 1988.)

[2] P.B. Bailey, W.N. Everitt and A. Zettl. 'Regular and singular Sturm-Liouville problems with coupled boundary conditions.' *Proc. Roy. Soc. Edinb.* (A) **126** (1996), 505-514.

[3] D. Buschmann, G. Stolz and J. Weidmann. 'One-dimensional Schrödinger operators with local point interactions.' *J. Reine Angew. Math.* (To appear.)

[4] C.S. Christ and G. Stolz. 'Spectral theory of one-dimensional Schrödinger operators with point interactions.' *J. Math. Anal. Appl.* **184** (1994), 491-516.

[5] E.A. Coddington and N. Levinson. *Theory of ordinary differential equations.* (McGraw-Hill, New York; 1995.)

[6] W.N. Everitt. 'A note on the self-adjoint domains of second-order differential equations'. *Quart. J. Math. (Oxford).* (2) **13** (1963), 41-45.

[7] W.N. Everitt. 'Some remarks on the Titchmarsh-Weyl m-coefficient and associated differential operators.' *Differential equations, dynamical systems and control science*; a Festschrift in Honour of Lawrence Markus; 33-53. (Lecture Notes in Pure and Applied Mathematics, **152**; Marcel Dekker, Inc., New York; 1994; edited by K.D. Elworthy, W.N. Everitt and E.B. Lee.)

[8] W.N. Everitt, G. Schöttler and P.L. Butzer. 'Sturm-Liouville boundary- value problems and Lagrange interpolation series.' *Rend. Mat. Appl.* (7) **14** (1994), 87-126.

[9] W.N. Everitt and A. Zettl. 'Sturm-Liouville differential operators in direct sum apaces.' *Rocky Mountain J. Math.* **16** (1986), 495-516.

[10] W.N. Everitt and A. Zettl. 'Differential operarors generated by a countable number of quasi-differential expressions on the real line.' *Proc. London Math. Soc.* (3) **64** (1992), 524-544.

[11] F. Gesztesy and W. Kirsch. 'One-dimensional Schrödinger operators with interactions on a discrete set.' *J. für Mathematik.* **362** (1985), 28-50.

[12] M.A. Naimark. *Linear differential operators;* **II**. (Ungar, New York, 1968; translated from the second Russian edition, 1964.)

[13] G. Stolz. 'Pure point spectra for Wannier-Stark hamiltonians.' (To appear in these Proceedings.)

[14] E.C. Titchmarsh. *Eigenfunctions expansions;* **I.** (Oxford University Press, second edition; 1962.)

[15] J. Weidmann. *Spectral theory of ordinary differential operators.* (Lecture notes in Mathematics, **1259**; Springer-Verlag, Berlin-Heidelberg; 1987.)