STURM-LIOUVILLE PROBLEMS AND DISCONTINUOUS EIGENVALUES

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Abstract. If a Sturm-Liouville problem is given in an open interval of the real line then regular boundary value problems can be considered on compact sub-intervals. For these regular problems, all with necessarily discrete spectra, the eigenvalues depend on both the end-points of the compact intervals, and upon the choice of the real separated boundary conditions at these end-points. These eigenvalues are not, in general, continuous functions of the end-points and boundary conditions. The paper shows the surprising form of these discontinuities. The results have applications to the approximations of singular Sturm-Liouville problems by regular problems, and to the theoretical aspects of the Sleign2 computer program.

1. Introduction

This paper follows on from the results in the previous paper in this subject [5]; however the present paper is self-contained. That earlier paper on Sturm-Liouville boundary value problems studies the continuous dependence of eigenvalues under the change of the separated boundary conditions at a fixed end-point; it is shown that whilst all eigenvalues are embedded in continuous curves, the indexing of the eigenvalues is subject to discontinuities along the length of these curves.

However the application of these ideas to the approximation theory used in the numerical code SLEIGN2, see [4], requires consideration of the continuous dependence of eigenvalues when both the separated boundary conditions and the end-points of the problem are varied. This could give the impression that this paper is concerned with a two parameter problem; however this is not the case as the SLEIGN2 code calls for the changes in the end-point and boundary condition to be linked, and this linkage reduces the dependence to the change of a single parameter. For technical reasons the single parameter is chosen to be the end-point of the interval.

2. The Sturm-Liouville background

Let I be an arbitrary open interval of the real line $\mathbb{R}$; let $\mathbb{C}$ denote the complex field, and AC and L absolute continuity and Lebesgue integration respectively.

Let $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ and $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.

1991 *Mathematics Subject Classification.* Primary: 34B05, 34B24; Secondary: 34F30, 34L05.

Key words and phrases. Sturm-Liouville, eigenvalues.

Let the coefficients p, q, w satisfy the basic conditions

$$\left.\begin{array}{ll} \text{(a)} & p, q, w : I \to \mathbb{R} \\ \text{(b)} & p^{-1}, q, w \in L^1_{\text{loc}}(I) \\ \text{(c)} & p(x) > 0 \text{ and } w(x) > 0 \text{ for almost all } x \in I. \end{array}\right\} \tag{2.1}$$

With $L^2(I : w)$ representing the classical, weighted, Hilbert function-space let the linear manifold $\Delta \subset L^2(I : w)$ be the maximal domain, defined by

$$\Delta := \{f : I \to \mathbb{C} \quad \text{with} \quad \begin{array}{ll} \text{(i)} & f \text{ and } pf' \in AC_{\text{loc}}(I) \\ \text{(ii)} & f \text{ and } w^{-1}\left(-(pf')' + qf\right) \in L^2(I : w)\}. \end{array} \tag{2.2}$$

A *boundary condition* function u on I satisfies the conditions

$$\left.\begin{array}{ll} \text{(i)} & u : I \to \mathbb{R} \\ \text{(ii)} & u \in \Delta \\ \text{(iii)} & |u(x)| + |(pu')(x)| > 0 \text{ for all } x \in I. \end{array}\right\} \tag{2.3}$$

The skew-hermitian form $[\cdot, \cdot](\cdot) : I \times \Delta \times \Delta \to \mathbb{C}$ is defined by

$$[f, g](x) := f(x).(p\bar{g}')(x) - (pf')(x).\bar{g}(x) \quad (\, x \in I \text{ and } f, g \in \Delta\,). \tag{2.4}$$

Let $\alpha \in I$; then a *separated* boundary condition at α on an element $f \in \Delta$ can be represented by, given u as in (2.3),

$$[f, u](\alpha) = 0. \tag{2.5}$$

Note that this reduces to the more usual form of a separated boundary condition

$$A_1 f(\alpha) + A_2(pf')(\alpha) = 0,$$

with $A_1, A_2 \in \mathbb{R}$ and not both zero, if $A_1 = (pu')(\alpha)$ and $A_2 = -u(\alpha)$.

3. Boundary value problems

Let $[\alpha, \beta] \subset I$ be a compact sub-interval of I; let u and v be boundary condition functions on I; then a *regular* boundary value problem on $[\alpha, \beta]$ is taken to be the differential equation, with a spectral parameter $\lambda \in \mathbb{C}$,

$$-(py')' + qy = \lambda wy \quad \text{on} \quad [\alpha, \beta] \tag{3.1}$$

and the two separated boundary conditions

$$[y, u](\alpha) = 0 \quad \text{and} \quad [y, v](\beta) = 0. \tag{3.2}$$

With the conditions and definitions (2.1) to (2.5) above the boundary value problem (3.1) and (3.2) has a discrete simple spectrum $\{\lambda_n : n \in \mathbb{N}_0\}$ with the properties, see [8, Chapter 13, Theorem 13.2],

$$\left.\begin{array}{ll} \text{(i)} & \lambda_n \in \mathbb{R} \quad (\, n \in \mathbb{N}_0\,) \\ \text{(ii)} & \lambda_n < \lambda_{n+1} \quad (\, n \in \mathbb{N}_0\,) \\ \text{(iii)} & \lim\limits_{n \to \infty} \lambda_n = +\infty. \end{array}\right\} \tag{3.3}$$

The eigenvalues $\{\, \lambda_n : n \in \mathbb{N}_0 \,\}$ depend on both the interval $[\alpha, \beta]$ and the boundary condition functions u and v; the dependence on the interval $[\alpha, \beta]$ is indicated by use of the notation $\{\, \lambda_n[\alpha, \beta] : n \in \mathbb{N}_0 \,\}$, when required.

Suppose that one or both of the end-points of I are singular for the differential equation

$$-(py')' + qy = \lambda wy \quad \text{on} \quad I; \tag{3.4}$$

then a *singular* boundary value problem requires, in general, boundary conditions at the end-points of I. In this the singular case one or both of these boundary conditions may be superfluous; this occurs if one or both of the end-points is in the limit-point condition; boundary conditions are required in the limit-circle case. For details see [3], [8].

The spectrum of the boundary value problem in the singular case is simple but may or may not be discrete.

It is shown in [2] that for a range of singular problems on I the spectrum can be approximated from within the collection of spectral sets $\{\{\lambda_n[\alpha, \beta] : n \in \mathbb{N}_0\}\}$ of regular eigenvalues.

4. The SLEIGN2 program

The FORTRAN program [4] computes numerical values of eigenvalues, and the associated eigenfunctions, of both regular and singular Sturm-Liouville problems. The numerical procedures are complicated but two of the principles involved are:

1. The computation of the eigenvalues $\{\lambda_n[\alpha, \beta] : n \in \mathbb{N}_0\}$ of regular problems using the Prüfer transformation of the differential equation (3.1).
2. The approximation of the spectrum of singular problems from spectral sets of regular problems as given at the end of the previous section.

¿From this information it follows that it is important, given a singular boundary value problem determined by (3.4) and boundary conditions, to consider the analytic behaviour of the eigenvalues $\{\lambda_n[\alpha, \beta] : n \in \mathbb{N}_0\}$ as the compact interval is "increased" to the interval I.

5. The boundary condition functions

Consider now the family of boundary value problems

$$-(py')' + qy = \lambda wy \quad \text{on} \quad [\alpha, \beta]$$

$$[y, u](\alpha) = 0 \quad \text{and} \quad [y, v](\beta) = 0 \tag{5.1}$$

for all $[\alpha, \beta] \subset I$.

The zeros of the boundary condition functions u and v play a significant role in the change of indexing of the eigenvalues as α decreases and as β increases.

In order to make the boundary conditions (5.1) more accessible to subsequent calculations we define $g_1, g_2 : I \to \mathbb{R}$ by

$$g_1(x) := \frac{(pu')(x)}{\sqrt{u^2(x) + (pu')^2(x)}} \text{ and } g_2(x) := \frac{u(x)}{\sqrt{u^2(x) + (pu')^2(x)}} \quad (x \in I); \tag{5.2}$$

these functions are well-defined in view of (iii) of (2.3). Define also $g : I \to \mathbb{R}$ by

$$g(x) := g_1(x) + ig_2(x) \quad (x \in I) \tag{5.3}$$

so that $|g(x)| = 1$ $(x \in I)$ and $g \in AC_{\text{loc}}(I)$. Hence there is a continuous function $\tilde{\gamma} : I \to \mathbb{R}$ such that

$$g(x) = \exp(i\tilde{\gamma}(x)) \quad (x \in I).$$

Note that $\tilde{\gamma}(\cdot)$ is not unique but can be made definite by prescribing the value of $\tilde{\gamma}(c)$ at some point c, say, of I.

The boundary condition (2.5) may now be written in the form

$$\cos(\tilde{\gamma}(\alpha))\, y(\alpha) - \sin(\tilde{\gamma}(\alpha))\, (py')(\alpha) = 0. \tag{5.4}$$

Similarly the second boundary condition in (3.2) is equivalent to

$$\cos(\tilde{\eta}(\beta))\, y(\beta) - \sin(\tilde{\eta}(\beta))\, (py')(\beta) = 0 \tag{5.5}$$

for a suitable continuous function $\tilde{\eta} : I \to \mathbb{R}$.

For the enumeration of the eigenvalues of (5.1) it is more convenient to introduce the function, where in this paragraph $[\cdot]$ denotes the integer part function,

$$\gamma(x) = \tilde{\gamma}(x) - \pi\, [\tilde{\gamma}(x)/\pi]\,,$$

which satisfies

$$0 \le \gamma(x) < \pi.$$

Denoting $\delta(x) = [\tilde{\gamma}(x)/\pi]$, we obtain

$$\exp(i\gamma(x)) = (-1)^{\delta(x)} \exp(i\tilde{\gamma}(x)),$$

and we can therefore replace $\tilde{\gamma}$ in (5.4) by γ. Similarly, we can replace $\tilde{\eta}$ in (5.5) by η, where

$$\eta(x) = \tilde{\eta}(x) + \pi + \pi\, [-\tilde{\eta}(x)/\pi]\,,$$

which satisfies

$$0 < \eta(x) \le \pi.$$

Note that γ and η are continuous except at those points x where $\tilde{\gamma}(x)$ or $\tilde{\eta}(x)$, respectively, are integer multiples of π, and $\tilde{\gamma}(x)$ or $\tilde{\eta}(x)$, respectively, have jumps there.

The two representations (5.4) and (5.5) emphasise the dependence of these boundary conditions on the end-points α and β; as these end-points are moved the boundary conditions are automatically changed; these changes are inherited from the initial choice of the boundary condition functions u and v. For this reason the boundary conditions in (5.1), equivalently (5.4) and (5.5) are called "inherited" in [3], from the boundary condition functions u and v.

We have

Lemma 1. *The set of zeros of the boundary condition function u is discrete in I, and u is strictly monotonic near each zero of u.*

There is a similar result for the boundary condition function v.

Proof. Let $a \in I$; since $u(a) = 0$ implies $(pu')(a) \neq 0$ the continuity of pu' implies that pu' is either positive or negative in a neighbourhood of a. Writing

$$u(x) = \int_a^x \frac{1}{p(t)} (pu')(t)\, dt \tag{5.6}$$

and use of (c) of (2.1) proves the monotonicity of u in that neighbourhood. The discreteness of the zeros of u is an immediate consequence of the result.

The proof for the function v is similar. □

Lemma 2. *Let $a \in I$ such that $u(a) = 0$. Then, for $x \in I$, $\tilde{\gamma}(x) < \tilde{\gamma}(a)$ if $x < a$ and $\tilde{\gamma}(x) > \tilde{\gamma}(a)$ if $x > a$.*

There is a similar result for the boundary condition function v.

Proof. Since $u(a) = 0$ is equivalent to $\tilde{\gamma}(a) \in \pi\mathbb{Z}$ and since by (5.6) there is $\varepsilon > 0$ such that

$$\tan\tilde{\gamma}(\alpha) = \frac{u(\alpha)}{(pu')(\alpha)} < 0 \text{ if } \alpha \in (a-\varepsilon, a),$$
$$\tan\tilde{\gamma}(\alpha) = \frac{u(\alpha)}{(pu')(\alpha)} > 0 \text{ if } \alpha \in (a, a+\varepsilon),$$

it follows that

$$\tilde{\gamma}(\alpha) < \tilde{\gamma}(a) \text{ if } \alpha \in (a-\varepsilon, a),$$
$$\tilde{\gamma}(\alpha) > \tilde{\gamma}(a) \text{ if } \alpha \in (a, a+\varepsilon).$$

Since this holds for each zero of u and since $\tilde{\gamma}$ is continuous, the result follows. □

6. The continuity and discontinuity of the eigenvalues.

Recall that we denote the eigenvalues of the boundary value problem (3.1) and (3.2) by $\{\, \lambda_n[\alpha, \beta] : n \in \mathbb{N}_0$ and $\alpha, \beta \in I$ with $\alpha < \beta \,\}$. We discuss the continuity of the mapping

$$\lambda_{(\cdot)}[\cdot,\cdot] : \mathbb{N}_0 \times I \times I \to \mathbb{R} \tag{6.1}$$

and the cases of discontinuity.

The result of Lemma 1 above shows that the zeros of the functions u and v form a grid on the set $I \times I$; we distinguish four cases for the mapping (6.1) depending on the position of a point $(a, b) \in I \times I$ relative to this grid.

Theorem 1. *Let $(a, b) \in I \times I$, $a < b$; if*

$$u(a) \neq 0 \textit{ and } v(b) \neq 0 \tag{6.2}$$

then for each $n \in \mathbb{N}_0$ the mapping $\lambda_n[\cdot,\cdot]$ is continuous at (a, b).

Proof. The functions $\gamma, \eta : I \to \mathbb{R}$ are defined in the previous section; from (6.2) we have that $0 < \gamma(a) < \pi$ and $0 < \eta(b) < \pi$. From the continuity of γ at a and η at b, and of u and v there exists an open neighbourhood $N = N_a \times N_b$ of (a, b) such that for $(\alpha, \beta) \in N$ we have

$$u(\alpha) \neq 0 \text{ and } v(\beta) \neq 0 \tag{6.3}$$

and

$$0 \leq \gamma(\alpha) < \pi \text{ and } 0 < \eta(\beta) \leq \pi. \tag{6.4}$$

It is well known, see [8, Chapter 13, Page 196], that every non-trivial solution y of (3.1) can be factored into

$$y(x) = \rho(x)\sin(\theta(x)) \quad (\, x \in I \,) \tag{6.5}$$

where $\rho(x) \neq 0$ ($x \in I$) and $\theta(\cdot)$ satisfies the first-order non-linear differential equation

$$\theta'(x) = \frac{1}{p(x)} \cos^2(\theta(x)) + (\lambda w(x) - q(x)) \sin^2(\theta(x)) \;\; (x \in I). \tag{6.6}$$

Let $\lambda \in \mathbb{R}$; from the basic theory of differential equations it is known that for each $\alpha \in N_a$ the initial value problem for (6.6), with initial value $\gamma(\alpha)$ at the point α, has a unique solution $\theta(\cdot;\alpha,\lambda)$ that is defined on all of I and is a continuous function of all variables (recall that $\gamma(\cdot)$ is continuous on N_a); this solution has the properties, for $\alpha < \beta$,

$$\left.\begin{array}{ll} \text{(i)} & \theta(\alpha;\alpha,\lambda) = \gamma(\alpha) \\ \text{(ii)} & \theta(\beta;\alpha,\cdot) \text{ is strictly increasing on } \mathbb{R} \\ \text{(iii)} & \lim_{\lambda\to+\infty} \theta(\beta;\alpha,\lambda) = +\infty \\ \text{(iv)} & \lim_{\lambda\to-\infty} \theta(\beta;\alpha,\lambda) = +0. \end{array}\right\} \tag{6.7}$$

For proofs of these results see the corresponding entries in [1]

(ii)	Theorem 8.4.3, Page 211
(iii)	Theorem 8.4.6, Pages 216-218
(iv)	Theorem 8.4.5, Page 212.

For $n \in \mathbb{N}_0$ the eigenvalue $\lambda_n[\alpha,\beta]$ of the problem (3.1) and (3.2) is the unique solution in λ of the equation

$$\theta(\beta;\alpha,\lambda) = \eta(\beta) + n\pi. \tag{6.8}$$

Let $n \in \mathbb{N}_0$; suppose that the mapping $\lambda_n[\cdot,\cdot] : N \to \mathbb{R}$ is not continuous at the point (a,b). In particular assume there exists a sequence $\{ (\alpha_k,\beta_k) : k \in \mathbb{N}_0 \}$, with $(\alpha_k,\beta_k) \in N$ ($k \in \mathbb{N}_0$), such that for some $\varepsilon > 0$

$$\begin{array}{ll} \text{(i)} & \lim_{k\to\infty} \alpha_k = a \text{ and } \lim_{k\to\infty} \beta_k = b \\ \text{(ii)} & \lambda_n[\alpha_k,\beta_k] \geq \lambda_n[a,b] + \varepsilon \quad (k \in \mathbb{N}_0). \end{array}$$

Then, using results given in (6.7),

$$\begin{aligned} \eta(b) + n\pi &= \lim_{k\to\infty} \eta(\beta_k) + n\pi \\ &= \lim_{k\to\infty} \theta(\beta_k;\alpha_k,\lambda_n[\alpha_k,\beta_k]) \\ &\geq \liminf_{k\to\infty} \theta(\beta_k;\alpha_k,\lambda_n[a,b]+\varepsilon) \\ &= \theta(b;a,\lambda_n[a,b]+\varepsilon) \\ &> \theta(b;a,\lambda_n[a,b]) \\ &= \eta(b) + n\pi \end{aligned}$$

and this is a contradiction.

There is a similar argument if ε is taken to be negative and the inequality in (ii) is reversed. □

In order to treat the remaining cases let the points be chosen so that

$$a < a_0 < b_0 < b$$

and let $m(u, a, a_0)$ be the number of zeros of u in the interval $(a, a_0]$ and $M(v, b_0, b)$ the number of zeros of v in $[b_0, b)$. Now define

$$\Omega_{a_0,b_0} := \{\, (a, b) \in I \times I : a < a_0 \text{ and } b > b_0 \,\}$$

and

$$\tilde{\lambda}_{n,a_0,b_0}[a, b] := \lambda_{n+m(u,a_0,a)+M(v,b_0,b)}[a, b] \quad (\, (a, b) \in \Omega_{a_0,b_0}\,).$$

We have

Theorem 2. *Let $n \in \mathbb{N}_0$ and $a_0, b_0 \in I$ be given, with $a_0 < b_0$; then $\tilde{\lambda}_{n,a_0,b_0}[\cdot, \cdot]$ is continuous on Ω_{a_0,b_0}.*

Proof. Observe that

$$\begin{aligned} \theta(b; a, \tilde{\lambda}_{n,a_0,b_0}[a, b]) &= \theta(b; a, \tilde{\lambda}_{n+m(u,a_0,a)+M(v,b_0,b)}[a, b]) \\ &= \eta(b) + [n + m(u, a_0, a) + M(v, b_0, b)]\pi. \end{aligned}$$

By definition of $\tilde{\eta}$ and η, $\tilde{\eta} - \eta$ takes values in $\pi\mathbb{Z}$, and together with Lemma 2 it follows that:

1. $\tilde{\eta} - \eta$ is constant on each open sub-interval of I which does not contain zeros of v.
2. $\tilde{\eta} - \eta$ is continuous from the left at all points $c \in I$ with $v(c) = 0$; for from Lemma 1 v is monotone in the neighbourhood of each of its zeros so that if $v(c) = 0$ then $v(x)/(pv')(x) < 0$ for $x < c$ in this neighbourhood, and so $\tilde{\eta}$ is increasing as x tends to c from the left.
3. $(\tilde{\eta}(c^+) - \eta(c^+)) - (\tilde{\eta}(c) - \eta(c)) = \pi$ for all points $c \in I$ with $v(c) = 0$.

Therefore

$$\begin{aligned} \eta(b) + M(v, b_0, b)\pi &= \eta(b) + (\tilde{\eta}(b) - \eta(b)) - (\tilde{\eta}(b_0) - \eta(b_0)) \\ &= \tilde{\eta}(b) - (\tilde{\eta}(b_0) - \eta(b_0)). \end{aligned}$$

Since adding an integer multiple of π to a solution θ of (6.6) gives again a solution of (6.6),

$$\tilde{\theta}(\cdot; a, \tilde{\lambda}_{n,a_0,b_0}[a, b]) := \theta(\cdot; a, \tilde{\lambda}_{n,a_0,b_0}[a, b]) - m(u, a_0, a)\pi$$

is the unique solution of (6.6) satisfying the initial condition

$$\tilde{\theta}(a; a, \tilde{\lambda}_{n,a_0,b_0}[a, b]) = \gamma(a) - m(u, a_0, a)\pi.$$

A reasoning similar to that for η and $\tilde{\eta}$ above gives

$$\begin{aligned} \gamma(a) - m(u, a_0, a)\pi &= \gamma(a) + (\tilde{\gamma}(a) - \gamma(a)) - (\tilde{\gamma}(a_0) - \gamma(a_0)) \\ &= \tilde{\gamma}(a) - (\tilde{\gamma}(a_0) - \gamma(a_0)). \end{aligned}$$

Altogether we have that $\tilde{\theta}(\cdot; a, \tilde{\lambda}_{n,a_0,b_0}[a, b])$ is the unique solution of (6.6) satisfying

$$\begin{aligned} \tilde{\theta}(a; a, \tilde{\lambda}_{n,a_0,b_0}[a, b]) &= \tilde{\gamma}(a) - (\tilde{\gamma}(a_0) - \gamma(a_0)), \\ \tilde{\theta}(b; a, \tilde{\lambda}_{n,a_0,b_0}[a, b]) &= \tilde{\eta}(b) - (\tilde{\eta}(b_0) - \eta(b_0)). \end{aligned}$$

Since $\tilde{\gamma}$ and $\tilde{\eta}$ are continuous on I, a reasoning similar to that in the proof of Theorem 1 gives that $\tilde{\lambda}_{n,a_0,b_0}$ is continuous on Ω_{a_0,b_0}. □

Theorem 3. *Let $a, b \in I$ such that $a < b$ and $u(a) = 0$. Then the following results hold*

$$\lim_{\alpha \to a-} \lambda_0[\alpha, b] = -\infty$$
$$\lim_{\alpha \to a+} \lambda_0[\alpha, b] = \lambda_0[a, b]$$

and for all $n \in \mathbb{N}_0$

$$\lim_{\alpha \to a-} \lambda_{n+1}[\alpha, b] = \lambda_n[a, b]$$
$$\lim_{\alpha \to a+} \lambda_{n+1}[\alpha, b] = \lambda_{n+1}[a, b].$$

Proof. Except for the behaviour of $\lambda_0[\alpha, b]$ as $\alpha \to a-$, all statements follow from Theorem 2. Assume that $\lambda_0[\alpha, b] \to -\infty$ as $\alpha \to a-$ does not hold. Then there is a sequence $\alpha_k \nearrow a$ such that $\lambda_0[\alpha_k, b] \to c \in (-\infty, \infty]$. Because of $\lambda_0[\alpha_k, b] < \widetilde{\lambda}_{0,a,b}[\alpha_k, b]$ it follows that $c \in (-\infty, \lambda_0[a, b]]$. Let $\tilde{\theta}_0(\cdot; \alpha, \lambda)$ be the solution of (6.6) satisfying $\tilde{\theta}_0(\alpha; \alpha, \lambda) = \tilde{\gamma}(\alpha)$. As above we can conclude that $\tilde{\theta}_0$ depends continuously on all three variables. We have

$$\begin{aligned}
\theta(\cdot; \alpha, \lambda) &= \tilde{\theta}_0(\cdot; \alpha, \lambda) - \tilde{\gamma}(\alpha) + \gamma(\alpha) \\
&= \tilde{\theta}_0(\cdot; \alpha, \lambda) + m(u, a, \alpha) - (\tilde{\gamma}(a) - \gamma(a)).
\end{aligned}$$

This gives

$$\begin{aligned}
\theta(b; a, c) &= \tilde{\theta}_0(b; a, c) - (\tilde{\gamma}(a) - \gamma(a)) \\
&= \lim_{k \to \infty} \tilde{\theta}_0(b; \alpha_k, \lambda_0[\alpha_k, b]) - (\tilde{\gamma}(a) - \gamma(a)) \\
&= \lim_{k \to \infty} \{\theta(b; \alpha_k, \lambda_0[\alpha_k, b]) - m(u, a, \alpha)\} \\
&= \eta(b) - \pi \le 0
\end{aligned}$$

in view of (6.8), which is impossible since $\theta(b, a, \lambda) \in (0, \infty)$ for all $\lambda \in \mathbb{R}$. □

Theorem 4. *Let $a, b \in I$ such that $a < b$ and $v(b) = 0$. Then the following results hold*

$$\lim_{\beta \to b+} \lambda_0[a, \beta] = -\infty$$
$$\lim_{\beta \to b-} \lambda_0[a, \beta] = \lambda_0[a, b]$$

and for all $n \in \mathbb{N}_0$

$$\lim_{\beta \to b+} \lambda_{n+1}[a, \beta] = \lambda_n[a, b]$$
$$\lim_{\beta \to b-} \lambda_{n+1}[a, \beta] = \lambda_{n+1}[a, b].$$

Proof. The proof is similar to the proof of Theorem 3 and therefore omitted. □

Theorem 5. *Let $a, b \in I$ such that $a < b$ and $u(a) = 0$, $v(b) = 0$. Then the following double limit results hold*

$$\lim_{\alpha \to a-,\, \beta \to b-} \lambda_0[\alpha, \beta] = -\infty$$
$$\lim_{\alpha \to a+,\, \beta \to b+} \lambda_0[\alpha, \beta] = -\infty$$
$$\lim_{\alpha \to a-,\, \beta \to b+} \lambda_0[\alpha, \beta] = -\infty$$
$$\lim_{\alpha \to a-,\, \beta \to b+} \lambda_1[\alpha, \beta] = -\infty$$

and for all $n \in \mathbb{N}_0$

$$\lim_{\alpha \to a+,\, \beta \to b-} \lambda_n[\alpha, \beta] = \lambda_n[a, b]$$
$$\lim_{\alpha \to a-,\, \beta \to b-} \lambda_{n+1}[\alpha, \beta] = \lambda_n[a, b]$$
$$\lim_{\alpha \to a+,\, \beta \to b+} \lambda_{n+1}[\alpha, \beta] = \lambda_n[a, b]$$
$$\lim_{\alpha \to a-,\, \beta \to b+} \lambda_{n+2}[\alpha, \beta] = \lambda_n[a, b].$$

Proof. The result on the finite jumps immediately follow from Theorem 2. For the other cases, if we assume that the limit is not $-\infty$, we obtain as in the proof of Theorem 3 that there is a sequence $(\alpha_k, \beta_k) \to (a, b)$ such that $\lambda_j[\alpha_k, \beta_k] \to c \in (-\infty, \lambda_j[a, b]]$ for the $j \in \{0, 1\}$ considered, and then

$$\theta(b; a, c) = \begin{cases} \eta(b) - \pi & \text{if } \alpha_k < a,\ \beta_k < b,\ j = 0 \\ \eta(b) - \pi & \text{if } \alpha_k > a,\ \beta_k > b,\ j = 0 \\ \eta(b) + (j-2)\pi & \text{if } \alpha_k < a,\ \beta_k > b,\ j = 0, 1, \end{cases}$$

which is impossible since $\theta(b; a, \lambda) \in (0, \infty)$ for all $\lambda \in \mathbb{R}$. □

Acknowledgment The authors thank the referee of this paper for constructive remarks and for drawing attention to a number of misprints in the original manuscript.

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