# REGULAR APPROXIMATIONS OF SINGULAR STURM-LIOUVILLE PROBLEMS WITH LIMIT-CIRCLE ENDPOINTS

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Abstract. For any self-adjoint realization $S$ of a singular Sturm-Liouville equation on an interval $(a,b)$ with limit-circle endpoints, we construct a family of self-adjoint realizations $S_r, r \in (0,\infty)$, of this equation on subintervals $(a_r, b_r)$ of $(a,b)$ such that every eigenvalue of $S$ is the limit of a continuous eigenvalue branch of this family. Of particular interest are the cases when at least one endpoint is oscillatory or the leading coefficient function changes sign. In these cases, we show that the index determining each continuous eigenvalue branch has an infinite number of jump discontinuities and give an explicit characterization of these discontinuities.

## 1. Introduction

Given a self-adjoint realization $S$ of a singular Sturm-Liouville equation, how can its spectrum be approximated by eigenvalues of regular Sturm-Liouville problems (SLP's)? This question has been studied by many authors, for some recent results see [1, 2, 8, 9, 10]. In [2], Bailey, Everitt, and Zettl constructed an algorithm for computing the eigenvalues of singular SLP's with positive leading coefficient, limit circle endpoints, and separated boundary condition (BC), see also [10]. In [1] the above question was studied for general singular self-adjoint SLP's with either limit-circle or limit-point endpoints and either separated or coupled BC's. Convergence properties were established in [1] using the abstract theory of strong resolvent and norm resolvent, particularly the Hilbert-Schmidt norm resolvent, convergence of self-adjoint operators in Hilbert spaces. Although these very general and powerful abstract methods provide some of the theoretical underpinnings for the Fortran code SLEIGN2 [3], they do not yield explicit and constructive algorithms which can be numerically implemented.

In this paper we investigate the general limit-circle SLP's with separated or coupled BC's where the leading coefficient may change sign. We establish a theory for continuous eigenvalue branches of a one-parameter family of "induced realizations" $S_r, r \in (0,\infty)$, to the eigenvalues of $S$. This allows us to utilize and extend the results on continuous eigenvalue branches obtained in [16] for regular SLP's to singular SLP's. We show that in the simplest case when the spectrum of $S$ is bounded below, there exists $r^* \in (0,\infty)$ such that for each $n \in \mathbb{N}_0 := \{0,1,2,\dots\}$, the $n$-th eigenvalue of $S_r$, $\lambda_n(S_r)$ for $r \in (r^*,\infty)$, constitute a continuous eigenvalue branch to $\lambda_n(S)$, and the minimum value of such $r^*$ is found; in the case when at least one endpoint is oscillatory, the index function $\tilde{n}(r)$ of a continuous eigenvalue branch $\lambda_{\tilde{n}(r)}(S_r)$ to $\lambda_n(S)$ has an infinite number of jump discontinuities. Thus, in the latter case, it is important to give a careful and systematic analysis of the jump behavior of the index $\tilde{n}(r)$ along a continuous eigenvalue branch. We give an explicit characterization of where the index $\tilde{n}(r)$ changes and show exactly how it changes. We also extend these results to the much more complicated case where the leading coefficient function $p$ changes sign. In this case, our indexing scheme for eigenvalues of the associated regular problems is adopted from the recent paper of Binding and Volkmer [4] for separated BC's and from [5] for coupled BC's. The results in this paper can be used to construct algorithms for the numerical

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computation of eigenvalues of SLP's for the above cases. However, in this paper, we do not pursue the numerical implementation of these algorithms. This work will be done in a subsequent paper.

Our approach is elementary in the sense that no abstract theory of convergence of self-adjoint operators in Hilbert spaces is used. The main tool is a transformation which transforms the SLP of a singular second order scalar equation to a boundary value problem (BVP) of a regular first order system. Also, a critical role is played by the jump set, i.e., the set of regular self-adjoint BC's where the indexed eigenvalues as functions of the BC's have discontinuities. For a systematic discussion of the jump set and the discontinuous behavior of the index for the continuous eigenvalue branches for regular SLP's with $p > 0$, see [17]. The singular problems with $p > 0$ were studied in [18], and the regular problems with $p$ changing sign were investigated in [5].

This paper is organized as follows: Section 2 contains a basic discussion of limit-circle SLP's. The main results are stated in Section 3. The proofs together with some technical lemmas are given in Section 4.

## 2. Limit-circle SLP's

Consider the Sturm-Liouville equation

$$-(py')' + qy = \lambda wy \quad \text{on } J = (a, b), \tag{2.1}$$

where

$$-\infty \le a < b \le \infty, \quad 1/p, q, w \in L_{loc}(J, \mathbb{R}) \ \text{ and } \ w > 0 \text{ a.e. on } J,$$

and $L_{loc}(J, \mathbb{R})$ is the set of real-valued functions which are Lebesgue integrable on any compact subset of $J$. Here, either $p > 0$ a.e. on $J$; or $p$ changes sign on $J$, i.e., there exist subsets (not necessarily subintervals) $J_1, J_2$ of $J$ with positive or infinite Lebesgue measures such that $p > 0$ on $J_1$ and $p < 0$ on $J_2$.

For any subinterval $J^*$ of $J$, let $L^2(J^*, w)$ be the set of complex-valued measurable functions $f$ on $J^*$ such that $\int_{J^*} |f|^2 w < \infty$. The endpoint $a$ of $J$ is called a *limit-circle* endpoint (or simply $a$ is LC) if all solutions of equation (2.1) are in $L^2((a, c), w)$ for some $c \in J$, and it is called a *limit-point* endpoint (or simply $a$ is LP) otherwise. The endpoint $a$ is said to be *oscillatory* (or simply $a$ is O) if every real-valued solution has an infinite number of zeros in $(a, c)$ for any $c \in J$, and it is *nonoscillatory* (or simply $a$ is NO) otherwise. LCO means LC and O, LCNO means LC and NO. Similar definitions are made for the endpoint $b$. It is well-known that the LC and LP classification is independent of $\lambda$ in $\mathbb{C}$; and for the case where both $a$ and $b$ are LC and $p > 0$, the O and NO classification is also independent of $\lambda$ in $\mathbb{R}$. Recall that the endpoint $a$ is called regular if for some $c \in J$, $1/p, q, w \in L(a, c)$, the space of Lebesgue integrable functions on $(a, c)$. Similarly for the endpoint $b$. Note that in this paper a regular endpoint may be finite or infinite and is included in the LC classification. Thus our results below apply when both endpoints are singular LC, when both are regular, and when one is singular LC and the other is regular.

*Throughout this paper we assume that both $a$ and $b$ are LC*, but each of them may be O or NO.

We define the maximal domain $\mathcal{D}$ by

$$\mathcal{D} = \left\{ f \in L^2(J, w) : \ f, \ pf' \in AC_{loc}(J), \ [-(pf')' + qf]/w \in L^2(J, w) \right\},$$

where $AC_{loc}(J)$ denotes the set of complex-valued functions which are absolutely continuous on all compact subintervals of $J$. Recall that for any $f, g \in \mathcal{D}$, the Lagrange bracket of $f, g$ is given by

$$[f, g] = f(p\bar{g}') - \bar{g}(pf')$$

and has finite limits at both the endpoints $a$ and $b$.

**Definition 2.1.** A pair of functions $\{y_1, y_2\}$ is said to be a *BC basis* if it satisfies the following conditions:

(i) $y_1, y_2$ are real solutions of equation (2.1) in a right-neighborhood $\mathcal{N}_a$ of $a$ for some $\lambda = \lambda_a$,

(ii) $y_1, y_2$ are real solutions of equation (2.1) in a left-neighborhood $\mathcal{N}_b$ of $b$ for some $\lambda = \lambda_b$,

(iii) $[y_1, y_2](a) = [y_1, y_2](b) = -1$.

The existence of such BC bases can be established using 'the Naimark Patching Lemma', see Lemma 2, p. 63 in [22]. Without loss of generality we may assume that $\mathcal{N}_a \cap \mathcal{N}_b = \emptyset$. We use such a BC basis $y_1, y_2$ to generate the self-adjoint BC's for equation (2.1)

$$A \begin{pmatrix} [y, y_1] \\ [y, y_2] \end{pmatrix} (a) + B \begin{pmatrix} [y, y_1] \\ [y, y_2] \end{pmatrix} (b) = 0, \tag{2.2}$$

where $A, B \in \mathbb{C}^{2\times 2}$ satisfy

$$\text{rank}(A|B) = 2 \quad \text{and} \quad AEA^* = BEB^* \quad \text{for} \quad E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \tag{2.3}$$

here $A^*, B^*$ are the complex conjugate transposes of $A, B$, respectively. The self-adjoint BC's given by (2.2) can be classified into two categories: the separated BC's having the canonical form

$$\begin{aligned} &\cos\alpha\ [y, y_1](a) - \sin\alpha\ [y, y_2](a) = 0, \quad \alpha \in [0, \pi), \\ &\cos\beta\ [y, y_1](b) - \sin\beta\ [y, y_2](b) = 0, \quad \beta \in (0, \pi], \end{aligned} \tag{2.4}$$

and the coupled BC's having the canonical form

$$\begin{pmatrix} [y, y_1] \\ [y, y_2] \end{pmatrix} (b) = e^{i\theta} K \begin{pmatrix} [y, y_1] \\ [y, y_2] \end{pmatrix} (a), \tag{2.5}$$

where

$$0 \le \theta < \pi, \quad K \in SL_2(\mathbb{R}) := \{K \in \mathbb{R}^{2\times 2} : \det K = 1\}.$$

In this paper, for fixed BC basis $\{y_1, y_2\}$, the BC's in (2.2), (2.4), and (2.5) may be abbreviated by $[A|B]$, $\mathbf{S}_{\alpha,\beta}$, and $[e^{i\theta}K| - I]$, respectively.

Note that the representations of a self-adjoint BC depend on the BC basis. The following shows the correspondence between two pairs of coefficient matrices for the same BC when two different pairs of BC bases are used, see Theorem 3.3 in [18].

**Lemma 2.1.** *Assume $\{\tilde{y}_1, \tilde{y}_2\}$ is any BC basis. Then the system*

$$A_1 \begin{pmatrix} [y, \tilde{y}_1] \\ [y, \tilde{y}_2] \end{pmatrix} (a) + B_1 \begin{pmatrix} [y, \tilde{y}_1] \\ [y, \tilde{y}_2] \end{pmatrix} (b) = 0$$

*and (2.2) represent the same self-adjoint BC if and only if*

$$A_1 = A \begin{pmatrix} [y_1, \tilde{y}_2] & [\tilde{y}_1, y_1] \\ [y_2, \tilde{y}_2] & [\tilde{y}_1, y_2] \end{pmatrix} (a) \quad \text{and} \quad B_1 = B \begin{pmatrix} [y_1, \tilde{y}_2] & [\tilde{y}_1, y_1] \\ [y_2, \tilde{y}_2] & [\tilde{y}_1, y_2] \end{pmatrix} (b).$$

Let $Y = \begin{pmatrix} y \\ py' \end{pmatrix}$, $P = \begin{pmatrix} 0 & 1/p \\ q & 0 \end{pmatrix}$, and $W = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}$. Then we obtain the system form of equation (2.1)

$$Y' = (P - \lambda W)Y. \tag{2.6}$$

Let $U = \begin{pmatrix} u_1 & u_2 \\ pu_1' & pu_2' \end{pmatrix}$ be a real fundamental matrix solution of equation (2.6) for $\lambda = \lambda^* \in \mathbb{R}$ normalized to satisfy $\det U(t) \equiv 1$ for $t \in J$. Consider the BVP consisting of the equation

$$Z' = (\lambda^* - \lambda)GZ \quad \text{on } J, \tag{2.7}$$

where

$$G = U^{-1}WU = \begin{pmatrix} -u_1u_2w & -u_2^2w \\ u_1^2w & u_1u_2w \end{pmatrix}, \tag{2.8}$$

and the BC

$$\tilde{A}Z(a) + \tilde{B}Z(b) = 0, \tag{2.9}$$

where

$$\tilde{A} = -A \begin{pmatrix} [y_1, u_1] & [y_1, u_2] \\ [y_2, u_1] & [y_2, u_2] \end{pmatrix} (a) \quad \text{and} \quad \tilde{B} = -B \begin{pmatrix} [y_1, u_1] & [y_1, u_2] \\ [y_2, u_1] & [y_2, u_2] \end{pmatrix} (b). \tag{2.10}$$

**Theorem 2.1.** *(i) For each $\lambda \in \mathbb{C}$, the BVP (2.7), (2.9) is a regular problem, i.e., $G \in L(J, \mathbb{R}^{2\times 2})$ and the $\tilde{A}, \tilde{B}$ given by (2.10) are well-defined.*

*(ii) For each $\lambda \in \mathbb{C}$, $Z(t, \lambda)$ is a non-trivial solution of the BVP (2.7), (2.9) if and only if*

$$Z(t, \lambda) = U^{-1}(t) Y(t, \lambda) \tag{2.11}$$

*for some non-trivial solution $Y(t, \lambda) = \begin{pmatrix} y \\ py' \end{pmatrix}(t, \lambda)$ of the BVP (2.6), (2.2). Hence these two BVP's have exactly the same eigenvalues.*

*Proof.* (i) By the Schwartz inequality it is easy to see that $G \in L(J, \mathbb{R}^{2\times 2})$. Hence equation (2.7) is regular on $J$. Note that

$$[y, y_i] = \begin{pmatrix} py_i' & -y_i \end{pmatrix} \begin{pmatrix} y \\ py' \end{pmatrix}, \quad i = 1, 2,$$

and

$$\begin{pmatrix} py_1' & -y_1 \\ py_2' & -y_2 \end{pmatrix} U = - \begin{pmatrix} [y_1, u_1] & [y_1, u_2] \\ [y_2, u_1] & [y_2, u_2] \end{pmatrix}.$$

Thus the BC (2.9) for $Z$ is equivalent to the BC (2.2) for $y$.

(ii) This follows from a simple computation. □

For each $\lambda \in \mathbb{C}$, let $\Phi(t, \lambda)$ be the fundamental matrix solution of (2.7) satisfying $\Phi(a, \lambda) = I$, where $I$ is the identity matrix. Define

$$\Delta(\lambda) = \det\left(\tilde{A} + \tilde{B}\Phi(b, \lambda)\right). \tag{2.12}$$

Singular initial value problems at LC endpoints have been studied in [25], see Section 3.8. The analytic dependence of solutions of singular initial value problems and their quasi-derivatives, on the spectral parameter $\lambda$ has been studied in [12]. The next theorem constructs a characteristic function for singular limit-circle boundary value problems.

**Theorem 2.2.** *(i) $\Delta(\lambda)$ is an entire function of $\lambda$.*

*(ii) $\lambda$ is an eigenvalue of BVP (2.7), (2.9), and hence of SLP (2.1), (2.2), if and only if $\Delta(\lambda) = 0$.*

*Proof.* It follows from the assumption that each endpoint of $J = (a, b)$ is LC that each component of $G$, given by (2.8), is in $L^2(J, w)$. Hence the system (2.7) is regular on $J$ for each $\lambda$ in $\mathbb{C}$. Note that this holds regardless of whether the endpoints $a, b$ of $J$ are finite or infinite. (Here we use "regular" in the sense of Zettl [25].) It follows from the theory of regular systems [25] that the fundamental matrix $\Phi(b, \lambda)$ is well defined and is an entire function of $\lambda$. It also follows from the hypothesis that each endpoint is LC that all the Lagrange brackets $[\cdot, \cdot]$ in (2.10) exist as finite limits at each endpoint $a, b$. Hence the matrices $\tilde{A}, \tilde{B}$ in (2.10) are well defined and (2.9) is a regular boundary condition for the system (2.7). Note that $y_j, u_j$, $j = 1, 2$ and hence $\tilde{A}, \tilde{B}$ are independent of $\lambda$. Therefore $\Delta(\lambda)$ defined by (2.12) is an entire function of $\lambda$. It follows from the well known standard theory of boundary value problems for regular systems that $\lambda$ is an eigenvalue of (2.7), (2.9) if and only if $\Delta(\lambda) = 0$.

Finally we note that the scalar SLP (2.1), (2.2) and the system BVP (2.7), (2.9) are equivalent. □

**Remark 2.1.** We comment on the definition of the geometric and the algebraic multiplicity of an eigenvalue $\lambda$. The term 'geometric multiplicity' of an eigenvalue is universally understood to mean the dimension of the eigenspace of $\lambda$, i.e. the number of linearly independent eigenfunctions of $\lambda$.

This applies to both linear algebra and functional analysis. In linear algebra the term 'algebraic multiplicity' of an eigenvalue $\lambda$ means [13], [14] the multiplicity of $\lambda$ as a zero of the characteristic polynomial. On the other hand, in functional analysis [6], [15], the algebraic multiplicity $A(T,\lambda)$ of an eigenvalue $\lambda$ of a linear operator $T$ acting in a Hilbert space $H$ is defined by

$$A(T,\lambda) = dim M(T,\lambda) \tag{2.13}$$

where $M(T,\lambda)$ is the union over all $n \in \mathbb{N}$ of the nullspace of $(T-\lambda I)^n$.

As mentioned in the Introduction our approach in this paper is elementary in the sense that it does not depend on operator theory. Following [7], [20], [11], we construct an entire function whose zeros are precisely the eigenvalues of the problem and then define the algebraic multiplicity of an eigenvalue as its multiplicity as a zero of this entire function which we call the characteristic function of the boundary value problem. Thus the characteristic functions constructed in these papers, for both regular and singular Sturm-Liouville problems, play the role of the characteristic polynomials in linear algebra. Since the characteristic functions are entire functions and thus 'infinite polynomials' this terminology seems reasonable.

The referee has suggested that the term 'analytic multiplicity' would be more appropriate for eigenvalues of Sturm-Liouville and other boundary value problems which can be represented by a linear operator in a Hilbert space and therefore have the above mentioned notion of algebraic multiplicity associated with them from the functional analysis point of view. While we agree with the referee we do not pursue this point further in this paper since, as previously mentioned, our approach here is elementary making no direct use of functional analysis. We are greatful to the referee for bringing this point to our attention and agree that it warrants further consideration.

Theorem 2.1 shows that SLP (2.1), (2.2) and BVP (2.7), (2.9) have exactly the same eigenvalues. We define the algebraic multiplicity (see the three paragraphs above) of an eigenvalue $\lambda$ of (2.1), (2.2) to be the algebraic multiplicity of $\lambda$ as an eigenvalue of (2.7), (2.9), i.e., the order of $\lambda$ as a root of the entire function $\Delta(\lambda)$. Then as shown in [20], the so defined algebraic multiplicity of $\lambda$ is independent of the choice of $\lambda^*$ and $u, v$, and the algebraic and geometric multiplicities of $\lambda$ as an eigenvalue of (2.1), (2.2) are equal. We simply call them *the multiplicity of* $\lambda$. The multiplicity of $\lambda$ is either 1 or 2. We say that $\lambda$ is simple in the former case and double in the latter case.

## 3. Eigenvalue approximations

The following results on the eigenvalues of SLP (2.1), (2.2) are well known, see [21] and Sections 4 and 5 in [25].

**Proposition 3.1.** *Assume that $p > 0$ a.e. on $J$ and $a$ and $b$ are LCNO. Then SLP (2.1), (2.2) has discrete spectrum consisting of an infinite number of eigenvalues $\{\lambda_n : n \in \mathbb{N}_0\}$, which are all real, unbounded from above and bounded from below. They can be indexed to satisfy*

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

*with only double eigenvalues appearing twice. Strict inequalities hold everywhere whenever the BC is separated or non-real coupled.*

**Proposition 3.2.** *Assume $a$ and $b$ are LC. If*

*(i) $p > 0$ a.e. on $J$ and at least one of $a$ and $b$ is O, or*

*(ii) $p$ changes sign on $J$,*

*then SLP (2.1), (2.2) has discrete spectrum consisting of an infinite number of eigenvalues $\{\lambda_n : n \in \mathbb{Z}\}$, which are all real, unbounded from above and below. They can be indexed to satisfy*

$$\cdots \leq \lambda_{-2} \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

*with only double eigenvalues appearing twice. Strict inequalities hold everywhere whenever the BC is separated or non-real coupled.*

**Remark 3.1.** For the cases of Proposition 3.2, the choice of $\lambda_0$ can be arbitrary. In this paper, we adopt the convention that $\lambda_0$ is the first nonnegative eigenvalue of the SLP (2.1), (2.2). This convention determines the index $n$ of $\lambda_n$ uniquely.

Let $\mathcal{N}_a$ be the right-neighborhood of $a$ and $\mathcal{N}_b$ the left-neighborhood of $b$ given in Definition 2.1. Let $a_r, b_r : (0,\infty) \to J$ be continuous functions of $r$ such that $a < a_r < b_r < b$, $a_r \in \mathcal{N}_a$, $b_r \in \mathcal{N}_b$, $a_r$ is strictly decreasing in $r$, $b_r$ is strictly increasing in $r$, and

$$\lim_{r\to\infty} a_r = a \quad \text{and} \quad \lim_{r\to\infty} b_r = b.$$

We observe that in this assumption, there is no restriction on the rates of changes of $a_r, b_r$, and on the relation between the rates of convergence $a_r \to a$ and $b_r \to b$. Following [1], see Section 2, we define the *induced problem* of SLP (2.1), (2.2) on $J_r = (a_r, b_r)$, $r \in (0,\infty)$, to be the regular SLP consisting of the equation

$$-(py')' + qy = \lambda wy \quad \text{on } J_r \tag{3.1}$$

and the BC

$$A\begin{pmatrix} [y,y_1] \\ [y,y_2] \end{pmatrix}(a_r) + B\begin{pmatrix} [y,y_1] \\ [y,y_2] \end{pmatrix}(b_r) = 0. \tag{3.2}$$

Observe that the conditions for $y_1$ and $y_2$ in Definition 2.1 imply that $[y_1, y_2](t) = -1$ for all $t \in \mathcal{N}_a \cup \mathcal{N}_b$, hence BC (3.2) is self-adjoint for all $r \in (0,\infty)$. It is easy to verify that BC (3.2) can be written in the form

$$A_r Y(a_r) + B_r Y(b_r) = 0, \tag{3.3}$$

where

$$Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \; A_r = A\begin{pmatrix} py_1' & -y_1 \\ py_2' & -y_2 \end{pmatrix}(a_r), \text{ and } B_r = B\begin{pmatrix} py_1' & -y_1 \\ py_2' & -y_2 \end{pmatrix}(b_r),$$

and $A_r, B_r$ satisfy the self-adjointness condition (2.3) with $A = A_r, B = B_r$. Note that $A_r$ and $B_r$ depend on $r$ continuously, and BC (3.3) is separated (resp. coupled) if and only if BC (2.2) is separated (resp. coupled). In fact, if BC (2.2) is separated, i.e., $[A|B] = \mathbf{S}_{\alpha,\beta}$, then the induced BC (3.3) becomes

$$\begin{aligned} c_\alpha(a_r)\, y(a_r) - d_\alpha(a_r)\, (py')(a_r) = 0, \\ c_\beta(b_r)\, y(b_r) - d_\beta(b_r)\, (py')(b_r) = 0, \end{aligned} \tag{3.4}$$

where for $\tau \in \{\alpha, \beta\}$ and $t \in \{a_r, b_r\}$

$$\begin{aligned} c_\tau(t) &= \cos\tau\, (py_1')(t) - \sin\tau\, (py_2')(t), \\ d_\tau(t) &= \cos\tau\, y_1(t) - \sin\tau\, y_2(t). \end{aligned} \tag{3.5}$$

If BC (2.2) is coupled, i.e., $[A|B] = [e^{i\theta}K| - I]$, then the induced BC (3.3) becomes

$$Y(b_r) = e^{i\theta}\tilde{K}(a_r, b_r)Y(a_r), \tag{3.6}$$

where

$$\tilde{K}(a_r, b_r) = \begin{pmatrix} -y_2 & y_1 \\ -py_2' & py_1' \end{pmatrix}(b_r) K \begin{pmatrix} py_1' & -y_1 \\ py_2' & -y_2 \end{pmatrix}(a_r) \in SL(2,\mathbb{R}). \tag{3.7}$$

**Remark 3.2.** (i) Assume $p > 0$ a.e. on $J_r$. Then the regular SLP (3.1), (3.2) has an infinite number of eigenvalues $\{\lambda_n(r) : n \in \mathbb{N}_0\}$, which are all real, unbounded from above and bounded from below. They can be indexed to satisfy

$$\lambda_0(r) \le \lambda_1(r) \le \lambda_2(r) \le \cdots$$

with only double eigenvalues appearing twice. Furthermore, we have the following, see [17] for details.

(a) If (3.2) is separated, then each $\lambda_n(r)$ is simple, the index is determined by the Prüfer angle of any corresponding eigenfunction, and each eigenfunction has exactly $n$ zeros in $J_r$. See [24] for the definition of the Prüfer angle and its relationship to the eigenfunctions.
(b) If (3.2) is real coupled, then each $\lambda_n(r)$ may be simple or double, and the number of zeros of any real eigenfunction on $[a_r, b_r)$ is 0 or 1 if $n = 0$, and $n - 1$ or $n$ or $n + 1$ if $n \geq 1$.
(c) If (3.2) is non-real coupled, then each $\lambda_n(r)$ is simple. Let $u_n$ be a corresponding eigenfunction. Then the number of zeros of $\Re u_n$ and $\Im u_n$ on $[a_r, b_r)$ is 0 or 1 if $n = 0$, and $n - 1$ or $n$ or $n + 1$ if $n \geq 1$. Moreover, $u_n$ has no zero on $[a_r, b_r]$.

(ii) Assume $p$ changes sign on $J_r$. Then SLP (3.1), (3.2) has an infinite number of eigenvalues $\{\lambda_n(r) : n \in \mathbb{Z}\}$ which are all real, unbounded from above and below, see [21] and [25]. Based on the recent work [4], the eigenvalues for separated BC's can be indexed using a generalized Prüfer angle. Consequently, the eigenvalues for coupled BC's can be indexed according to the eigenvalue inequalities established in [5]. Although the numbers of zeros of eigenfunctions for this case are different from (i), they can be determined based on the sign changes of $p$ on $J$, see [5] for details.

In the sequel, we will use the indexing scheme for eigenvalues given in Remark 3.2, and investigate the approximation of the eigenvalues of SLP (2.1), (2.2) by those of the family of regular SLP's (3.1), (3.2).

The next lemma extends, for the case $n = 2$, the "Continuation Principle", see Theorems 3.2 and 3.3 in [16].

**Lemma 3.1.** *Let $\mathcal{O}$ be a bounded open set in $\mathbb{R}$ such that its boundary points do not contain eigenvalues of SLP (2.1), (2.2), and let $k \geq 0$ be the number of eigenvalues (counting multiplicity) of (2.1), (2.2) in $\mathcal{O}$. Then there exists $r^* \in (0, \infty)$ such that SLP (3.1), (3.2) has exactly $k$ eigenvalues in $\mathcal{O}$ for all $r > r^*$.*

*Proof.* The transformation (2.11) transforms the singular SLP (2.1), (2.2) to the regular BVP (2.7), (2.9), while the induced SLP (3.1), (3.2) is transformed to the regular BVP consisting of the equation

$$Z' = (\lambda^* - \lambda)GZ \quad \text{on } J_r$$

and the BC

$$A_r Z(a) + B_r Z(b) = 0,$$

where

$$\tilde{A}_r = A \left[ \begin{pmatrix} py_1' & -y_1 \\ py_2 & -y_2 \end{pmatrix} U \right] (a_r) \quad \text{and} \quad \tilde{B}_r = B \left[ \begin{pmatrix} py_1' & -y_1 \\ py_2 & -y_2 \end{pmatrix} U \right] (b_r).$$

Then by the same arguments as in the proof of Theorem 3.2 [16], we obtain the conclusion. □

**Remark 3.3.** For each eigenvalue $\lambda$ of SLP (2.1), (2.2) with multiplicity $k = 1$ or 2, let $\mathcal{O}$ be a bounded neighborhood of $\lambda$ such that the closure of $\mathcal{O}$ does not contain any other eigenvalue of (2.1), (2.2). By applying Lemma 3.1 and Theorem 3.2 in [16] we see that there exist an $r^* \in (0, \infty)$ and $k$ eigenvalues $\Lambda_j(r), 1 \leq j \leq k$, of SLP (3.1), (3.2) in $\mathcal{O}$ for $r \in (r^*, \infty)$, not necessarily distinct when $k = 2$, which are continuous in $r$ and satisfy

$$\lim_{r \to \infty} \Lambda_j(r) = \lambda, \quad 1 \leq j \leq k.$$

**Such eigenvalue functions $\Lambda_j(r), 1 \leq j \leq k$, are called continuous eigenvalue branches to $\lambda$.**

Next we recall some work in Bailey, Everitt, Weidmann and Zettl [1] where equation (2.1) is considered under the assumptions that $p > 0$ a.e. on $J$ and each of $a$ and $b$ is either LC or LP. Using the abstract spectral theory on Hilbert spaces results were obtained on the approximations of the spectrum of any self-adjoint operator associated with equation (2.1) by that of a sequence of induced operators. Applying those results to the case where $p > 0$ a.e. on $J$ and $a$ and $b$ are LC we have the following (stated in our notation).

**Proposition 3.3.** *([1], 15-16) Let $p > 0$ a.e. on $J$ and suppose $a$ and $b$ are LC, and $r \in \mathbb{N}$.*
*(i) Assume $a$ and $b$ are NO. Then for each $n \in \mathbb{N}_0$, $\lambda_n(r) \to \lambda_n$ as $r \to \infty$.*
*(ii) Assume either $a$ or $b$ is O. Then for each $n \in \mathbb{N}_0$ we have*

$$\lambda_n(r) \to -\infty \quad \text{as } r \to \infty.$$

*Nevertheless, for each $n \in \mathbb{Z}$ there exists an index sequence $\tilde{n}(r) : \mathbb{N} \to \mathbb{N}_0$, which depends on $r$, such that*

$$\lambda_{\tilde{n}(r)}(r) \to \lambda_n \quad \text{as } r \to \infty.$$

However, Proposition 3.3 does not yield an explicit construction of continuous eigenvalue branches, and for case (ii), it does not tell us for which values of $r$ the index $\tilde{n}(r)$ changes and how it changes at each such value. Here, with an elementary approach, we will show that for case (i) $\lambda_n(r)$ for $r \in (r^*, \infty)$ forms a continuous eigenvalue branch to $\lambda_n$, where $r^* \in (0, \infty)$ can be found explicitly in terms of the BC basis $\{y_1, y_2\}$; and for case (ii) we will identify the set of $r$ in $(0, \infty)$ where the index function $\tilde{n}(r)$ for a continuous eigenvalue branch to $\lambda_n$ has a jump discontinuity and show how it changes. The results will be further extended to the case where $p$ changes sign on $J$.

The following are the main results of this paper.

**Theorem 3.1.** *Assume that $p > 0$ a.e. on $J$ and $a$ and $b$ are LCNO. Then there exists $r^* \in (0, \infty)$ such that for each eigenvalue $\lambda_n$ of SLP (2.1), (2.2), $n \in \mathbb{N}_0$, the $n$-th eigenvalue of SLP (3.1), (3.2), $\lambda_n(r)$ for $r \in (r^*, \infty)$, constitute a continuous eigenvalue branch to $\lambda_n$.*

*Furthermore, for the separated problem (3.1), (3.4), there are at most a finite number of values of $r$ in $(0, \infty)$ satisfying*

$$\cos\alpha \, y_1(a_r) - \sin\alpha \, y_2(a_r) = 0 \quad \text{or} \quad \cos\beta \, y_1(b_r) - \sin\beta \, y_2(b_r) = 0, \tag{3.8}$$

*and $r^*$ can be chosen to be the largest value of $r$ in $(0, \infty)$ such that (3.8) holds or $r^* = 0$ if no such $r$ exists; and for the coupled problem (3.1), (3.6), there are at most a finite number of values of $r$ in $(0, \infty)$ satisfying*

$$(-y_2(b_r), y_1(b_r)) \, K \begin{pmatrix} -y_1(a_r) \\ -y_2(a_r) \end{pmatrix} = 0, \tag{3.9}$$

*and $r^*$ can be chosen to be the largest value of $r$ in $(0, \infty)$ such that (3.9) holds or $r^* = 0$ if no such $r$ exists.*

For the cases where either at least one of $a$ and $b$ is O or $p$ changes sign, the continuous eigenvalue branches do not have fixed indices. To present the results, we need the following definitions. For more details of the jump set in Definition 3.1, see [19].

**Definition 3.1.** Let $\mathcal{B}$ be the set of all self-adjoint BC's given by (3.2). Define a subset $\mathcal{J}$ of $\mathcal{B}$ by

$$\begin{aligned} \mathcal{J} &= \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \end{bmatrix} \in \mathbb{R}^{2\times 4} : a_2 b_2 = 0 \right\} \\ &\bigcup \left\{ \left[ e^{i\theta} K | - I \right] : K \in SL(2, \mathbb{R}), k_{12} = 0 \right\}. \end{aligned} \tag{3.10}$$

This set $\mathcal{J}$ is called the *jump set* in the self-adjoint BC space $\mathcal{B}$.

**Definition 3.2.** Let $\tilde{n} : (r^*, \infty) \to \mathbb{Z}$ be a function with $r^* \in [0, \infty)$ and $k \in \mathbb{Z}$. $\tilde{n}(r)$ is said to have a *k-jump* at $r = l \in (r^*, \infty)$ if $\tilde{n}(r)$ is discontinuous at $r = l$ and

$$\lim_{r \to l+} \tilde{n}(r) - \lim_{r \to l-} \tilde{n}(r) = k.$$

**Theorem 3.2.** *Assume that $p > 0$ a.e. on $J$, $a$ and $b$ are LC, and at least one of them is O. Then for each eigenvalue $\lambda_n$ of SLP (2.1), (2.2), $n \in \mathbb{Z}$, there exist an $r_n \in (0, \infty)$ and an index function $\tilde{n} : (r_n, \infty) \to \mathbb{N}_0$ such that $\tilde{n}(r) \to \infty$ as $r \to \infty$ and the eigenvalues of SLP (3.1), (3.2), $\lambda_{\tilde{n}(r)}(r)$ for $r \in (r_n, \infty)$, constitute a continuous eigenvalue branch to $\lambda_n$.*

*Furthermore, $\tilde{n}(r)$ is piecewise constant and always continuous from the left on $(r_n, \infty)$. Each of its infinite number of jumps is a 1-jump or a 2-jump. The jump discontinuities occur if and only if BC (3.2) crosses the jump set $\mathcal{J}$ in the self-adjoint BC space $\mathcal{B}$. More precisely, we have the following:*

*(I) Consider the separated BC case, i.e., (2.2) and (3.2) are replaced by (2.4) and (3.4), respectively.*

*Assume $a$ is O and $b$ is NO. Then $\tilde{n}(r)$ has 1-jumps only. It has a 1-jump at $r = l > r_n$ if and only if*

$$\cos\alpha\, y_1(a_l) - \sin\alpha\, y_2(a_l) = 0. \tag{3.11}$$

*Assume $a$ is NO and $b$ is O. Then $\tilde{n}(r)$ has 1-jumps only. It has a 1-jump at $r = l > r_n$ if and only if*

$$\cos\beta\, y_1(b_l) - \sin\beta\, y_2(b_l) = 0. \tag{3.12}$$

*Assume $a$ and $b$ are both O. Then $\tilde{n}(r)$ has a 1-jump at $r = l > r_n$ if and only if exactly one of (3.11) and (3.12) holds; it has a 2-jump at $r = l > r_n$ if and only if both (3.11) and (3.12) hold.*

*(II) Consider the coupled BC case, i.e., (2.2) and (3.2) are replaced by (2.5) and (3.6), respectively. Then $\tilde{n}(r)$ has 1-jumps only. It has a 1-jump at $r = l > r_n$ if and only if*

$$(-y_2(b_l), y_1(b_l))\, K \begin{pmatrix} -y_1(a_l) \\ -y_2(a_l) \end{pmatrix} = 0. \tag{3.13}$$

The next theorem gives results parallel to Theorems 3.1 and 3.2 for the case where $p$ changes sign on $J$.

**Definition 3.3.** For any $t \in J$, we say that the function $p$ *does not change sign* at $t$ if there exists a neighborhood $\mathcal{N}_t$ of $t$ in $J$ such that either $p > 0$ a.e. on $\mathcal{N}_t$ or $p < 0$ a.e. on $\mathcal{N}_t$. Otherwise, we say that $p$ *changes sign* at $t$.

**Theorem 3.3.** *Assume that $p$ changes sign on $J$ and $a$ and $b$ are LC. Define*

$$\mathbb{T} = \{t \in (a, b) : p \text{ changes sign at } t\}.$$

*Assume $a_r$ is not an accumulation point of $\mathbb{T}$ whenever $a_r \in \mathcal{N}_a$ such that (3.11) or (3.13) holds with $l = r$, and $b_r$ is not an accumulation point of $\mathbb{T}$ whenever $b_r \in \mathcal{N}_b$ such that (3.12) or (3.13) holds with $l = r$.*

*Then there exists an $r^* \in [0, \infty)$ and for each eigenvalue $\lambda_n$ of SLP (2.1), (2.2), $n \in \mathbb{Z}$, there exists an index function $\tilde{n} : (r^*, \infty) \to \mathbb{Z}$ such that the eigenvalues of SLP (3.1), (3.2), $\lambda_{\tilde{n}(r)}(r)$ for $r \in (r^*, \infty)$, constitute a continuous eigenvalue branch to $\lambda_n$.*

*Furthermore, $\tilde{n}(r)$ is piecewise constant and may have $k$-jumps with $k = 0, \pm 1, \pm 2$ only. The jump discontinuities occur only if BC (3.2) crosses the jump set $\mathcal{J}$ of the self-adjoint BC space $\mathcal{B}$. More precisely, we have the following:*

*(I) Consider the separated BC case, i.e., (2.2) and (3.2) are replaced by (2.4) and (3.4), respectively. Then*

*(i) $\tilde{n}(r)$ has a 1-jump at $r = l > r^*$ if and only if*
- *(a) (3.11), but not (3.12), holds, and $p > 0$ a.e. in a neighborhood of $a_l$, or*
- *(b) (3.12), but not (3.11), holds, and $p > 0$ a.e. in a neighborhood of $b_l$, or*
- *(c) both (3.11) and (3.12) hold, $b_l \in \mathbb{T}$, and $p > 0$ a.e. in a neighborhood of $a_l$, or*

*(d) both (3.11) and (3.12) hold, $a_l \in \mathbb{T}$, and $p > 0$ a.e. in a neighborhood $\mathcal{N}_{b_l}$ of $b_l$;*

*(ii) $\tilde{n}(r)$ has a 2-jump at $r = l > r^*$ if and only if both (3.11) and (3.12) hold and $p > 0$ a.e. in neighborhoods of $a_l$ and $b_l$;*

*(iii) $\tilde{n}(r)$ has a $(-1)$-jump at $r = l > r^*$ if and only if*

*(a) (3.11), but not (3.12), holds, and $p < 0$ a.e. in a neighborhood of $a_l$, or*
*(b) (3.12), but not (3.11), holds, and $p < 0$ a.e. in a neighborhood of $b_l$, or*
*(c) both (3.11) and (3.12) hold, $b_l \in \mathbb{T}$, and $p < 0$ a.e. in a neighborhood of $a_l$, or*
*(d) both (3.11) and (3.12) hold, $a_l \in \mathbb{T}$, and $p < 0$ a.e. in a neighborhood of $b_l$;*

*(iv) $\tilde{n}(r)$ has a $(-2)$-jump at $r = l > r^*$ if and only if both (3.11) and (3.12) hold and $p < 0$ a.e. in neighborhoods of $a_l$ and $b_l$;*

*(v) $\tilde{n}(r)$ has a 0-jump at $r = l > r^*$ with $a_l, b_l \notin \mathbb{T}$ if and only if both (3.11) and (3.12) hold and $p$ has opposite signs a.e. in a neighborhood $a_l$ and in a neighborhood of $b_l$.*

*(II) Consider the coupled BC's, i.e., (2.2) and (3.2) are replaced by (2.5) and (3.6), respectively. Then $\tilde{n}(r)$ may have $k$-jumps with $k = 0, \pm 1$ only.*

*(i) $\tilde{n}(r)$ has a 1-jump at $r = l > r^*$ if and only if (3.13) holds and $p > 0$ a.e. in neighborhoods of $a_l$ and $b_l$;*

*(ii) $\tilde{n}(r)$ has a $(-1)$-jump at $r = l > r^*$ if and only if (3.13) holds and $p < 0$ a.e. in neighborhoods of $a_l$ and $b_l$;*
*(iii) $\tilde{n}(r)$ has a 0-jump at $r = l > r^*$ if and only if (3.13) holds and $p$ has opposite signs a.e. in a neighborhood of $a_l$ and in a neighborhood of $b_l$.*

*For all the above cases in (I) and (II) and $n \in \mathbb{Z}$, $\tilde{n}(r)$ is continuous from the left at $l$ whenever a $k$-jump occurs with $k > 0$, it is continuous from the right at $l$ whenever a $k$-jump occurs with $k < 0$.*

*Finally, $\tilde{n}(r)$ may be continuous or have a 0-jump at $r = l > r^*$ provided*

*(a) (3.11), but not (3.12), holds with $a_l \in \mathbb{T}$, or*
*(b) (3.12), but not (3.11), holds with $b_l \in \mathbb{T}$, or*
*(c) both (3.11) and (3.12) hold with $a_l, b_l \in \mathbb{T}$, or*
*(d) (3.13) holds with $a_l, b_l \in \mathbb{T}$.*

*The index function $\tilde{n}(r)$ may be continuous or have either 0-jump, 1-jump, or $(-1)$-jump at $r = l > r^*$ provided either $a_l$ or $b_l$, but not both, are in $\mathbb{T}$ and (3.13) holds.*

**Remark 3.4.** It is not difficult to demonstrate that each type of jump in Theorems 3.2 and 3.3 actually occurs with a specific SLP (2.1), (2.2) under the given assumptions, but we omit the details.

**Remark 3.5.** Note that $\tilde{n}(r)$ has a 0-jump at $r = l$ means that $\tilde{n}(r)$ is discontinuous at $r = l$; in fact, it may be discontinuous from either side, or discontinuous from both sides at $r = l$. This is determined by the relations among $\tilde{n}(l)$, $\lim_{r\to l^+} \tilde{n}(r)$, and $\lim_{r\to l^-} \tilde{n}(r)$. For the case that $p$ changes sign on $J$, a systematic classification of the discontinuity of $\tilde{n}(r)$ at $r = l$ where a 0-jump occurs can be established with the ideas outlined in the proof of Theorem 3.3. We omit the details because the 0-jumps are less significant in applications, especially in the numerical approximations of the eigenvalues of SLP (2.1), (2.2).

**Remark 3.6.** For the case where $p > 0$ a.e. near $a$ and $b$, the conclusions in Theorem 3.3 reduce to those in Theorem 3.2.

## 4. Proofs of the main results

1. The case when $p > 0$

We assume $p > 0$ a.e. on $J$, and consider the regular SLP consisting of the equation

$$-(py')' + qy = \lambda wy \quad \text{on } J' = (a', b') \tag{4.1}$$

and the self-adjoint BC

$$A \begin{pmatrix} y \\ py' \end{pmatrix} (a') + B \begin{pmatrix} y \\ py' \end{pmatrix} (b') = 0, \tag{4.2}$$

where $a < a' < b' < b$, and $A, B$ satisfy (2.2). The following lemmas characterize the continuity and discontinuity of the $n$-th eigenvalue $\lambda_n$ of the SLP (4.1), (4.2) when the endpoints $a', b'$ and the BC $[A|B]$ change, see [23] and Theorem 2.1, Lemma 3.32, and Theorem 3.39 in [17] for proofs. The topologies involved are the usual topologies inherited from $\mathbb{R}$ and $\mathbb{C}^{2\times 4}$, respectively.

**Lemma 4.1.** *Assume $p > 0$ a.e. on $J$. For a fixed BC $[A|B] \in \mathcal{B}$ and an $n \in \mathbb{N}_0$, the $n$-th eigenvalue $\lambda_n$ of SLP (4.1), (4.2) depends continuously on $a'$ and on $b'$ respectively.*

**Lemma 4.2.** *Assume $p > 0$ a.e. on $J$.*

*(i) For fixed $a', b' \in (a, b)$ and $n \in \mathbb{N}_0$, the $n$-th eigenvalue $\lambda_n$ of SLP (4.1), (4.2) as a function of the BC $[A|B]$ in $\mathcal{B}$ is continuous at each point of $\mathcal{B} \setminus \mathcal{J}$.*

*(ii) For the separated BC $[A|B] = \mathbf{S}_{\alpha,\beta}$ and $n \in \mathbb{N}_0$, $\lambda_n(\mathbf{S}_{\alpha,\beta})$ is continuous for $(\alpha, \beta) \in [0, \pi) \times (0, \pi]$, strictly decreasing in $\alpha$ and strictly increasing in $\beta$. Moreover, for each $\alpha \in [0, \pi)$*

$$\lim_{\beta \to 0+} \lambda_0(\mathbf{S}_{\alpha,\beta}) = -\infty, \quad \lim_{\beta \to 0+} \lambda_n(\mathbf{S}_{\alpha,\beta}) = \lambda_{n-1}(\mathbf{S}_{\alpha,\pi}) \;\; \textit{for } n \in \mathbb{N}; \tag{4.3}$$

*for each $\beta \in [0, \pi)$*

$$\lim_{\alpha \to \pi-} \lambda_0(\mathbf{S}_{\alpha,\beta}) = -\infty, \quad \lim_{\alpha \to \pi-} \lambda_n(\mathbf{S}_{\alpha,\beta}) = \lambda_{n-1}(\mathbf{S}_{0,\beta}) \;\; \textit{for } n \in \mathbb{N}; \tag{4.4}$$

*and*

$$\lim_{\substack{\alpha \to \pi- \\ \beta \to 0+}} \lambda_0(\mathbf{S}_{\alpha,\beta}) = \lim_{\substack{\alpha \to \pi- \\ \beta \to 0+}} \lambda_1(\mathbf{S}_{\alpha,\beta}) = -\infty, \quad \lim_{\substack{\alpha \to \pi- \\ \beta \to 0+}} \lambda_n(\mathbf{S}_{\alpha,\beta}) = \lambda_{n-2}(\mathbf{S}_{0,\pi}) \;\; \textit{for } n \geq 2. \tag{4.5}$$

*(iii) For the coupled BC $[A|B] = [e^{i\theta}K| - I]$, $\lambda_0$ is discontinuous at each point of $\mathcal{J}$, $\lambda_n, n \in \mathbb{N}$, is continuous on $\mathcal{J}$ whenever $\lambda_n = \lambda_{n+1}$ and discontinuous on $\mathcal{J}$ otherwise. Moreover, let*

$$\mathcal{F}_- = \{[e^{i\theta}K| - I] : K \in SL(2, \mathbb{R}), k_{11}k_{12} \leq 0\} \tag{4.6}$$

*and*

$$\mathcal{F}_+ = \{[e^{i\theta}K| - I] : K \in SL(2, \mathbb{R}), k_{11}k_{12} > 0\}. \tag{4.7}$$

*Then the restriction of $\lambda_n$ to $\mathcal{F}_-$ is continuous for each $n \in \mathbb{N}_0$, and for any $\mathbf{A} \in \mathcal{J}$*

$$\lim_{\mathcal{F}_+ \ni \mathbf{B} \to \mathbf{A}} \lambda_0(\mathbf{B}) = -\infty, \quad \lim_{\mathcal{F}_+ \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = \lambda_{n-1}(\mathbf{A}) \;\; \textit{for } n \in \mathbb{N}. \tag{4.8}$$

**Remark 4.1.** For the case when $p > 0$ a.e. on $J$, it has been further proved that for $n \in \mathbb{N}_0$, the $n$-th eigenvalue $\lambda_n = \lambda_n(a', b', [A|B])$ of SLP (4.1), (4.2) as a function of the endpoints $a', b' \in (a, b)$ and the BC $[A|B] \in \mathcal{B}$ is continuous whenever $[A|B] \in \mathcal{B} \setminus \mathcal{J}$; and the continuous and discontinuous behavior of $\lambda_n(a', b', [A|B])$ at a point where $[A|B] \in \mathcal{J}$ is exactly the same as characterized in Lemma 4.2, no matter how $a', b'$ change near this point. Therefore, the jump discontinuity is solely determined by the BC. See Theorem 3.76 in [17] for details.

To prove Theorem 3.1, we need the following lemmas and corollary. For the proofs of Lemmas 4.3 and 4.4, see [23]. Here, $\mathcal{N}_a$ and $\mathcal{N}_b$ are the same as in Definition 2.1, but $\lambda_a'$ and $\lambda_b'$ may be different from $\lambda_a$ and $\lambda_b$ defined there.

**Lemma 4.3.** *Assume that $p > 0$ a.e. on $J$ and $a$ and $b$ are LCNO. Then there exist real-valued functions $u$ and $v$ in the maximal domain satisfying the following conditions:*

*a) $v > 0$ on $J = (a, b)$;*

*b) for some fixed real $\lambda = \lambda'_a$, $u$ is a principal solution and $v$ is a non-principal solution of equation (2.1) on $\mathcal{N}_a$;*

*c) for some fixed real $\lambda = \lambda'_b$, $u$ is a principal solution and $v$ is a non-principal solution of equation (2.1) on $\mathcal{N}_b$;*

*d) $[u, v](a) = [u, v](b) = -1$.*

**Corollary 4.1.** *Let $u, v$ be defined as in Lemma 4.3. Let $\tilde{p} = pv^2$, and for any real function $y \in \mathcal{D}$ define $z = y/v$. Then on $\mathcal{N}_a \cup \mathcal{N}_b$,*

$$[v, y] = \tilde{p}z' \quad \text{and} \quad [y, u] = z - \frac{u}{v}\tilde{p}z'.$$

*Therefore, $[y, u](c) = z(c)$ and $[v, y](c) = (\tilde{p}z')(c)$ for $c \in \{a, b\}$.*

*Proof.* Since $u$ and $v$ are solutions of (2.1) with $\lambda = \lambda_a$ on $\mathcal{N}_a$ and with $\lambda = \lambda_b$ on $\mathcal{N}_b$, respectively, we have that $[u, v] = -1$ on $\mathcal{N}_a \cup \mathcal{N}_b$. Hence on $\mathcal{N}_a \cup \mathcal{N}_b$,

$$\tilde{p}z' = pv^2 \frac{y'v - yv'}{v^2} = [v, y],$$

and

$$\begin{aligned} [y, u] &= y(pu') - u(py') \\ &= \frac{y}{v}[v(pu') - u(pv')] - \frac{u}{v}[v(py') - y(pv')] \\ &= \frac{y}{v}[v, u] - \frac{u}{v}[v, y] = z - \frac{u}{v}\tilde{p}z'. \end{aligned}$$

The last part of the corollary follows from the fact that $u$ and $v$ are principal and nonprincipal solutions of (2.1) in $\mathcal{N}_a$ for $\lambda = \lambda_a$ and in $\mathcal{N}_b$ for $\lambda = \lambda_b$, respectively. □

**Lemma 4.4.** *Assume that $p > 0$ a.e. on $J$ and $a$ and $b$ are LCNO. Let $u, v$ be defined as in Lemma 4.3 and let $\{y_1, y_2\} := \{u, v\}$ be the BC basis in (2.2). Then the transformation $z(\cdot, \lambda) = y(\cdot, \lambda)/v$ transforms SLP (2.1), (2.2) to the SLP consisting of the equation*

$$-(\tilde{p}z')' + \tilde{q}z = \lambda\tilde{w}z \quad \text{on } J \tag{4.9}$$

*and the BC*

$$A \begin{pmatrix} z \\ \tilde{p}z' \end{pmatrix}(a) + B \begin{pmatrix} z \\ \tilde{p}z' \end{pmatrix}(b) = 0, \tag{4.10}$$

*where*

$$\tilde{p} = pv^2, \tilde{q} = (-(pv')' + qv)v, \text{ and } \tilde{w} = v^2 w. \tag{4.11}$$

*Moreover,*

$$\tilde{p} > 0,\ \tilde{w} > 0 \text{ a.e. on } J, \quad 1/\tilde{p},\ \tilde{q},\ \tilde{w}\ \in L(J, \mathbb{R}),$$

*and consequently, SLP (4.9), (4.10) is regular.*

*Proof of Theorem 3.1:* By Lemma 2.1, BC (2.2) is equivalent to the BC

$$A_1 \begin{pmatrix} [y, u] \\ [y, v] \end{pmatrix}(a) + B_1 \begin{pmatrix} [y, u] \\ [y, v] \end{pmatrix}(b) = 0, \tag{4.12}$$

where

$$A_1 = A \begin{pmatrix} [v, y_1] & [y_1, u] \\ [v, y_2] & [y_2, u] \end{pmatrix}(a) \quad \text{and} \quad B_1 = B \begin{pmatrix} [v, y_1] & [y_1, u] \\ [v, y_2] & [y_2, u] \end{pmatrix}(b).$$

Similarly, the induced BC (3.2) is equivalent to the BC

$$A_1(r)\begin{pmatrix}[y,u]\\ [y,v]\end{pmatrix}(a_r)+B_1(r)\begin{pmatrix}[y,u]\\ [y,v]\end{pmatrix}(b_r)=0, \tag{4.13}$$

where

$$A_1(r)=A\begin{pmatrix}[v,y_1] & [y_1,u]\\ [v,y_2] & [y_2,u]\end{pmatrix}(a_r) \quad \text{and} \quad B_1(r)=B\begin{pmatrix}[v,y_1] & [y_1,u]\\ [v,y_2] & [y_2,u]\end{pmatrix}(b_r). \tag{4.14}$$

Let $\tilde{p}=pv^2$, $z(\cdot,\lambda)=y(\cdot,\lambda)/v$, and $z_i=y_i/v,\ i=1,2$, on $(a,b)$. By Corollary 4.1 we have that on $\mathcal{N}_a\cup\mathcal{N}_b$

$$[v,y]=\tilde{p}z', \quad [y,u]=z-\frac{u}{v}\tilde{p}z'; \tag{4.15}$$

and

$$[v,y_i]=\tilde{p}z_i', \quad [y_i,u]=z_i-\frac{u}{v}\tilde{p}z_i', \quad i=1,2. \tag{4.16}$$

Therefore, from Lemma 4.4, the transformation $z(\cdot,\lambda)=y(\cdot,\lambda)/v$ transforms SLP (2.1), (2.2) into the SLP consisting of equation (4.9) and the BC

$$A_2\begin{pmatrix}z\\ \tilde{p}z'\end{pmatrix}(a)+B_2\begin{pmatrix}z\\ \tilde{p}z'\end{pmatrix}(b)=0, \tag{4.17}$$

where

$$A_2=A\begin{pmatrix}\tilde{p}{z_1}' & -z_1\\ \tilde{p}{z_2}' & -z_2\end{pmatrix}(a) \quad \text{and} \quad B_2=B\begin{pmatrix}\tilde{p}{z_1}' & -z_1\\ \tilde{p}{z_2}' & -z_2\end{pmatrix}(b);$$

and it transforms the induced problem (3.1), (3.2) into the SLP consisting of the equation

$$-(\tilde{p}z')'+\tilde{q}z=\lambda\tilde{w}z \quad \text{on } J_r \tag{4.18}$$

and the BC

$$A_2(r)\begin{pmatrix}z\\ \tilde{p}z'\end{pmatrix}(a_r)+B_2(r)\begin{pmatrix}z\\ \tilde{p}z'\end{pmatrix}(b_r)=0, \tag{4.19}$$

where $\tilde{p},\tilde{q},\tilde{w}$ are defined as in Lemma 4.4, and

$$A_2(r)=A\begin{pmatrix}\tilde{p}{z_1}' & -z_1\\ \tilde{p}{z_2}' & -z_2\end{pmatrix}(a_r) \quad \text{and} \quad B_2(r)=B\begin{pmatrix}\tilde{p}{z_1}' & -z_1\\ \tilde{p}{z_2}' & -z_2\end{pmatrix}(b_r).$$

In fact, from (4.13), (4.14), and (4.16)

$$\begin{aligned}
A_2(r) &= A_1(r)\begin{pmatrix}1 & -\frac{u}{v}\\ 0 & -1\end{pmatrix}(a_r)\\
&= A\begin{pmatrix}[v,y_1] & [y_1,u]\\ [v,y_2] & [y_2,u]\end{pmatrix}(a_r)\begin{pmatrix}1 & -\frac{u}{v}\\ 0 & -1\end{pmatrix}(a_r)\\
&= A\begin{pmatrix}[v,y_1] & -\frac{u}{v}[v,y_1]-[y_1,u]\\ [v,y_2] & -\frac{u}{v}[v,y_2]-[y_2,u]\end{pmatrix}(a_r)\\
&= A\begin{pmatrix}\tilde{p}{z_1}' & -z_1\\ \tilde{p}{z_2}' & -z_2\end{pmatrix}(a_r).
\end{aligned}$$

Similarly for $B_2(r)$.

Clearly, SLP (4.9), (4.17) and its induced problem (4.18), (4.19) are both regular problems, and they have exactly the same eigenvalues as SLP (2.1), (2.2) and its induced problem (3.1), (3.2), respectively.

Since $y_i,\ i=1,2$, are linearly independent solutions of (2.1) in $\mathcal{N}_a$ for $\lambda=\lambda_a$, by definition, $z_i,\ i=1,2$, are linearly independent solutions of (4.9) in $\mathcal{N}_a$ for $\lambda=\lambda_a$.

(i) Consider the case where (2.2) is the separated BC (2.4), i.e., $[A|B] = \mathbf{S}_{\alpha,\beta}$. In this case, define

$$\tilde{z}_1(t) = \cos\alpha\; z_1(t) - \sin\alpha\; z_2(t) \qquad \text{for } t \in \mathcal{N}_a.$$

Then $\tilde{z}_1(t)$ is a nontrivial solution of (4.9) in $\mathcal{N}_a$ for $\lambda = \lambda_a$. Note that equation (2.1) is NO implies that equation (4.9) is NO. Hence $\tilde{z}_1(a_r)$ has at most a finite number of zeros in $(0,\infty)$, and there exists an $r^* \in (0,\infty)$ such that $\tilde{z}_1(a_r) \neq 0$ for all $r \in (r^*,\infty)$. Similarly, if we define

$$\tilde{z}_2(t) = \cos\beta\; z_1(t) - \sin\beta\; z_2(t) \qquad \text{for } t \in \mathcal{N}_b,$$

then $\tilde{z}_2(b_r)$ has at most a finite number of zeros in $(0,\infty)$, and for sufficiently large $r^* \in (0,\infty)$, $\tilde{z}_2(b_r) \neq 0$ for all $r \in (r^*,\infty)$. This implies that the BC given in (4.19) is never in $\mathcal{J}$ for all $r \in (r^*,\infty)$. By Lemmas 4.1, 4.2 and Remark 4.1, for each $n \in \mathbb{N}_0$, $\lambda_n(r)$ for $r \in (r^*,\infty)$ constitute a continuous eigenvalue branch. ¿From the fact that $y_i$ has the same sign as $z_i$ for $i = 1,2$, it is easy to see that there are at most a finite number of values of $r$ in $(0,\infty)$ such that (3.8) holds, and $r^*$ can be chosen to be the largest such value, or 0 if such an $r$ does not exist.

Now we show that for $n \in \mathbb{N}_0$, $\lambda_n(r) \to \lambda_n$ as $r \to \infty$. For $\tau \in \{\alpha,\beta\}$ and $t \in [a,b]$ define

$$\tilde{c}_\tau(t) = \cos\tau\; (\tilde{p}z_1')(t) - \sin\tau\; (\tilde{p}z_2')(t), \quad \tilde{d}_\tau(t) = \cos\tau\; z_1(t) - \sin\tau\; z_2(t).$$

The BC (4.17) becomes

$$\tilde{c}_\alpha(a)z(a) - \tilde{d}_\alpha(a)(\tilde{p}z')(a) = 0, \quad \tilde{c}_\beta(b)z(b) - \tilde{d}_\beta(b)(\tilde{p}z')(b) = 0; \tag{4.20}$$

and BC (4.19) becomes

$$\tilde{c}_\alpha(a_r)z(a_r) - \tilde{d}_\alpha(a_r)(\tilde{p}z')(a_r) = 0, \quad \tilde{c}_\beta(b_r)z(b_r) - \tilde{d}_\beta(b_r)(\tilde{p}z')(b_r) = 0. \tag{4.21}$$

If BC (4.20) is not in the jump set $\mathcal{J}$, then the conclusion follows from Lemmas 4.1, 4.2 and Remark 4.1. Otherwise, we have that $\tilde{d}_\alpha(a)\tilde{d}_\beta(b) = 0$. Without loss of generality assume $\tilde{d}_\alpha(a) = 0$. Then $\tilde{d}_\alpha(a_r) \neq 0$ for $r \in (r^*,\infty)$. Hence for $r \in (r^*,\infty)$

$$\tilde{d}_\alpha(a_r)\tilde{c}_\alpha(a_r) = \tilde{d}_\alpha(a_r)(\tilde{p}\tilde{d}_\alpha')(a_r) > 0$$

since $\tilde{p} > 0$ a.e. on $J$. This, together with Lemmas 4.1, 4.2 and Remark 4.1, implies that the BC (4.21) for $r \in (r^*,\infty)$ is on the continuous side of BC (4.20).

(ii) Consider the case where (2.2) is the coupled BC (2.5), i.e., $[A|B] = [e^{i\theta}K| - I]$. In this case, BC (4.19) becomes $[e^{i\theta}\tilde{K}(a_r,b_r)| - I]$, where

$$\tilde{K}(t,s) = \begin{pmatrix} \tilde{k}_{11} & \tilde{k}_{12} \\ \tilde{k}_{21} & \tilde{k}_{22} \end{pmatrix}(t,s)$$

and

$$\tilde{k}_{11}(t,s) = (-z_2,z_1)(s)K\begin{pmatrix} \tilde{p}z_1' \\ \tilde{p}z_2' \end{pmatrix}(t), \qquad \tilde{k}_{12}(t,s) = (-z_2,z_1)(s)K\begin{pmatrix} -z_1 \\ -z_2 \end{pmatrix}(t),$$

$$\tilde{k}_{21}(t,s) = (-\tilde{p}z_2',\tilde{p}z_1')(s)K\begin{pmatrix} \tilde{p}z_1' \\ \tilde{p}z_2' \end{pmatrix}(t), \quad \tilde{k}_{22}(t,s) = (-\tilde{p}z_2',\tilde{p}z_1')(s)K\begin{pmatrix} -z_1 \\ -z_2 \end{pmatrix}(t).$$

We claim that there are at most a finite number of values of $r$ in $(0,\infty)$ such that $\tilde{k}_{12}(a_r,b_r) = 0$. If not, there exists an increasing sequence $r_n \to \infty$ such that $\tilde{k}_{12}(a_{r_n},b_{r_n}) = 0$, $a_{r_n} \to a' \in \mathcal{N}_a$, and $b_{r_n} \to b' \in \mathcal{N}_b$. By continuity, $\tilde{k}_{12}(a',b') = 0$. Hence $\tilde{k}_{11}(a',b')\tilde{k}_{22}(a',b') > 0$ since $\tilde{K}(a',b') \in SL_2(\mathbb{R})$. By continuity, there exist a neighborhood $\mathcal{N}_{a'}$ of $a'$ in $\mathcal{N}_a$ and a neighborhood $\mathcal{N}_{b'}$ of $b'$ in $\mathcal{N}_b$ such that

$$\tilde{k}_{11}(t,s)\tilde{k}_{22}(t,s) > 0 \qquad \text{for all } t \in \mathcal{N}_{a'} \text{ and } s \in \mathcal{N}_{b'}. \tag{4.22}$$

On the other hand, for a fixed $s \in \mathcal{N}_b$, $\tilde{k}_{12}(\cdot,s)$ is a solution of (4.9) with $\lambda = \lambda_a$ on $\mathcal{N}_a$, and for a fixed $t \in \mathcal{N}_a$, $\tilde{k}_{12}(t,\cdot)$ is a solution of (4.9) with $\lambda = \lambda_b$ on $\mathcal{N}_b$. Therefore, $\tilde{p}(t)\dfrac{\partial}{\partial t}\tilde{k}_{12}(t,s)$ and

$\tilde{p}(s)\dfrac{\partial}{\partial s}\tilde{k}_{12}(t,s)$ exist and are continuous for all $t \in \mathcal{N}_a$ and $s \in \mathcal{N}_b$. Similarly, $\tilde{k}_{11}(t,s)$ and $\tilde{k}_{22}(t,s)$ are continuous for all $t \in \mathcal{N}_a$ and $s \in \mathcal{N}_b$. Furthermore, for $t \in \mathcal{N}_a$ and $s \in \mathcal{N}_b$

$$\tilde{p}(t)\frac{\partial}{\partial t}\tilde{k}_{12}(t,s) = -\tilde{k}_{11}(t,s) \quad \text{and} \quad \tilde{p}(s)\frac{\partial}{\partial s}\tilde{k}_{12}(t,s) = \tilde{k}_{22}(t,s). \tag{4.23}$$

Thus, (4.22) implies that

$$\left[\tilde{p}(t)\frac{\partial}{\partial t}\tilde{k}_{12}(t,s)\right]\left[\tilde{p}(s)\frac{\partial}{\partial s}\tilde{k}_{12}(t,s)\right] < 0 \quad \text{for all } t \in \mathcal{N}_{a'} \text{ and } s \in \mathcal{N}_{b'}.$$

Choose $n$ sufficiently large such that $a_{r_n} \in \mathcal{N}_{a'}$ and $b_{r_n} \in \mathcal{N}_{b'}$. Then

$$\begin{aligned} 0 &= \tilde{k}_{12}(a',b') - \tilde{k}_{12}(a_{r_n},b_{r_n}) \\ &= [\tilde{k}_{12}(a',b') - \tilde{k}_{12}(a_{r_n},b')] + [\tilde{k}_{12}(a_{r_n},b') - \tilde{k}_{12}(a_{r_n},b_{r_n})] \\ &= \int_{a_{r_n}}^{a'} \frac{1}{\tilde{p}(t)}\left[\tilde{p}(t)\frac{\partial}{\partial t}\tilde{k}_{12}(t,b')\right] dt + \int_{b_{r_n}}^{b'} \frac{1}{\tilde{p}(s)}\left[\tilde{p}(s)\frac{\partial}{\partial s}\tilde{k}_{12}(a_{r_n},s)\right] ds \neq 0 \end{aligned}$$

since $(a' - a_{r_n})(b' - b_{r_n}) < 0$. Thus we reach a contradiction.

This claim implies that there exists an $r^* \in (0,\infty)$ such that the BC given in (4.19) is never in $\mathcal{J}$ for all $r \in (r^*,\infty)$. By Lemmas 4.1, 4.2 and Remark 4.1, for each $n \in \mathbb{N}_0$, $\lambda_n(r)$ for $r \in (r^*,\infty)$ constitute a continuous eigenvalue branch. ¿From the fact that $y_i$ has the same sign as $z_i$ for $i = 1,2$, it is easy to see that there are at most a finite number of values of $r$ in $(0,\infty)$ such that (3.9) holds, and $r^*$ can be chosen to be the largest such value, or 0 if such an $r$ does not exist.

Now we show that for $n \in \mathbb{N}_0$, $\lambda_n(r) \to \lambda_n$ as $r \to \infty$. If the BC $[e^{i\theta}\tilde{K}(a,b)| - I]$ is not in the jump set $\mathcal{J}$, then the conclusion follows from Lemmas 4.1, 4.2 and Remark 4.1. Otherwise, we have $\tilde{k}_{12}(a,b) = 0$. By the above argument, (4.23) and (4.22) hold. Without loss of generality we may assume $\tilde{k}_{11}(t,s) > 0$ and $\tilde{k}_{22}(t,s) > 0$ for $t \in \mathcal{N}'_a$ and $s \in \mathcal{N}'_b$. From (4.23) we see that

$$\frac{\partial}{\partial t}\tilde{k}_{12}(t,s) < 0 \quad \text{and} \quad \frac{\partial}{\partial s}\tilde{k}_{12}(t,s) > 0$$

a.e. for $t \in \mathcal{N}'_a$ and $s \in \mathcal{N}'_b$. Consequently, $\tilde{k}_{12}(t,s) < 0$ for $t \in \mathcal{N}'_a$ and $s \in \mathcal{N}'_b$. This, together with Lemmas 4.1, 4.2 and Remark 4.1, implies that the BC $[e^{i\theta}\tilde{K}(a_r,b_r)| - I]$ for $r \in (r^*,\infty)$ is on the continuous side of the BC $[e^{i\theta}\tilde{K}(a,b)| - I]$ and hence completes the proof. ∎

*Proof of Theorem 3.2:* The existence of continuous eigenvalue branches follows from Lemma 3.1. Note that the continuous eigenvalue branches passing through an eigenvalue $\lambda$ of SLP (3.1), (3.2) with $r \in (0,\infty)$ may not be unique. From the latter part of this proof we can see that the set of $r$ such that BC (3.2) is in $\mathcal{J}$ is a discrete set. Hence by Lemmas 4.1, 4.2 and Remark 4.1, for any $l \in (0,\infty)$ and any eigenvalue $\lambda$ of SLP (3.1), (3.2) with $r = l$, we can define a continuous eigenvalue branch $\lambda_{\tilde{n}(r)}(r)$ passing through this $\lambda$ such that the index $\tilde{n}(r) : (r^*,\infty) \to \mathbb{N}_0$ for some $r^* \in [0,\infty)$ satisfies that $\lambda$ is the $\tilde{n}(l)$-th eigenvalue of (3.1), (3.2) with $r = l$, $\tilde{n}(r)$ is continuous at an $r$ if BC (3.2) with this $r$ is outside of $\mathcal{J}$ and has a jump discontinuity consistent with the index change in Lemma 4.2 if (3.2) is in $\mathcal{J}$. We call such continuous eigenvalue branches the natural continuous eigenvalue branches. Clearly, any eigenvalue of (3.1), (3.2) is contained in one of the natural continuous eigenvalue branches, and any two natural continuous eigenvalue branches do not cross each other. It is also easy to see that for each $n \in \mathbb{Z}$, there exists a natural continuous eigenvalue branch $\lambda_{\tilde{n}(r)}(r)$ to $\lambda_n$. For otherwise, there exist $n_* \in \mathbb{Z}$ and $r_* \in (0,\infty)$ such that some eigenvalues on the continuous eigenvalue branch to $\lambda_{n_*}$ for $r \in (r_*,\infty)$ do not belong to any natural continuous eigenvalue branch, which is impossible. We note that $a_r \in \mathcal{N}_a$ and $b_r \in \mathcal{N}_b$ for $r > r_n$.

(I) For the separated BC case, (3.2) becomes (3.4) where $c_\tau(t)$ and $d_\tau(t)$, $\tau = \alpha,\beta$, are given by (3.5). We observe that $d_\tau(t)$ is a nontrivial solution of (2.1) in $\mathcal{N}_a$ for $\lambda = \lambda_a$ and $(pd'_\alpha)(t) = c_\alpha(t)$, and the same for $d_\beta(t)$ and $c_\beta(t)$.

Assume $a$ is O and $b$ is NO. Then we can choose $r_n$ sufficiently large such that $d_\beta(t) \neq 0$ on $(b_{r_n}, b)$. Hence for $r = l > r_n$, BC (3.4) is in $\mathcal{J}$ if and only if

$$d_\alpha(a_l) = -\cos\alpha\, y_1(a_l) + \sin\alpha\, y_2(a_l) = 0.$$

In this case, $(pd'_\alpha)(a_l) \neq 0$, and hence for $t$ a.e. in a neighborhood of $a_l$

$$d_\alpha(t)c_\alpha(t) = d_\alpha(t)(pd'_\alpha)(t) \begin{array}{l} > 0 \text{ for } t > a_l \\ < 0 \text{ for } t < a_l \end{array}$$

since $p > 0$ a.e. on $J$. This, together with (4.4) in Lemma 4.2 and Remark 4.1, implies that $\tilde{n}(r)$ has a 1-jump at $r = l$ and is continuous from the left at $r = l$. Since $d_\alpha(t)$ is oscillatory at $a$, $d_\alpha(a_r) = 0$ for a sequence $r = l_k, k \in \mathbb{N}$. Therefore, $\tilde{n}(r) \to \infty$.

The conclusion for the case that $b$ is O and $a$ is NO follows from (4.3) in Lemma 4.2, and the conclusion for the case that $a$ and $b$ are both O follows from (4.5) in Lemma 4.2. We omit the details.

(II) For the coupled BC case, (3.2) becomes (3.6) where

$$\tilde{K}(a_r, b_r) = \begin{pmatrix} \tilde{k}_{11} & \tilde{k}_{12} \\ \tilde{k}_{21} & \tilde{k}_{22} \end{pmatrix} (a_r, b_r)$$

is given by (3.7). Here for any $r \in (0, \infty)$, $\tilde{K}(a_r, b_r) \in SL_2(\mathbb{R})$, and

$$\tilde{k}_{11}(a_r, b_r) = (-y_2, y_1)(b_r) K \begin{pmatrix} py'_1 \\ py'_2 \end{pmatrix} (a_r),$$
$$\tilde{k}_{12}(a_r, b_r) = (-y_2, y_1)(b_r) K \begin{pmatrix} -y_1 \\ -y_2 \end{pmatrix} (a_r),$$
$$\tilde{k}_{21}(a_r, b_r) = (-py'_2, py'_1)(b_r) K \begin{pmatrix} py'_1 \\ py'_2 \end{pmatrix} (a_r),$$
$$\tilde{k}_{22}(a_r, b_r) = (-py'_2, py'_1)(b_r) K \begin{pmatrix} -y_1 \\ -y_2 \end{pmatrix} (a_r).$$

For fixed $b_r$, $\tilde{k}_{12}(t, b_r)$ is a nontrivial solution of (2.1) in $\mathcal{N}_a$ for $\lambda = \lambda_a$, and $p\tilde{k}'_{12}(t, b_r) = -\tilde{k}_{11}(t, b_r)$. Similarly, for fixed $a_r$, $\tilde{k}_{12}(a_r, t)$ is a nontrivial solution of (2.1) in $\mathcal{N}_b$ for $\lambda = \lambda_b$, and $p\tilde{k}'_{12}(a_r, t) = \tilde{k}_{22}(a_r, t)$.

Note that BC (3.6) is in $\mathcal{J}$ for some $r = l$ if and only if $\tilde{k}_{12}(a_l, b_l) = 0$, i.e., (3.13) holds. In this case, $\tilde{k}_{11}(a_l, b_l)\tilde{k}_{22}(a_l, b_l) > 0$ since $\tilde{K}(a_l, b_l) \in SL_2(\mathbb{R})$. Without loss of generality we assume $\tilde{k}_{11}(a_l, b_l) > 0$ and $\tilde{k}_{22}(a_l, b_l) > 0$. Hence

$$(p\tilde{k}'_{12})(t, b_l)|_{t=a_l} < 0 \text{ and } (p\tilde{k}'_{12})(a_l, t)|_{t=b_l} > 0 \quad \text{for } (a_l, b_l) \subset (a, b). \tag{4.24}$$

This implies that $\tilde{k}_{12}(a_r, b_r) < 0$ as $r \to l-$ and $\tilde{k}_{12}(a_r, b_r) > 0$ as $r \to l+$. Thus, BC (3.7) is in $\mathcal{F}_-$ as $r \to l-$ and in $\mathcal{F}_+$ as $r \to l+$, where $\mathcal{F}_-$ and $\mathcal{F}_+$ are defined as in Lemma 4.2. By Lemma 4.2, (iii) and Remark 4.1, $\tilde{n}(r)$ has a 1-jump at $r = l$ and is continuous from the left at $r = l$. Since either $\tilde{k}_{12}(t, b_r)$ is oscillatory at $a$ or $\tilde{k}_{12}(a_r, t)$ is oscillatory at $b$, $\tilde{k}_{12}(a_r, b_r) = 0$ for a sequence $r = l_k, k \in \mathbb{N}$. Therefore, $\tilde{n}(r) \to \infty$. ∎

2. The case when $p$ changes sign

The following are analogues of Lemmas 4.1, 4.2 and Remark 4.1 for the case where $p$ changes sign, see Theorem 3.10 in [5].

**Lemma 4.5.** *Assume $p$ changes sign on $J$ and $a < a' < b' < b$. For fixed BC $[A|B] \in \mathcal{B}$ and $n \in \mathbb{Z}$, the $n$-th eigenvalue $\lambda_n$ of SLP (4.1), (4.2) depends continuously on $a'$ and on $b'$, respectively.*

**Lemma 4.6.** *Assume $p$ changes sign on $J$.*

*(i) For fixed $a', b' \in (a,b)$ and $n \in \mathbb{Z}$, the $n$-th eigenvalue $\lambda_n$ of SLP (4.1), (4.2) as a function of the BC $[A|B]$ in $\mathcal{B}$ is continuous at each point of $\mathcal{B} \setminus \mathcal{J}$.*

*(ii) For the separated BC $\mathbf{S}_{\alpha,\beta}$ and $n \in \mathbb{Z}$, $\lambda_n(\mathbf{S}_{\alpha,\beta})$ is continuous for $(\alpha,\beta) \in [0,\pi) \times (0,\pi]$, strictly decreasing in $\alpha$ and strictly increasing in $\beta$. Moreover, for each $\alpha \in [0,\pi)$*

$$\lim_{\beta \to 0+} \lambda_n(\mathbf{S}_{\alpha,\beta}) = \lambda_{n-1}(\mathbf{S}_{\alpha,\pi}),$$

*for each $\beta \in (0,\pi]$*

$$\lim_{\alpha \to \pi-} \lambda_n(\mathbf{S}_{\alpha,\beta}) = \lambda_{n-1}(\mathbf{S}_{0,\beta}),$$

*and*

$$\lim_{\substack{\alpha \to \pi- \\ \beta \to 0+}} \lambda_n(\mathbf{S}_{\alpha,\beta}) = \lambda_{n-2}(\mathbf{S}_{0,\pi}).$$

*(iii) For the coupled BC $[e^{i\theta}K| - I]$, let $\mathcal{F}_-$ and $\mathcal{F}_+$ be given by (4.6) and (4.7), respectively. Then for each $n \in \mathbb{Z}$, $\lambda_n$ is continuous on $\mathcal{J}$ whenever $\lambda_n = \lambda_{n+1}$ and discontinuous at any other points of $\mathcal{J}$. Moreover, the restriction of $\lambda_n$ to $\mathcal{F}_-$ is continuous, and for any $\mathbf{A} \in \mathcal{J}$*

$$\lim_{\mathcal{F}_+ \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = \lambda_{n-1}(\mathbf{A}).$$

**Remark 4.2.** For the case when $p$ changes sign on $J$, it has been further proved in [5] that for $n \in \mathbb{N}_0$, the $n$-th eigenvalue $\lambda_n = \lambda_n(a', b', [A|B])$ of SLP (4.1), (4.2) as a function of the endpoints $a', b' \in (a,b)$ and the BC $[A|B] \in \mathcal{B}$ is continuous whenever $[A|B] \in \mathcal{B} \setminus \mathcal{J}$; and the continuous and discontinuous behavior of $\lambda_n(a', b', [A|B])$ at a point where $[A|B] \in \mathcal{J}$ is exactly the same as characterized in Lemma 4.6, no matter how $a', b'$ change near this point. Therefore, the jump discontinuity is solely determined by the BC.

*Proof of Theorem 3.3:* The existence of continuous eigenvalue branches follows from Lemma 3.1. Let $r^* \in [0,\infty)$ be such that $p$ changes sign on $[a_{r^*}, b_{r^*}]$. Then each such branch $\Lambda(r)$ exists on $(r^*, \infty)$ since otherwise, there is an $r_* \in (r^*, \infty)$ such that the spectrum of SLP (3.1), (3.2) with $r = r_*$ is either bounded below or bounded above, contradicting Proposition 3.2. Similar to the proof of Theorem 3.2, by Lemmas 4.5, 4.6 and Remark 4.2, for each $n \in \mathbb{Z}$, we can choose a continuous branch $\lambda_{\tilde{n}(r)}(r)$ to $\lambda_n$ such that the index $\tilde{n} : (r^*, \infty) \to \mathbb{N}_0$ is continuous at $r = l$ if the BC (3.2) with $r = l$ is outside $\mathcal{J}$ and has a jump discontinuity consistent with the index change in Lemma 4.6 if (3.2) is in $\mathcal{J}$. Note that $a_r \in \mathcal{N}_a$ and $b_r \in \mathcal{N}_b$ for $r > r^*$.

The rest of the proof is similar to, though much more complicated than, that of Theorem 3.2. For simplicity, we only present the proof for case (II) under the assumption that $a_l, b_l \notin \mathbb{T}$ whenever (3.13) holds. In this case, $\tilde{k}_{11}(a_l,b_l)\tilde{k}_{22}(a_l,b_l) > 0$ since $\tilde{K}(a_l,b_l) \in SL_2(\mathbb{R})$. Without loss of generality we assume $\tilde{k}_{11}(a_l,b_l) > 0$ and $\tilde{k}_{22}(a_l,b_l) > 0$. Hence (4.24) holds. From the assumptions, there are three possibilities for the signs of $p$ on a neighborhood $\mathcal{N}_{a_l}$ of $a_l$ and on a neighborhood $\mathcal{N}_{b_l}$ of $b_l$:

(i) $p > 0$ a.e. on $\mathcal{N}_{a_l}$ and on $\mathcal{N}_{b_l}$,
(ii) $p < 0$ a.e. on $\mathcal{N}_{a_l}$ and on $\mathcal{N}_{b_l}$,
(iii) $p$ has different signs a.e. on $\mathcal{N}_{a_l}$ and on $\mathcal{N}_{b_l}$.

For case (i), we can show that BC (3.6) is in $\mathcal{F}_-$ as $r \to l-$ and in $\mathcal{F}_+$ as $r \to l+$. By Lemma 4.6, (iii) and Remark 4.2, $\tilde{n}(r)$ has a 1-jump at $r = l$ and is continuous from the left at $r = l$.

For case (ii), (4.24) implies that $\tilde{k}_{12}(a_r,b_r) > 0$ as $r \to l-$ and $\tilde{k}_{12}(a_r,b_r) < 0$ as $r \to l+$. Thus, BC (3.6) is in $\mathcal{F}_+$ as $r \to l-$ and in $\mathcal{F}_-$ as $r \to l+$. By Lemma 4.6, (iii) and Remark 4.2, $\tilde{n}(r)$ has a $(-1)$-jump at $r = l$ and is continuous from the right at $r = l$.

For case (iii), (4.24) implies that either $\tilde{k}_{12}(a_r,b_r) < 0$ or $\tilde{k}_{12}(a_r,b_r) > 0$ as $r \to l$ from both sides of $l$ and $r \neq l$. For the former, $\tilde{n}(r)$ is in $\mathcal{F}_-$ as $r \to l$. By Lemma 4.6, (iii) and Remark 4.2, $\tilde{n}(r)$

has a 0-jump at $r = l$ and is continuous at $r = l$. For the latter, $\tilde{n}(r)$ is in $\mathcal{F}_+$ as $r \to l$ and $r \neq l$. By Lemma 4.6, (iii) and Remark 4.2,

$$\lim_{r \to l} \tilde{n}(r) = n(l) - 1.$$

This means that $\tilde{n}(r)$ has a 0-jump and is continuous from neither side of $r = l$. ∎

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