# A New Proof of the Inequalities among Sturm-Liouville Eigenvalues

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**Abstract.** We consider self-adjoint regular Sturm-Liouville problems with positive leading coefficients and weight functions. A new proof of the inequalities among the eigenvalues for separated boundary conditions and those for coupled boundary conditions established recently by three of the authors with M. S. P. Eastham is given. This new proof does not assume that any special case of the inequalities has been proven.

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Consider a self-adjoint regular Sturm-Liouville problem (SLP) with a positive leading coefficient and a positive weight function, i.e.,

$$-(py')' + qy = \lambda wy \text{ on } (a,b), \tag{0.1}$$

$$(A\,|\,B)\begin{pmatrix} y(a) \\ (py')(a) \\ y(b) \\ (py')(b) \end{pmatrix} = 0, \tag{0.2}$$

where

$$-\infty \le a < b \le \infty, \qquad 1/p,\ q,\ w \in \mathrm{L}((a,b),\mathbb{R}), \qquad p, w > 0 \text{ a.e. on } (a,b), \tag{0.3}$$

$$(A\,|\,B) \in \mathrm{M}^*_{2\times 4}(\mathbb{C}), \qquad A\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}A^* = B\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}B^*, \tag{0.4}$$

and $\lambda \in \mathbb{C}$ is the so called spectral parameter of (0.1). Here $\mathrm{L}((a,b),\mathbb{R})$ denotes the space of Lebesgue integrable real functions on $(a,b)$, $\mathrm{M}^*_{2\times 4}(\mathbb{C})$ stands for the set of 2 by 4 matrices over $\mathbb{C}$ with rank 2, and $A^*$ is the complex conjugate transpose of the complex matrix $A$. It is well-known that the eigenvalues of the problem can be ordered to form a non-decreasing sequence

$$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \cdots \tag{0.5}$$

approaching $+\infty$ so that the number of times an eigenvalue appears in the sequence is equal to its multiplicity. Hence, when the differential equation (0.1) satisfying (0.3) is fixed, there is a function $\lambda_n$ defined on the space of self-adjoint boundary conditions (BC's) for each $n \in \mathbb{N}_0 =: \{0, 1, 2, \cdots\}$.

In [4], for each coupled self-adjoint BC, two separated self-adjoint BC's were identified and some inequalities among the eigenvalues for these three BC's were established. These inequalities generalize the well known classical ones among the eigenvalues for the periodic BC, the Dirichlet BC and the Neumann BC (see, for example, [3] and [2] for the case of smooth coefficients and weight functions and [9] for the general case of integrable coefficients and weight functions). Moreover, they play an important role in the forthcoming update of the Fortran code SLEIGN2, in which they are used not only to bracket the eigenvalues for coupled BC's but also to determine their indices.

In this paper, we present a new proof of these inequalities. This new proof, unlike the original one given in [4], does not use the fact that the inequalities are already proved for the periodic case, and hence shorten the whole proof of the inequalities. To obtain the new proof, we combine some continuity results about the eigenvalues considered as functions on the the space of self-adjoint BC's shown in [7] and some relations among the eigenvalues for coupled BC's and those for separated BC's. These relations are direct consequences of the manifold structure on the space of self-adjoint BC's introduced in [6]. The main idea of this new proof is as follows: one first shows the inequalities for a coupled self-adjoint BC very close to the set of separated self-adjoint BC's, then the inequalities for a general coupled self-adjoint BC can be deduced from the connectedness of some sets of coupled self-adjoint BC's.

This paper is organized as follows. In Section 1, we mention some basic results, describe the space of self-adjoint BC's, and recall certain continuity results about eigenvalues. Section 2 is devoted to the new proof of the inequalities among eigenvalues.

Throughout this paper, we always fix the differential equation (0.1) and assume that it satisfies (0.3).

## §1. Notation and Prerequisite Results

For each $\lambda \in \mathbb{C}$, let $\phi_{11}(\cdot, \lambda)$ and $\phi_{12}(\cdot, \lambda)$ be the solutions to (0.1) determined by the initial conditions

$$(1.1) \qquad \phi_{11}(a,\lambda) = 1,\ (p\phi_{11}')(a,\lambda) = 0 \quad \text{and} \quad \phi_{12}(a,\lambda) = 0,\ (p\phi_{12}')(a,\lambda) = 1,$$

respectively. We will denote $p\phi_{11}'$ by $\phi_{21}$ and $p\phi_{12}'$ by $\phi_{22}$. Set

$$(1.2) \qquad \Phi(t,\lambda) = \begin{pmatrix} \phi_{11}(t,\lambda) & \phi_{12}(t,\lambda) \\ \phi_{21}(t,\lambda) & \phi_{22}(t,\lambda) \end{pmatrix}, \quad t \in [a,b],\ \lambda \in \mathbb{C}.$$

Here the values of $\Phi(\cdot, \lambda)$ at $a$ and $b$ are defined by its right limit at $a$ and left limit at $b$, respectively. For each $t \in [a,b]$, $\Phi(t,\lambda)$ is an entire matrix function of $\lambda$. Moreover, $\Phi(t,\lambda) \in \mathrm{SL}(2,\mathbb{R})$ for $t \in [a,b]$ and $\lambda \in \mathbb{R}$. The following result is well-known (see, for example, [10] or [4]).

**Theorem 1.3.** *The self-adjoint regular Sturm-Liouville problem consisting of (0.1) and (0.2) has an infinite number of eigenvalues, and they are all real and bounded from below. Moreover, the eigenvalues are the zeros of the* characteristic function

$$(1.4) \qquad \Delta(\lambda) =: \det(A + B\Phi(b,\lambda))$$

*of the problem and hence do not have a finite accumulation point.*

Thus, as mentioned in the introduction, the eigenvalues of the problem can be ordered to form a non-decreasing sequence

$$(1.5) \qquad \lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \cdots$$

approaching $+\infty$ so that the number of times an eigenvalue appears in the sequence is equal to its multiplicity. Note that by Theorem 4.16 in [6], the algebraic and geometric multiplicities of each eigenvalue of the SLP consisting of (0.1) and (0.2) are equal. So, in this paper we are not going to distinguish these two concepts and the word multiplicity will be used for either of them. Moreover, when counting the eigenvalues of the problem in a given interval, we will always assume that the eigenvalues are counted according to their multiplicities.

Following [6], we will take the quotient space

$$\mathrm{GL}(2,\mathbb{C})\backslash \mathrm{M}^*_{2\times 4}(\mathbb{C}) \tag{1.6}$$

as the space of BC's, i.e., *each BC is an equivalence class of coefficient matrices of linear systems such as* (0.2), and the BC represented by the linear system (0.2) will be denoted by $[A\,|\,B]$. Note here that square brackets, not parentheses, are used. Usual bold faced capital Latin letters, such as $\boldsymbol{A}$, will also be used for BC's. The space $\mathcal{B}^{\mathbb{R}}_{\mathrm{S}}$ of self-adjoint real BC's consists of the separated real BC's and the coupled real BC's of the form $[K\,|\,-I]$ with $K\in \mathrm{SL}(2,\mathbb{R})$. By Theorem 2.18 in [6], $\mathcal{B}^{\mathbb{R}}_{\mathrm{S}}$ is a connected and compact analytic 3-dimensional manifold. It can be obtained by "gluing" the open sets

$$\mathcal{O}^{\mathbb{R}}_{1,\mathrm{S}}=\mathcal{O}^{\mathbb{R}}_{6,\mathrm{S}}=\{[K\,|\,-I];\ K\in\mathrm{SL}(2,\mathbb{R})\}, \tag{1.7}$$

$$\mathcal{O}^{\mathbb{R}}_{2,\mathrm{S}}=\left\{\begin{bmatrix}1 & a_{12} & 0 & a_{22}\\ 0 & a_{22} & -1 & b_{22}\end{bmatrix};\ a_{12},a_{22},b_{22}\in\mathbb{R}\right\}, \tag{1.8}$$

$$\mathcal{O}^{\mathbb{R}}_{3,\mathrm{S}}=\left\{\begin{bmatrix}1 & a_{12} & -a_{22} & 0\\ 0 & a_{22} & b_{21} & -1\end{bmatrix};\ a_{12},a_{22},b_{21}\in\mathbb{R}\right\}, \tag{1.9}$$

$$\mathcal{O}^{\mathbb{R}}_{4,\mathrm{S}}=\left\{\begin{bmatrix}a_{11} & 1 & 0 & -a_{21}\\ a_{21} & 0 & -1 & b_{22}\end{bmatrix};\ a_{11},a_{21},b_{22}\in\mathbb{R}\right\}, \tag{1.10}$$

$$\mathcal{O}^{\mathbb{R}}_{5,\mathrm{S}}=\left\{\begin{bmatrix}a_{11} & 1 & a_{21} & 0\\ a_{21} & 0 & b_{21} & -1\end{bmatrix};\ a_{11},a_{21},b_{21}\in\mathbb{R}\right\} \tag{1.11}$$

via the coordinate transformations among these open sets. Note that the topology on $\mathrm{SL}(2,\mathbb{R})$ is the one induced from the usual topology on the set $\mathrm{M}_{2\times 2}(\mathbb{R})$ of $2\times 2$ matrices over $\mathbb{R}$, and each of the four open sets in (1.8)–(1.11) can be identified with $\mathbb{R}^3$. A complex BC $[A\,|\,B]$ is self-adjoint if and only if either $[A\,|\,B]$ is real with $\det A=\det B$ or $[A\,|\,B]=[e^{i\theta}K\,|\,-I]$ with $\theta\in(0,\pi)$ and $K\in\mathrm{SL}(2,\mathbb{R})$. By Theorem 2.25 in [6], the space $\mathcal{B}^{\mathbb{C}}_{\mathrm{S}}$ of self-adjoint complex BC's is a connected and compact analytic 4-dimensional real manifold.

When $n\in\mathbb{N}_0$, we will use the notation $\lambda_n(\boldsymbol{A})$ for $\boldsymbol{A}\in\mathcal{B}^{\mathbb{C}}_{\mathrm{S}}$ and $\lambda_n(e^{i\theta}K)$ for $\theta\in(-\pi,\pi]$ and $K\in\mathrm{SL}(2,\mathbb{R})$.

For $K\in\mathrm{SL}(2,\mathbb{R})$, we use $\{\mu_n(K);\ n\in\mathbb{N}_0\}$ to denote the eigenvalues for the separated self-adjoint BC

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & k_{22} & -k_{12}\end{bmatrix}, \tag{1.12}$$

and $\{\nu_n(K);\ n\in\mathbb{N}_0\}$ the eigenvalues for the separated self-adjoint BC

$$\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -k_{21} & k_{11}\end{bmatrix}. \tag{1.13}$$

Here and in the rest of this paper, when a capital Latin letter stands for a matrix, the entrices of the matrix are denoted by the corresponding lower case letter with two indices. We remark that $(k_{22}, -k_{12}) \neq (0,0) \neq (-k_{21}, k_{11})$, since $\det K = 1$. Thus, (1.12) and (1.13) are well-defined BC's. Note also that $\mu_n(K) = \mu_n(-K)$ and $\nu_n(K) = \nu_n(-K)$ for any $n \in \mathbb{N}_0$.

The following results are implied by Lemmas 2.1 and 4.1 in [ 4 ].

**Lemma 1.14** *Let $K \in \mathrm{SL}(2,\mathbb{R})$ and denote by $\Delta_K$ the characteristic function of the Sturm-Liouville problem consisting of (0.1) and the boundary condition $[K \mid -I]$.*

a) *For any $\theta \in (-\pi, \pi]$, the eigenvalues $\{\lambda_n(e^{i\theta}K);\ n \in \mathbb{N}_0\}$ are the roots of the real equation*

$$\Delta_K(\lambda) = 2 - 2\cos\theta. \tag{1.15}$$

b) *For any $\lambda_* \in \mathbb{R}$ such that $0 < \Delta_K(\lambda_*) < 4$, we have $\Delta'_K(\lambda_*) \neq 0$.*

c) *For any $\lambda \in \{\nu_0(K), \nu_1(K), \nu_2(K), \cdots, \mu_0(K), \mu_1(K), \mu_2(K), \cdots\}$, we have*

$$\Delta_K(\lambda) \leq 0 \qquad \text{or} \qquad \Delta_K(\lambda) \geq 4. \tag{1.16}$$

The following result is from Lemma 3.1 in [ 7 ].

**Lemma 1.17** *For any two positive constants $c$ and $\epsilon$, there exists a $\lambda_*$ such that for any $\lambda \leq \lambda_*$,*

$$\phi_{11}(b,\lambda) \geq c, \quad \phi_{12}(b,\lambda) > 0, \quad \phi_{21}(b,\lambda) \geq c, \quad \phi_{22}(b,\lambda) > 0, \tag{1.18}$$

$$\frac{\phi_{11}(b,\lambda)}{\phi_{21}(b,\lambda)} \leq \epsilon, \quad \frac{\phi_{12}(b,\lambda)}{\phi_{11}(b,\lambda)} \leq \epsilon, \quad \frac{\phi_{12}(b,\lambda)}{\phi_{21}(b,\lambda)} \leq \epsilon, \quad \frac{\phi_{22}(b,\lambda)}{\phi_{21}(b,\lambda)} \leq \epsilon. \tag{1.19}$$

The following continuity results about $\lambda_n$ on $\mathcal{B}^{\mathbb{R}}_{\mathrm{S}}$ are from Propositions 3.10 and 3.18 in [ 7 ] and are used there to characterize the discontinuity of $\lambda_n$.

**Lemma 1.20.** *Let $n \in \mathbb{N}_0$. Then, as a function on the space $\mathcal{B}^{\mathbb{R}}_{\mathrm{S}}$ of self-adjoint real boundary conditions, $\lambda_n$ is continuous at each point not in*

$$\begin{aligned} \mathcal{J}^{\mathbb{R}} = \{[K \mid -I];&\quad K \in \mathrm{SL}(2,\mathbb{R}),\ k_{12} = 0\} \\ &\bigcup \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \end{bmatrix} \in \mathcal{B}^{\mathbb{R}}_{\mathrm{S}};\quad a_2 b_2 = 0 \right\}, \end{aligned} \tag{1.21}$$

*and its restriction to*

$$\mathcal{F}^{\mathbb{R}}_{-} = \{[K \mid -I]; \quad K \in \mathrm{SL}(2,\mathbb{R}),\ k_{11}k_{12} \le 0\} \tag{1.22}$$

*is also continuous.*

For each $\alpha \in [0,\pi)$ and $\beta \in (0,\pi]$, let

$$\boldsymbol{S}_{\alpha,\beta} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \end{bmatrix}. \tag{1.23}$$

Then, the set $\mathcal{T}$ of separated real BC's consists of these $\boldsymbol{S}_{\alpha,\beta}$'s and is topologically a torus. The following result is the main part of the theorem in [5], which is proved using some derivative formulas for continuous eigenvalue branches in [8] and the Prüfer transformation.

**Lemma 1.24.** *As a function of* $(\alpha,\beta)$, $\lambda_n(\boldsymbol{S}_{\alpha,\beta})$ *is continuous on* $[0,\pi)\times(0,\pi]$, *strictly decreasing in* $\alpha$, *and strictly increasing in* $\beta$. *Moreover, for each* $\alpha \in [0,\pi)$,

$$\lim_{\beta\to 0^+} \lambda_0(\boldsymbol{S}_{\alpha,\beta}) = -\infty, \qquad \lim_{\beta\to 0^+} \lambda_n(\boldsymbol{S}_{\alpha,\beta}) = \lambda_{n-1}(\boldsymbol{S}_{\alpha,\pi}) \text{ for } n \in \mathbb{N}, \tag{1.25}$$

*and for each* $\beta \in (0,\pi]$,

$$\lim_{\alpha\to \pi^-} \lambda_0(\boldsymbol{S}_{\alpha,\beta}) = -\infty, \qquad \lim_{\alpha\to \pi^-} \lambda_n(\boldsymbol{S}_{\alpha,\beta}) = \lambda_{n-1}(\boldsymbol{S}_{0,\beta}) \text{ for } n \in \mathbb{N}. \tag{1.26}$$

## §2. A New Proof of the Inequalities

In this section, we give a new proof of the following results from [4]. In these results and the rest of this paper, for any integer $k \ge 2$ and any $k$ numbers $c_1$, $c_2$, ..., $c_k$, the notation $\boldsymbol{\{}c_1, c_2, \cdots, c_k\boldsymbol{\}}$ with bold faced braces means each of $c_1$, $c_2$, ..., $c_k$.

**Theorem 2.1.** *Fix the differential equation* (0.1), *assume that it satisfies* (0.3), *and let* $K \in \mathrm{SL}(2,\mathbb{R})$.

a) *If* $k_{11} > 0$ *and* $k_{12} \le 0$, *then for any* $\theta \in (-\pi,\pi)$, $\theta \ne 0$, *we have*

$$\begin{aligned} \nu_0(K) \le \lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) &\le \boldsymbol{\{}\mu_0(K), \nu_1(K)\boldsymbol{\}} \\ \le \lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) &\le \boldsymbol{\{}\mu_1(K), \nu_2(K)\boldsymbol{\}} \\ \le \lambda_2(K) < \lambda_2(e^{i\theta}K) < \lambda_2(-K) &\le \boldsymbol{\{}\mu_2(K), \nu_3(K)\boldsymbol{\}} \\ \le \lambda_3(-K) < \lambda_3(e^{i\theta}K) < \lambda_3(K) &\le \boldsymbol{\{}\mu_3(K), \nu_4(K)\boldsymbol{\}} \le \cdots . \end{aligned} \tag{2.2}$$

b) *If* $k_{11} \le 0$ *and* $k_{12} < 0$*, then for any* $\theta \in (-\pi, \pi)$*,* $\theta \ne 0$*, we have*

$$\begin{aligned}\lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) \le \{\mu_0(K), \nu_0(K)\} \le \\ \lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) \le \{\mu_1(K), \nu_1(K)\} \le \\ \lambda_2(K) < \lambda_2(e^{i\theta}K) < \lambda_2(-K) \le \{\mu_2(K), \nu_2(K)\} \le \\ \lambda_3(-K) < \lambda_3(e^{i\theta}K) < \lambda_3(K) \le \{\mu_3(K), \nu_3(K)\} \le \cdots .\end{aligned} \tag{2.3}$$

c) *If neither Part a) nor Part b) applies to* $K$*, then either Part a) or Part b) applies to* $-K$*.*

Proof. We only need to prove (2.2), since then we can obtain (2.3) from (2.2) (see [4]). Moreover, by Parts a) and b) of Lemma 1.14, we only need to show that for any $K$ in

$$\mathcal{K} =: \{L \in \mathrm{SL}(2, \mathbb{R});\ l_{11} > 0,\, l_{12} \le 0\}, \tag{2.4}$$

one has

$$\begin{aligned}\nu_0(K) \le \lambda_0(K) < \lambda_0(-K) \le \{\mu_0(K), \nu_1(K)\} \\ \le \lambda_1(-K) < \lambda_1(K) \le \{\mu_1(K), \nu_2(K)\} \\ \le \lambda_2(K) < \lambda_2(-K) \le \{\mu_2(K), \nu_3(K)\} \\ \le \lambda_3(-K) < \lambda_3(K) \le \{\mu_3(K), \nu_4(K)\} \le \cdots .\end{aligned} \tag{2.5}$$

Motivated by both Lemma 1.20 and Lemma 1.24, we consider

$$K_h = \begin{pmatrix} 1/h & -1/h \\ 0 & h \end{pmatrix} \in \mathcal{K} \tag{2.6}$$

for $h \in (0, 1]$. Then, using the notation $\boldsymbol{S}_{\alpha,\beta}$ defined by (1.23), we have that as $h \to 0^+$,

$$[K_h \,|\, -I] = \begin{bmatrix} 1 & -1 & -h & 0 \\ 0 & h & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \boldsymbol{S}_{\pi/4,\pi/2}, \tag{2.7}$$

$$[-K_h \,|\, -I] = \begin{bmatrix} 1 & -1 & h & 0 \\ 0 & -h & 0 & -1 \end{bmatrix} \to \boldsymbol{S}_{\pi/4,\pi/2}. \tag{2.8}$$

From (2.7), (2.8) and Lemma 1.20 we deduce that for each $j \in \mathbb{N}_0$,

$$\lim_{h\to 0^+} \lambda_j(K_h) = \lambda_j(\boldsymbol{S}_{\pi/4,\pi/2}) = \lim_{h\to 0^+} \lambda_j(-K_h). \tag{2.9}$$

Note that $\{\mu_j(K_h);\ j \in \mathbb{N}_0\}$ are the eigenvalues for the BC

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & h & 1/h \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -h^2 & -1 \end{bmatrix}, \tag{2.10}$$

which converges to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \boldsymbol{S}_{0,\pi/2} \tag{2.11}$$

as $h \to 0^+$, while $\{\nu_j(K_h);\ j \in \mathbb{N}_0\}$ are the eigenvalues for the BC

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/h \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \boldsymbol{S}_{\pi/2,\pi/2}. \tag{2.12}$$

From (2.10), (2.11) and Lemma 1.24 we then obtain that for each $j \in \mathbb{N}_0$,

$$\lim_{h \to 0^+} \mu_j(K_h) = \lambda_j(\boldsymbol{S}_{0,\pi/2}). \tag{2.13}$$

Lemma 1.24 also implies that

$$\begin{aligned} &\lambda_0(\boldsymbol{S}_{\pi/2,\pi/2}) < \lambda_0(\boldsymbol{S}_{\pi/4,\pi/2}) < \lambda_0(\boldsymbol{S}_{0,\pi/2}) < \\ &\lambda_1(\boldsymbol{S}_{\pi/2,\pi/2}) < \lambda_1(\boldsymbol{S}_{\pi/4,\pi/2}) < \lambda_1(\boldsymbol{S}_{0,\pi/2}) < \cdots . \end{aligned} \tag{2.14}$$

Fix an odd integer $n \geq 5$. Then (2.9), (2.12), (2.13) and (2.14) together yield that when $h$ is positive and sufficiently close to 0,

$$\begin{aligned} &\nu_0(K_h) < \{\lambda_0(K_h), \lambda_0(-K_h)\} < \mu_0(K_h) < \\ &\nu_1(K_h) < \{\lambda_1(K_h), \lambda_1(-K_h)\} < \mu_1(K_h) < \cdots < \\ &\nu_{n-1}(K_h) < \{\lambda_{n-1}(K_h), \lambda_{n-1}(-K_h)\} < \mu_{n-1}(K_h) < \\ &\nu_n(K_h) < \{\lambda_n(K_h), \lambda_n(-K_h)\} < \mu_n(K_h) < \\ &\nu_{n+1}(K_h) < \lambda_{n+1}(K_h). \end{aligned} \tag{2.15}$$

For any $K \in \mathcal{K}$, from

$$\Delta_K(\lambda) = 2 - k_{22}\phi_{11}(b,\lambda) + k_{21}\phi_{12}(b,\lambda) + k_{12}\phi_{21}(b,\lambda) - k_{11}\phi_{22}(b,\lambda) \tag{2.16}$$

and Lemma 1.17 we see that

$$\lim_{\lambda \to -\infty} \Delta_K(\lambda) = -\infty, \tag{2.17}$$

and hence

$$\Delta_K(\lambda) < 0 \text{ on } (-\infty, \lambda_0(K)). \tag{2.18}$$

Thus, by (2.15), (2.18) and Part a) of Lemma 1.14, we have that when $h$ is positive and sufficiently close to 0,

$$\begin{aligned}
&\nu_0(K_h) < \lambda_0(K_h) < \lambda_0(-K_h) < \mu_0(K_h) < \\
&\nu_1(K_h) < \lambda_1(-K_h) < \lambda_1(K_h) < \mu_1(K_h) < \cdots < \\
&\nu_{n-1}(K_h) < \lambda_{n-1}(K_h) < \lambda_{n-1}(-K_h) < \mu_{n-1}(K_h) < \\
&\nu_n(K_h) < \lambda_n(-K_h) < \lambda_n(K_h) < \mu_n(K_h) < \\
&\nu_{n+1}(K_h) < \lambda_{n+1}(K_h),
\end{aligned} \tag{2.19}$$

which together with (2.18) and Parts a) and c) of Lemma 1.14 imply

$$\Delta_{K_h}(\lambda) \le 0 \tag{2.20}$$

for any $\lambda \in \{\nu_0(K_h), \nu_2(K_h), \cdots, \nu_{n+1}(K_h), \mu_1(K_h), \mu_3(K_h), \cdots, \mu_n(K_h)\}$ and

$$\Delta_{K_h}(\lambda) \ge 4 \tag{2.21}$$

for any $\lambda \in \{\nu_1(K_h), \nu_3(K_h), \cdots, \nu_n(K_h), \mu_0(K_h), \mu_2(K_h), \cdots, \mu_{n-1}(K_h)\}$. By (1.12), (1.13) and Lemma 1.24, for each $j \in \mathbb{N}_0$, $\mu_j$ and $\nu_j$ are continuous functions on $\mathcal{K}$. Thus, from (2.20), (2.21), Part c) of Lemma 1.14 and the connectedness of $\mathcal{K}$ we deduce that for each $K \in \mathcal{K}$,

$$\Delta_K(\lambda) \le 0 \tag{2.22}$$

for any $\lambda \in \{\nu_0(K), \nu_2(K), \cdots, \nu_{n+1}(K), \mu_1(K), \mu_3(K), \cdots, \mu_n(K)\}$ and

$$\Delta_K(\lambda) \ge 4 \tag{2.23}$$

for any $\lambda \in \{\nu_1(K), \nu_3(K), \cdots, \nu_n(K), \mu_0(K), \mu_2(K), \cdots, \mu_{n-1}(K)\}$. By Lemma 1.20, for each $j \in \mathbb{N}_0$, $\lambda_j$ is continuous on $\mathcal{K}$. So, (2.19), (2.22), (2.23), the continuity of $\nu_j$, $\lambda_j$ and

$\mu_j$ on $\mathcal{K}$ and the connectedness of $\mathcal{K}$ then yield

$$\begin{aligned} \{\nu_0(K), \lambda_0(K)\} &< \{\lambda_0(-K), \mu_0(K), \nu_1(K), \lambda_1(-K)\} \\ &< \{\lambda_1(K), \mu_1(K), \nu_2(K), \lambda_2(K)\} \\ &< \cdots \\ &< \{\lambda_{n-1}(-K), \mu_{n-1}(K), \nu_n(K), \lambda_n(-K)\} \\ &< \{\lambda_n(K), \mu_n(K), \nu_{n+1}(K), \lambda_{n+1}(K)\} \end{aligned} \tag{2.24}$$

for any $K \in \mathcal{K}$. Since the odd integer $n \geq 5$ is arbitrary, we have that for any such $K$,

$$\begin{aligned} \{\nu_0(K), \lambda_0(K)\} &< \{\lambda_0(-K), \mu_0(K), \nu_1(K), \lambda_1(-K)\} \\ &< \{\lambda_1(K), \mu_1(K), \nu_2(K), \lambda_2(K)\} \\ &< \{\lambda_2(-K), \mu_2(K), \nu_3(K), \lambda_3(-K)\} \\ &< \{\lambda_3(K), \mu_3(K), \nu_4(K), \lambda_4(K)\} < \cdots, \end{aligned} \tag{2.25}$$

which together with (2.18), (2.22) and (2.23) imply (2.5). ∎

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