Geometric Aspects of Sturm-Liouville Problems

I. Structures on Spaces of Boundary Conditions

Qingkai Kong, Hongyou Wu and Anton Zettl

Abstract. We consider some geometric aspects of regular Sturm-Liouville problems. First, we clarify a natural geometric structure on the space of boundary conditions. This structure is the base for studying the dependence of Sturm-Liouville eigenvalues on the boundary condition, and reveals many new properties of these eigenvalues. In particular, the eigenvalues for separated boundary conditions and those for coupled boundary conditions, or the eigenvalues for self-adjoint boundary conditions and those for non-self-adjoint boundary conditions, are closely related under this structure. Then, we give complete characterizations of several subsets of boundary conditions such as the set of self-adjoint boundary conditions that have a given real number as an eigenvalue, and determine their shapes. The shapes are shown to be independent of the differential equation in question. Moreover, we investigate the differentiability of continuous eigenvalue branches under this structure, and discuss the relationships between the algebraic and geometric multiplicities of an eigenvalue.

§1. Introduction

A regular Sturm-Liouville problem (SLP) consists of an ordinary differential equation of the form

$$(1.1) \qquad -(py')' + qy = \lambda wy \text{ on } (a, b)$$

and a complex boundary condition (BC), i.e.,

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$$(A\,|\,B)\begin{pmatrix} y(a) \\ (py')(a) \\ y(b) \\ (py')(b) \end{pmatrix} = 0, \tag{1.2}$$

where

$$\begin{aligned} -\infty \le a < b \le \infty, &\qquad 1/p,\, q,\, w \in \mathrm{L}^1((a,b),\mathbb{R}), \\ w \ne 0 \text{ a.e. on } (a,b), &\qquad (A\,|\,B) \in \mathrm{M}^*_{2\times 4}(\mathbb{C}), \end{aligned} \tag{1.3}$$

and $\lambda \in \mathbb{C}$ is the so called spectral parameter. Here $\mathrm{L}^1((a,b),\mathbb{R})$ denotes the space of Lebesgue integrable real functions on (a,b), while $\mathrm{M}^*_{2\times 4}(\mathbb{C})$ stands for the set of 2 by 4 matrices over $\mathbb{C}$ with rank 2. Each value of λ for which the equation (1.1) has a non-trivial solution satisfying the BC (1.2) is called an eigenvalue of the SLP consisting of (1.1) and (1.2) and such a solution is called an eigenfunction for this eigenvalue.

In this series of papers, we want to address some geometric aspects in the study of SLP's and their applications. These investigations may serve as the beginning of interplay between differential geometry and SLP's. A few observations made from the geometric point of view are quite new, and we believe that they will be proven important.

In this paper, we first clarify a natural geometric structure on the space of complex BC's and on the space of real BC's, i.e., the Grassmann manifold structure. Under this structure, the separated BC's and the coupled ones, or the self-adjoint BC's and the non-self-adjoint ones, are mutually related, which makes it possible to obtain information about SLP's with BC's of one type from information about SLP's with BC's of the other type. For example, from the simplicity of the eigenvalues for separated real BC's one deduces the simplicity of the eigenvalues in an arbitrary bounded domain in $\mathbb{C}$ for any BC sufficiently close to a separated real one. This geometric structure plays an important role in the complete characterization [6] of the discontinuity of the n-th eigenvalue and a new proof [4] of the inequalities among eigenvalues established recently in [2]. More applications of similar flavor will be given in subsequent papers.

Then, we characterize the following subsets of BC's: the set of complex BC's that have a given complex number λ as an eigenvalue of geometric multiplicity 2, the set of complex BC's that have λ as an eigenvalue, the set of real BC's that have a given real number λ as an eigenvalue, the set of self-adjoint complex BC's that have λ as an eigenvalue and the set of self-adjoint real BC's that have λ as an eigenvalue. It turns out that the first set consists of a single coupled BC. This BC varies as λ changes to form a complex curve in the space of complex BC's. The part of this curve corresponding to the real λ's is called the real characteristic curve of the SLP. Using the real characteristic curve, it is proved that when $p, w > 0$ a.e. on (a, b), the eigenvalues for the separated real BC's determine the eigenvalues for any complex boundary condition. We also figure out the shapes of the other sets. It is proved that the shapes do not depend on the concrete differential equation in question. For example, the set of self-adjoint real BC's that have a real number λ as an eigenvalue is always diffeomorphic to the 2-sphere with two points glued together. The reason for this phenomenon is the following: these sets are always images under natural Lie group actions of some sets that are universal to all the regular SLP's. More geometric information about these sets and its applications will appear in later papers.

In [7], Kong and Zettl proved the continuous differentiability (with respect to the BC) of certain continuous eigenvalue branches and obtained formulas for their differentials in several cases. The third purpose of this paper is to prove the analyticity of any continuous simple eigenvalue branch under the manifold structure. Our proof is both elementary and very short. Moreover, the main idea in our proof is used to show that when $w > 0$ a.e. on (a, b), the algebraic and geometric multiplicities of an eigenvalue for a separated real BC are equal. This result and a theorem in [2] together imply that *when $w > 0$ a.e. on (a, b), the algebraic and geometric multiplicities of an eigenvalue for an arbitrary self-adjoint BC are always equal.* We also give an example to demonstrate that in general, the algebraic and geometric multiplicities of an eigenvalue are not equal.

§2. Notation and Prerequisite Results

By a solution to (1.1) we mean a function y on (a, b) such that y and py' are absolutely continuous on all compact subintervals of (a, b) and satisfy (1.1) a.e.. The second condition in (1.3) guarantees that for any solution y to (1.1), y and py' are absolutely continuous on the interval (a, b), hence, one can define $y(a)$, $(py')(a)$, $y(b)$ and $(py')(b)$ via appropriate limits. Thus, the BC (1.2) is always well defined. From now on, we will denote py' by $y^{[1]}$ for any solution y to (1.1).

For each $\lambda \in \mathbb{C}$, let $\phi_{11}(\cdot, \lambda)$ and $\phi_{12}(\cdot, \lambda)$ be the solutions to (1.1) determined by the initial conditions

(2.1) $$\phi_{11}(a, \lambda) = 1,\ \phi_{11}^{[1]}(a, \lambda) = 0, \qquad \phi_{12}(a, \lambda) = 0,\ \phi_{12}^{[1]}(a, \lambda) = 1.$$

Then any solution to (1.1) is a linear combination of $\phi_{11}(\cdot,\lambda)$ and $\phi_{12}(\cdot,\lambda)$. We will denote $\phi_{11}^{[1]}$ and $\phi_{12}^{[1]}$ by ϕ_{21} and ϕ_{22}, respectively. Set

$$\Phi(t,\lambda)=\begin{pmatrix}\phi_{11}(t,\lambda) & \phi_{12}(t,\lambda)\\ \phi_{21}(t,\lambda) & \phi_{22}(t,\lambda)\end{pmatrix},\quad t\in[a,b],\ \lambda\in\mathbb{C}. \tag{2.2}$$

Then $\Phi(t,\lambda)$ satisfies the matrix form of (1.1), i.e.,

$$\Phi'(t,\lambda)=\begin{pmatrix}0 & 1/p(t)\\ q(t)-\lambda w(t) & 0\end{pmatrix}\Phi(t,\lambda), \tag{2.3}$$

and $\Phi(a,\lambda)=I$. It is known [9] that for each $t\in[a,b]$, $\Phi(t,\lambda)$ is an entire matrix function of λ. Moreover, $\Phi(t,\lambda)\in \mathrm{SL}(2,\mathbb{R})$ for $t\in[a,b]$ and $\lambda\in\mathbb{R}$. The following result says that $\Phi(b,\lambda)$ determines the eigenvalues of the SLP.

Lemma 2.1. *A number $\lambda\in\mathbb{C}$ is an eigenvalue of the Sturm-Liouville problem consisting of (1.1) and (1.2) if and only if*

$$\Delta(\lambda)=:\det(A+B\Phi(b,\lambda))=0. \tag{2.4}$$

Therefore, either all the complex numbers are eigenvalues or the eigenvalues are isolated and do not have an accumulation point in $\mathbb{C}$.

We will call the function Δ the *characteristic function* of the SLP. The *algebraic multiplicity* (or just *multiplicity*) of an isolated eigenvalue is the order of the eigenvalue as a zero of Δ. An eigenvalue is said to be *simple* if it has multiplicity 1, while the eigenvalues of multiplicity 2 are called *double* eigenvalues. When we count the (isolated) eigenvalues in a domain in $\mathbb{C}$ of an SLP, their multiplicities will be taken into account. The linear space spanned by the eigenfunctions for an eigenvalue is called the *eigenspace* for the eigenvalue. The *geometric multiplicity* of an eigenvalue is defined to be the dimension of its eigenspace, which is either 1 or 2. The relation between the two multiplicities of an eigenvalue will be discussed in Section 5. The following result is a slight generalization of Theorem 3.1 in [7] or Theorem 3.2 in [5] applied to the variation of the BC in an SLP only. It requires a norm $||\cdot||$ on the space $\mathrm{M}_{2\times 2}(\mathbb{C})$ of 2 by 2 matrices over $\mathbb{C}$ and can be proved using Rouché's Theorem [1].

Theorem 2.2. *Let $\mathcal{N}\subset\mathbb{C}$ be a bounded open set such that its boundary does not contain any eigenvalue of the Sturm-Liouville problem consisting of (1.1) and (1.2), and $n\geq 0$*

the number of eigenvalues in $\mathcal{N}$ of the problem. Then there exists a $\delta > 0$ such that the Sturm-Liouville problem consisting of (1.1) and an arbitrary boundary condition

$$(C \mid D) \begin{pmatrix} y(a) \\ y^{[1]}(a) \\ y(b) \\ y^{[1]}(b) \end{pmatrix} = 0 \tag{2.5}$$

satisfying

$$||A - C|| + ||B - D|| < \delta \tag{2.6}$$

also has exactly n eigenvalues in $\mathcal{N}$.

The following formula has appeared in [2] and can be verified directly using the ordinary differential equation about $\partial_\lambda \Phi(t, \lambda)$ derived from (2.3) and the initial condition $\partial_\lambda \Phi(a, \lambda) = 0$.

$$\partial_\lambda \Phi(t, \lambda) = \Phi(t, \lambda) \begin{pmatrix} \alpha_{12}(t, \lambda) & \alpha_{22}(t, \lambda) \\ -\alpha_{11}(t, \lambda) & -\alpha_{12}(t, \lambda) \end{pmatrix}, \tag{2.7}$$

where

$$\begin{aligned} \alpha_{11}(t, \lambda) &= \int_a^t \phi_{11}(s, \lambda)\phi_{11}(s, \lambda)w(s)\,\mathrm{d}s, \\ \alpha_{12}(t, \lambda) &= \int_a^t \phi_{11}(s, \lambda)\phi_{12}(s, \lambda)w(s)\,\mathrm{d}s, \\ \alpha_{22}(t, \lambda) &= \int_a^t \phi_{12}(s, \lambda)\phi_{12}(s, \lambda)w(s)\,\mathrm{d}s. \end{aligned} \tag{2.8}$$

The reality of p, q in (1.1) and $\Phi(b, \lambda)$ for $\lambda \in \mathbb{R}$ implies the following result.

Lemma 2.3. *The non-real eigenvalues for a real boundary condition appear in conjugate pairs. Each such pair share the same multiplicity and the same geometric multiplicity.*

BC's that can be written into the form

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} y(a) \\ y^{[1]}(a) \\ y(b) \\ y^{[1]}(b) \end{pmatrix} = 0 \tag{2.9}$$

are called *separated* ones. Any eigenvalue for a separated BC has geometric multiplicity 1. A BC that is not separated and not one of the *degenerated* BC's (actually the trivial initial conditions)

$$y(a) = 0 = y^{[1]}(a) \tag{2.10}$$

and

$$y(b) = 0 = y^{[1]}(b) \tag{2.11}$$

is called a *coupled* one. Note that there is no eigenvalue for each of the degenerated BC's. The BC (1.2) is said to be *self-adjoint* if

$$A \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A^* = B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B^*, \tag{2.12}$$

where A^* is the complex conjugate transpose of A. The following result is well-known, see [9] or [2].

Theorem 2.4. *Assume that $p, w > 0$ a.e. on (a, b) and the boundary condition (1.2) is self-adjoint. Then the Sturm-Liouville problem consisting of (1.1) and (1.2) has an infinite number of eigenvalues, and they are real and bounded from below.*

By Lemma 2.1 and Theorem 2.4, when $p, w > 0$ a.e. on (a, b) and the BC (1.2) is self-adjoint, the eigenvalues for (1.2) can be ordered to form a non-decreasing sequence

$$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \cdots \tag{2.13}$$

approaching $+\infty$ so that the number of times an eigenvalue appears in the sequence is equal to its multiplicity. Therefore, for each $n \in \mathbb{N}_0$, λ_n is a function on the space of SLP's with positive leading coefficient and positive weight.

When $w > 0$ a.e. on (a, b), the eigenvalues for a self-adjoint BC are always real. Moreover, we have the following result due to Möller [8].

Theorem 2.5. *Assume that $w > 0$ a.e. on (a, b), p changes sign on (a, b), i.e., both $\{t \in (a, b);\ p(t) > 0\}$ and $\{t \in (a, b);\ p(t) < 0\}$ have positive Lebesgue measures, and the boundary condition (1.2) is self-adjoint. Then the eigenvalues of the Sturm-Liouville problem consisting of (1.1) and (1.2) are neither bounded from below nor bounded from above.*

Throughout this paper, a capital English letter other than Y stands for a 2 by 2 matrix, while the entrices of the matrix are denoted by the corresponding lower case letter with two indices.

§3. Spaces of Boundary Conditions

In this section, we discuss a natural geometric structure on spaces of BC's, give the general continuous dependence of eigenvalues on BC under this geometric structure, and then present some actions of Lie groups on spaces of BC's.

As mentioned in the introduction, a complex BC is just a system of two linearly independent homogeneous equations on $y(a)$, $y^{[1]}(a)$, $y(b)$ and $y^{[1]}(b)$ with complex coefficients, i.e.,

$$(A\,|\,B)\begin{pmatrix} Y(a) \\ Y(b) \end{pmatrix} = 0 \tag{3.1}$$

with $(A\,|\,B) \in \mathrm{M}^*_{2\times 4}(\mathbb{C})$. Here we have used the notation

$$Y(t) = \begin{pmatrix} y(t) \\ y^{[1]}(t) \end{pmatrix}, \qquad t \in [a, b]. \tag{3.2}$$

Two systems

$$(A\,|\,B)\begin{pmatrix} Y(a) \\ Y(b) \end{pmatrix} = 0 \qquad \text{and} \qquad (C\,|\,D)\begin{pmatrix} Y(a) \\ Y(b) \end{pmatrix} = 0 \tag{3.3}$$

represent the same complex BC if and only if there exists a matrix $T \in \mathrm{GL}(2, \mathbb{C})$ such that

$$(C\,|\,D) = (TA\,|\,TB). \tag{3.4}$$

Thus, the space $\mathcal{B}^{\mathbb{C}}$ of complex BC's is just the quotient space

$$\mathrm{GL}(2, \mathbb{C})\backslash\mathrm{M}^*_{2\times 4}(\mathbb{C}). \tag{3.5}$$

The complex BC represented by the system (3.1) will be denoted by $[A\,|\,B]$. For example, in this notation, the two degenerated BC's (2.10) and (2.11) can be written as $[I\,|\,0]$ and $[0\,|\,-I]$, respectively. Usual bold faced capital English letters will also be used to denote BC's. We give the space $\mathrm{M}_{2\times 4}(\mathbb{C})$ of 2 by 4 complex matrices the usual topology on $\mathbb{C}^8$, then $M^*_{2\times 4}(\mathbb{C})$ is an open subset of $M_{2\times 4}(\mathbb{C})$. In this way, $\mathcal{B}^{\mathbb{C}}$ inherits a topology, the quotient topology.

Theorem 3.1. *The space $\mathcal{B}^{\mathbb{C}}$ of complex boundary conditions is a connected and compact complex manifold of complex dimension* 4.

Proof. $\mathcal{B}^{\mathbb{C}}$ is also the space of complex 2-planes in $\mathbb{C}^4$ through the origin, so, it is the well-known Grassmann manifold $G_2(\mathbb{C}^4)$ (see, for example, [3]). ∎

For use in the sequel, here we mention that $\mathcal{B}^{\mathbb{C}}$ has the following canonical atlas of local coordinate systems:

$$
\begin{aligned}
\mathcal{O}_1^{\mathbb{C}} &= \left\{ \begin{bmatrix} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{bmatrix};\ b_{11}, b_{12}, b_{21}, b_{22} \in \mathbb{C} \right\}, \\
\mathcal{O}_2^{\mathbb{C}} &= \left\{ \begin{bmatrix} 1 & a_{12} & 0 & b_{12} \\ 0 & a_{22} & -1 & b_{22} \end{bmatrix};\ a_{12}, a_{22}, b_{12}, b_{22} \in \mathbb{C} \right\}, \\
\mathcal{O}_3^{\mathbb{C}} &= \left\{ \begin{bmatrix} 1 & a_{12} & b_{11} & 0 \\ 0 & a_{22} & b_{21} & -1 \end{bmatrix};\ a_{12}, a_{22}, b_{11}, b_{21} \in \mathbb{C} \right\}, \\
\mathcal{O}_4^{\mathbb{C}} &= \left\{ \begin{bmatrix} a_{11} & 1 & 0 & b_{12} \\ a_{21} & 0 & -1 & b_{22} \end{bmatrix};\ a_{11}, a_{21}, b_{12}, b_{22} \in \mathbb{C} \right\}, \\
\mathcal{O}_5^{\mathbb{C}} &= \left\{ \begin{bmatrix} a_{11} & 1 & b_{11} & 0 \\ a_{21} & 0 & b_{21} & -1 \end{bmatrix};\ a_{11}, a_{21}, b_{11}, b_{21} \in \mathbb{C} \right\}, \\
\mathcal{O}_6^{\mathbb{C}} &= \left\{ \begin{bmatrix} a_{11} & a_{12} & -1 & 0 \\ a_{21} & a_{22} & 0 & -1 \end{bmatrix};\ a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{C} \right\},
\end{aligned}
\tag{3.6}
$$

the so called *canonical coordinate systems* on $\mathcal{B}^{\mathbb{C}}$.

Remark 3.2. Note that $\mathcal{B}^{\mathbb{C}} \setminus \{[I\,|\,0], [0\,|\,-I]\}$ is not compact. This is the reason for including $[I\,|\,0]$ and $[0\,|\,-I]$ in $\mathcal{B}^{\mathbb{C}}$.

Similarly, the space $\mathcal{B}^{\mathbb{R}}$ of real BC's is just ${}_{GL(2,\mathbb{R})}\backslash M^*_{2\times 4}(\mathbb{R})$, and we have the following result.

Theorem 3.3. *The space $\mathcal{B}^{\mathbb{R}}$ of real boundary conditions is a connected and compact analytic manifold of dimension* 4.

Remark 3.4. Geometrically, $\mathcal{B}^{\mathbb{R}}$ is also the space of 2-planes in $\mathbb{R}^4$ through the origin, i.e., the Grassmann manifold $G_2(\mathbb{R}^4)$. It has a canonical atlas $\{\mathcal{O}_j^{\mathbb{R}};\ 1 \le j \le 6\}$ of local coordinate systems, the so called *canonical coordinate systems* on $\mathcal{B}^{\mathbb{R}}$, whose definition is obtained from (3.6) by replacing $\mathbb{C}$ by $\mathbb{R}$.

Under the Grassmann manifold structure on $\mathcal{B}^{\mathbb{C}}$ and $\mathcal{B}^{\mathbb{R}}$, the coupled BC's are naturally related to the degenerated BC's and the separated BC's. Using the canonical coordinate systems on $\mathcal{B}^{\mathbb{C}}$ and $\mathcal{B}^{\mathbb{R}}$, it is easy to determine how close to each other any two given BC's are. Moreover, by applying Theorem 2.2 to each of $\mathcal{O}_1^{\mathbb{C}}$, $\mathcal{O}_2^{\mathbb{C}}$, $\cdots$, $\mathcal{O}_6^{\mathbb{C}}$, one deduces the following general version of the continuous dependence of eigenvalues on BC.

Theorem 3.5. *Let $\mathcal{N} \subset \mathbb{C}$ be a bounded open set whose boundary does not contain any eigenvalue of the Sturm-Liouville problem consisting of (1.1) and (1.2), and $n \geq 0$ the number of the problem's eigenvalues in $\mathcal{N}$. Then there exists a neighborhood $\mathcal{O}$ of the boundary condition (1.2) in $\mathcal{B}^{\mathbb{C}}$ such that the Sturm-Liouville problem consisting of (1.1) and an arbitrary boundary condition in $\mathcal{O}$ also has exactly n eigenvalues in $\mathcal{N}$.*

Remark 3.6. Theorem 3.5 implies that if λ_* is a simple eigenvalue for a BC $\boldsymbol{A} \in \mathcal{B}^{\mathbb{C}}$, then there is a continuous function $\Lambda : \mathcal{O} \to \mathbb{C}$ defined on a connected neighborhood $\mathcal{O}$ of $\boldsymbol{A}$ in $\mathcal{B}^{\mathbb{C}}$ such that

i) $\Lambda(\boldsymbol{A}) = \lambda_*$;

ii) for any $\boldsymbol{X} \in \mathcal{O}$, $\Lambda(\boldsymbol{X})$ is a simple eigenvalue for $\boldsymbol{X}$.

Any two such functions agree on the common part (still a neighborhood of $\boldsymbol{A}$ in $\mathcal{B}^{\mathbb{C}}$) of their domains. So, by the *continuous simple eigenvalue branch* through λ_* we will mean any such function. In general, by a *continuous eigenvalue branch* we mean a continuous function $\Lambda : \mathcal{O} \to \mathbb{C}$ defined on a connected open set $\mathcal{O} \subset \mathcal{B}^{\mathbb{C}}$ such that for each $\boldsymbol{A} \in \mathcal{O}$, $\Lambda(\boldsymbol{A})$ is an eigenvalue for $\boldsymbol{A}$. The concept of continuous eigenvalue branch has appeared in [7] and [5].

Remark 3.7. We may restrict our attention to the space $\mathcal{B}^{\mathbb{R}}$ of real BC's. There is a result for $\mathcal{B}^{\mathbb{R}}$ similar to Theorem 3.5. Moreover, the concepts of continuous eigenvalue branch over $\mathcal{B}^{\mathbb{R}}$ and continuous simple eigenvalue branch over $\mathcal{B}^{\mathbb{R}}$ have their clear meanings.

The following result demonstrates the importance of the concept of continuous simple eigenvalue branch in addition to implying existence of eigenvalues.

Theorem 3.8. *The values of a continuous simple eigenvalue branch over $\mathcal{B}^{\mathbb{R}}$ are either all real or all non-real.*

Proof. Let $\Lambda : \mathcal{O} \to \mathbb{C}$ be a continuous simple eigenvalue branch over $\mathcal{B}^{\mathbb{R}}$. Assume that $\Lambda(\boldsymbol{A}_1)$ is real and $\Lambda(\boldsymbol{A}_2)$ is non-real for some $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathcal{O}$. Consider a path $s \mapsto \boldsymbol{A}(s) \in \mathcal{O}$,

$1 \le s \le 2$, from $\boldsymbol{A}_1$ to $\boldsymbol{A}_2$. By the continuity of Λ, $\Lambda(\boldsymbol{A}(s))$ is non-real for s sufficiently close to 2. Without loss of generality, we can assume that $\Lambda(\boldsymbol{A}(s))$ is non-real for any $s \in (1,2]$. Then, for each $s \in (1,2]$, both $\Lambda(\boldsymbol{A}(s))$ and $\overline{\Lambda(\boldsymbol{A}(s))}$ are eigenvalues for $\boldsymbol{A}(s)$. Since the continuity of Λ also implies that

$$\lim_{s\to 1} \Lambda(\boldsymbol{A}(s)) = \Lambda(\boldsymbol{A}_1), \qquad \lim_{s\to 1} \overline{\Lambda(\boldsymbol{A}(s))} = \Lambda(\boldsymbol{A}_1), \tag{3.7}$$

the multiplicity of $\Lambda(\boldsymbol{A}_1)$ is at least 2. This is impossible. ∎

The space $\mathcal{B}_S^{\mathbb{R}}$ of self-adjoint real BC's consists of the separated real BC's

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & b_{21} & b_{22} \end{bmatrix} \tag{3.8}$$

and the coupled real BC's of the form $[K \mid -I]$ with $K \in \mathrm{SL}(2,\mathbb{R})$. Thus,

$$\mathcal{B}_S^{\mathbb{R}} = \{[A \mid B] \in \mathcal{B}^{\mathbb{R}};\ \det A = \det B\}. \tag{3.9}$$

Theorem 3.9. *The space $\mathcal{B}_S^{\mathbb{R}}$ of self-adjoint real boundary conditions is a connected and closed analytic 3-dimensional submanifold of $\mathcal{B}^{\mathbb{R}}$. Therefore, $\mathcal{B}_S^{\mathbb{R}}$ is also compact.*

Proof. The open subset

$$\mathcal{B}_S^{\mathbb{R}} \cap \mathcal{O}_6^{\mathbb{R}} = \{[K \mid -I]; K \in \mathrm{SL}(2,\mathbb{R})\} \tag{3.10}$$

of $\mathcal{B}_S^{\mathbb{R}}$ consists of the coupled BC's in $\mathcal{B}_S^{\mathbb{R}}$ and is clearly analytic. The separated BC's in $\mathcal{B}_S^{\mathbb{R}}$ are the separated real ones:

$$\begin{bmatrix} 1 & c & 0 & 0 \\ 0 & 0 & -1 & d \end{bmatrix}, \quad \begin{bmatrix} 1 & c & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & d \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{3.11}$$

where c, $d \in \mathbb{R}$, and have the neighborhoods

$$\begin{aligned}
&\left\{\begin{bmatrix} 1 & a_{12} & 0 & a_{22} \\ 0 & a_{22} & -1 & b_{22} \end{bmatrix};\ a_{12}, a_{22}, b_{22} \in \mathbb{R}\right\} = \mathcal{B}_S^{\mathbb{R}} \cap \mathcal{O}_2^{\mathbb{R}}, \\
&\left\{\begin{bmatrix} 1 & a_{12} & -a_{22} & 0 \\ 0 & a_{22} & b_{21} & -1 \end{bmatrix};\ a_{12}, a_{22}, b_{21} \in \mathbb{R}\right\} = \mathcal{B}_S^{\mathbb{R}} \cap \mathcal{O}_3^{\mathbb{R}}, \\
&\left\{\begin{bmatrix} a_{11} & 1 & 0 & -a_{21} \\ a_{21} & 0 & -1 & b_{22} \end{bmatrix};\ a_{11}, a_{21}, b_{22} \in \mathbb{R}\right\} = \mathcal{B}_S^{\mathbb{R}} \cap \mathcal{O}_4^{\mathbb{R}}, \\
&\left\{\begin{bmatrix} a_{11} & 1 & a_{21} & 0 \\ a_{21} & 0 & b_{21} & -1 \end{bmatrix};\ a_{11}, a_{21}, b_{21} \in \mathbb{R}\right\} = \mathcal{B}_S^{\mathbb{R}} \cap \mathcal{O}_5^{\mathbb{R}}
\end{aligned} \tag{3.12}$$

in $\mathcal{B}_S^{\mathbb{R}}$, respectively. These neighborhoods are analytic. So, $\mathcal{B}_S^{\mathbb{R}}$ is an analytic 3-dimensional submanifold of $\mathcal{B}^{\mathbb{R}}$.

Since $\mathrm{SL}(2,\mathbb{R})$ is connected and the separated BC's in $\mathcal{B}_S^{\mathbb{R}}$ can be connected, in the neighborhoods given above, to the coupled ones in $\mathcal{B}_S^{\mathbb{R}}$, $\mathcal{B}_S^{\mathbb{R}}$ is connected.

To see $\mathcal{B}_S^{\mathbb{R}}$ is closed, let $\big\{[A(n)\,|\,B(n)]\big\}_{n=1}^{+\infty}$ be a sequence in $\mathcal{B}_S^{\mathbb{R}}$ such that

$$[A(n)\,|\,B(n)] \to [C\,|\,D] \in \mathcal{B} \tag{3.13}$$

as $n \to +\infty$. Then $[C\,|\,D]$ is in $\mathcal{O}_j^{\mathbb{R}}$ for some j, $1 \le j \le 6$, and $[A(n)\,|\,B(n)]$ is also in $\mathcal{O}_j^{\mathbb{R}}$ when n is sufficiently large. Thus, we can assume that as $n \to +\infty$,

$$(A(n)\,|\,B(n)) \longrightarrow (C\,|\,D) \tag{3.14}$$

in $\mathrm{M}_{2\times 4}(\mathbb{R})$. Thus, from $\det A(n) = \det B(n)$ for each n we deduce $\det C = \det D$, i.e., $[C\,|\,D] \in \mathcal{B}_S^{\mathbb{R}}$. This completes the proof. ∎

Remark 3.10. $\mathcal{B}_S^{\mathbb{R}}$ is a compactification of $\mathrm{SL}(2,\mathbb{R})$.

A complex BC $[A\,|\,B]$ is self-adjoint if and only if either $[A\,|\,B]$ is real with $\det A = \det B$ or $[A\,|\,B] = [e^{i\theta}K\,|\,-I]$ with $\theta \in (0,\pi)$ and $K \in \mathrm{SL}(2,\mathbb{R})$. Equivalently, a complex BC is self-adjoint if and only if it can be written as $[z_1C\,|\,z_2D]$ for some complex numbers z_1, z_2 satisfying $|z_1| = |z_2| > 0$ and real matrices C, D satisfying $\det C = \det D$.

Theorem 3.11. *The space $\mathcal{B}_S^{\mathbb{C}}$ of self-adjoint complex boundary conditions is a connected, closed and analytic real submanifold of $\mathcal{B}^{\mathbb{C}}$ and has dimension 4. Therefore, $\mathcal{B}_S^{\mathbb{C}}$ is also compact.*

Proof. The coupled self-adjoint complex BC's have the neighborhood

$$\big\{[e^{i\theta}K\,|\,-I];\ \theta \in [0,\pi),\ K \in \mathrm{SL}(2,\mathbb{R})\big\} = \mathcal{B}_S^{\mathbb{C}} \cap \mathcal{O}_6^{\mathbb{C}} \tag{3.15}$$

in $\mathcal{B}_S^{\mathbb{C}}$; the separated self-adjoint complex BC's are listed in (3.11) and have the neighborhoods

$$\begin{aligned}
&\left\{\begin{bmatrix}1 & a_{12} & 0 & \bar z\\ 0 & z & -1 & b_{22}\end{bmatrix};\ a_{12}\in\mathbb{R}, z\in\mathbb{C}, b_{22}\in\mathbb{R}\right\} = \mathcal{B}_S^{\mathbb{C}} \cap \mathcal{O}_2^{\mathbb{C}},\\
&\left\{\begin{bmatrix}1 & a_{12} & -\bar z & 0\\ 0 & z & b_{21} & -1\end{bmatrix};\ a_{12}\in\mathbb{R}, z\in\mathbb{C}, b_{21}\in\mathbb{R}\right\} = \mathcal{B}_S^{\mathbb{C}} \cap \mathcal{O}_3^{\mathbb{C}},\\
&\left\{\begin{bmatrix}a_{11} & 1 & 0 & -\bar z\\ z & 0 & -1 & b_{22}\end{bmatrix};\ a_{11}\in\mathbb{R}, z\in\mathbb{C}, b_{22}\in\mathbb{R}\right\} = \mathcal{B}_S^{\mathbb{C}} \cap \mathcal{O}_4^{\mathbb{C}},\\
&\left\{\begin{bmatrix}a_{11} & 1 & \bar z & 0\\ z & 0 & b_{21} & -1\end{bmatrix};\ a_{11}\in\mathbb{R}, z\in\mathbb{C}, b_{21}\in\mathbb{R}\right\} = \mathcal{B}_S^{\mathbb{C}} \cap \mathcal{O}_5^{\mathbb{C}}
\end{aligned} \tag{3.16}$$

in $\mathcal{B}_S^{\mathbb{C}}$, respectively. These neighborhoods are real analytic. Thus, $\mathcal{B}_S^{\mathbb{C}}$ is an analytic real submanifold of $\mathcal{B}^{\mathbb{C}}$ and has dimension 4.

Since each non-real self-adjoint BC can be connected, in the neighborhood given in (3.15), to a self-adjoint real BC and the space $\mathcal{B}_S^{\mathbb{R}}$ of self-adjoint real BC's is connected, $\mathcal{B}_S^{\mathbb{C}}$ is connected.

To see $\mathcal{B}_S^{\mathbb{C}}$ is closed, let $\{[A_n \,|\, B_n]\}_{n=1}^{+\infty}$ be a sequence in $\mathcal{B}_S^{\mathbb{C}}$ that converges to $[A_* \,|\, B_*] \in \mathcal{B}^{\mathbb{C}}$. Without loss of generality, we can assume

$$[A_* \,|\, B_*] = [I \,|\, C] \in \mathcal{O}_1^{\mathbb{C}}. \tag{3.17}$$

Then for sufficiently large n, $[A_n \,|\, B_n] \in \mathcal{O}_1^{\mathbb{C}}$, and hence

$$[A_n \,|\, B_n] = [I \,|\, e^{-i\theta_n} D_n] \tag{3.18}$$

for some $\theta_n \in [0, \pi)$ and $D_n \in \mathrm{SL}(2, \mathbb{R})$. The convergence of $\{[A_n \,|\, B_n]\}_{n=1}^{+\infty}$ implies that

$$e^{-i\theta_n} D_n \longrightarrow C \tag{3.19}$$

in $\mathrm{M}_{2\times 2}(\mathbb{C})$ as $n \to +\infty$. So, $\{D_n\}_{n=1}^{+\infty}$ is bounded in $\mathrm{SL}(2, \mathbb{R})$. Hence, by using subsequences if necessary, we can assume that $\{e^{i\theta_n}\}_{n=1}^{+\infty}$ converges in $\mathbb{C}$, say to $e^{i\theta_*}$ with $\theta_* \in [0, \pi]$, and $\{D_n\}_{n=1}^{+\infty}$ converges in $\mathrm{SL}(2, \mathbb{R})$, say to D_*. Therefore,

$$[A_* \,|\, B_*] = [I \,|\, e^{-i\theta_*} D_*] \in \mathcal{B}_S^{\mathbb{C}}. \tag{3.20}$$

This completes the proof. ∎

Note that $\mathcal{B}_S^{\mathbb{C}}$ is not a complex submanifold of $\mathcal{B}^{\mathbb{C}}$. It is interesting to find out if $\mathcal{B}_S^{\mathbb{C}}$ has a complex structure compatible with its differential structure.

We will also use the concepts of continuous eigenvalue branch over $\mathcal{B}_S^{\mathbb{C}}$ and continuous eigenvalue branch over $\mathcal{B}_S^{\mathbb{R}}$. Combining the reality of the eigenvalues for a self-adjoint BC when $w > 0$ a.e. on (a, b) and Theorem 3.5 yields the following result.

Theorem 3.12. *Assume that $w > 0$ a.e. on (a, b) and the boundary condition (1.2) is self-adjoint. Let r_1 and r_2, $r_1 < r_2$, be any two real numbers such that none of them is an eigenvalue of the Sturm-Liouville problem consisting of (1.1) and (1.2), and $n \geq 0$ the number of eigenvalues in the interval (r_1, r_2) of the problem. Then there exists a neighborhood $\mathcal{O}$ of the boundary condition (1.2) in $\mathcal{B}_S^{\mathbb{C}}$ such that the Sturm-Liouville* *problem consisting of (1.1) and an arbitrary boundary condition in $\mathcal{O}$ also has exactly n eigenvalues in (r_1, r_2).*

REMARK 3.13. Assume that $w > 0$ a.e. on (a, b). Let λ_* be an eigenvalue for a BC $\boldsymbol{A} \in \mathcal{B}_S^{\mathbb{C}}$ and n its multiplicity. Pick a small $\epsilon > 0$ such that $\boldsymbol{A}$ has only n eigenvalues in the interval $[\lambda_* - \epsilon, \lambda_* + \epsilon]$. Then, by Theorem 3.12, there is a connected neighborhood $\mathcal{O}$ of $\boldsymbol{A}$ in $\mathcal{B}_S^{\mathbb{C}}$ such that each BC in $\mathcal{O}$ has only n eigenvalues in $(\lambda_* - \epsilon, \lambda_* + \epsilon)$. Thus, there are continuous functions $\Lambda_1, \cdots, \Lambda_n : \mathcal{O} \to \mathbb{C}$ defined on $\mathcal{O}$ such that

i) $\Lambda_1(\boldsymbol{A}) = \cdots = \Lambda_n(\boldsymbol{A}) = \lambda_*$;

ii) $\Lambda_1(\boldsymbol{X}) \le \cdots \le \Lambda_n(\boldsymbol{X})$ for any $\boldsymbol{X} \in \mathcal{O}$;

iii) for each $\boldsymbol{X} \in \mathcal{O}$, $\Lambda_1(\boldsymbol{X}), \cdots, \Lambda_n(\boldsymbol{X})$ are eigenvalues for $\boldsymbol{X}$.

We will see that $n \le 2$ and when $n = 2$, these are actually different functions on $\mathcal{O}$ and locally they are the only continuous eigenvalue branches over $\mathcal{B}_S^{\mathbb{C}}$ through λ_*, see Remark 5.7.

REMARK 3.14. There hold results for $\mathcal{B}_S^{\mathbb{R}}$ similar to Theorem 3.12 and Remark 3.13.

For use in the sequel, we mention that the set $\mathcal{T}$ of all separated real BC's can be written as

$$\mathcal{T} = \left\{ \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & \sin\beta \end{bmatrix};\ \alpha \in \mathbb{R}/(\pi\mathbb{Z}),\ \beta \in \mathbb{R}/(\pi\mathbb{Z}) \right\} \tag{3.21}$$

and geometrically is a smooth torus (in $\mathcal{B}_S^{\mathbb{R}}$, $\mathcal{B}_S^{\mathbb{C}}$, $\mathcal{B}^{\mathbb{R}}$ and $\mathcal{B}^{\mathbb{C}}$). The diagonal circle in $\mathcal{T}$ will always be denoted by $\mathcal{C}$, i.e.,

$$\mathcal{C} = \left\{ \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & -\cos\alpha & -\sin\alpha \end{bmatrix};\ \alpha \in \mathbb{R}/(\pi\mathbb{Z}) \right\}. \tag{3.22}$$

To end this section, let us discuss some group actions on spaces of BC's. Given

$$\begin{pmatrix} G & H \\ K & L \end{pmatrix} \in \mathrm{GL}(4, \mathbb{R}), \tag{3.23}$$

where $G, H, K, L \in \mathrm{M}_{2\times 2}(\mathbb{R})$, the well-defined map

$$[A \,|\, B] \longmapsto [AG + BK \,|\, AH + BL] \tag{3.24}$$

is a diffeomorphism of $\mathcal{B}^{\mathbb{R}}$ (onto itself). Thus, the group $\mathrm{GL}(4, \mathbb{R})$ acts on $\mathcal{B}^{\mathbb{R}}$ from the right. In particular, the subgroup

$$\left\{ \begin{pmatrix} G & 0 \\ 0 & L \end{pmatrix};\ G, L \in \mathrm{SL}(2, \mathbb{R}) \right\} \tag{3.25}$$

of $\mathrm{GL}(4,\mathbb{R})$ actually acts on $\mathcal{B}_S^{\mathbb{R}}$ as onto diffeomorphisms and also on $\mathcal{T}$ as onto diffeomorphisms. Moreover, for any $G \in \mathrm{GL}(2,\mathbb{R})$, the action of

$$\mathrm{diag}(G, I) =: \begin{pmatrix} G & 0 \\ 0 & I \end{pmatrix} \tag{3.26}$$

on $\mathcal{B}^{\mathbb{R}}$ leaves $\mathcal{O}_6^{\mathbb{R}}$ and $\mathcal{T}$ invariant; and for any $\Psi \in \mathrm{SL}(2,\mathbb{R})$, the action of $\mathrm{diag}(\Psi, I)$ on $\mathcal{B}_S^{\mathbb{R}}$ leaves the open and dense subset $\mathcal{B}_S^{\mathbb{R}} \cap \mathcal{O}_6^{\mathbb{R}}$ of $\mathcal{B}_S^{\mathbb{R}}$ invariant. When there is no confusion, the image of a real BC $[A \,|\, B]$ under the action of $\mathrm{diag}(G, I)$ will be abbreviated as $[A \,|\, B]_\bullet G$, while the image of a subset $\mathcal{S}$ of $\mathcal{B}^{\mathbb{R}}$ will be written as $\mathcal{S}_\bullet G$.

Similarly, the group $\mathrm{GL}(4,\mathbb{C})$ acts on $\mathcal{B}^{\mathbb{C}}$ from the right, the subgroup

$$\big\{\mathrm{diag}(zG, H);\ z \in \mathbb{C},\, |z| = 1,\, G, H \in \mathrm{SL}(2,\mathbb{R})\big\} \tag{3.27}$$

of $\mathrm{GL}(4,\mathbb{C})$ acts on $\mathcal{B}_S^{\mathbb{C}}$, and the notations $[A \,|\, B]_\bullet zG$, $\mathcal{S}_\bullet zG$ have their obvious meanings.

Note that from above, $\mathcal{T}_\bullet G = \mathcal{T}$ for any $G \in \mathrm{SL}(2,\mathbb{R})$. Moreover, there holds the following basic fact.

Proposition 3.15. *If G and H are in $\mathrm{SL}(2,\mathbb{R})$, then $\mathcal{C}_\bullet G = \mathcal{C}_\bullet H$ if and only if $G = \pm H$.*

Proof. The fact is equivalent to the claim that if G is in $\mathrm{SL}(2,\mathbb{R})$, then $\mathcal{C}_\bullet G = \mathcal{C}$ if and only if $G = \pm I$. The latter can be proved as follows: let $G \in \mathrm{SL}(2,\mathbb{R})$, then $\mathcal{C}_\bullet G = \mathcal{C}$ if and only if

$$(g_{12} - g_{21}) + (g_{12} + g_{21}) \cos(2\alpha) + (g_{22} - g_{11}) \sin(2\alpha) = 0 \qquad \text{on } [0, \pi), \tag{3.28}$$

which together with $G \in \mathrm{SL}(2,\mathbb{R})$ amount to $G = \pm I$. ∎

§4. Characteristic Curve and λ-Surfaces

In this section, we will characterize the set of complex BC's that have a complex number λ as an eigenvalue of geometric multiplicity 2, the set of complex BC's that have λ as an eigenvalue, the set of real BC's that have a real number λ as an eigenvalue, the set of self-adjoint complex BC's that have λ as an eigenvalue and the set of self-adjoint real BC's that have λ as an eigenvalue. Some direct applications using these sets are presented. We also give a first geometric description of each of these sets when it is not a point.

Theorem 4.1. *Let λ be a complex number. Then among all the complex boundary conditions, $[\Phi(b,\lambda)\,|\,-I]$ is the unique one that has λ as an eigenvalue of geometric multiplicity 2.*

Proof. A complex BC $[A\,|\,B]$ has λ as an eigenvalue of geometric multiplicity 2 if and only if

$$A = -B\Phi(b,\lambda), \tag{4.1}$$

which implies that $[A\,|\,B]$ is not a separated complex BC and that $\det B \neq 0$: if $\det B = 0$, i.e., if the two rows of B are linearly dependent, then we can assume that the second row of B is 0, and hence the second row of A is also 0, which is impossible. So, the only BC that has λ as an eigenvalue of geometric multiplicity 2 is the one $[\Phi(b,\lambda)\,|\,-I]$. ∎

Definition 4.2. We will call the complex curve

$$\lambda \longmapsto [\Phi(b,\lambda)\,|\,-I], \qquad \lambda \in \mathbb{C} \tag{4.2}$$

in $\mathcal{O}_6^{\mathbb{C}} \subset \mathcal{B}^{\mathbb{C}}$ the *complex characteristic curve* or *characteristic surface* for the equation (1.1) and denote it by $\mathcal{D}^{\mathbb{C}}$, while the analytic real curve

$$\lambda \longmapsto [\Phi(b,\lambda)\,|\,-I], \qquad \lambda \in \mathbb{R} \tag{4.3}$$

in $\mathcal{B}_S^{\mathbb{R}} \cap \mathcal{O}_6^{\mathbb{R}} \subset \mathcal{B}^{\mathbb{R}}$ will be called the *real characteristic curve* for the equation and given the notation $\mathcal{D}^{\mathbb{R}}$.

Theorem 4.1 implies that any complex BC not on $\mathcal{D}^{\mathbb{C}}$ only has eigenvalues of geometric multiplicity 1. Note that $\mathcal{B}^{\mathbb{C}}$ has complex dimension 4, while $\mathcal{D}^{\mathbb{C}} \subset \mathcal{B}^{\mathbb{C}}$ is just an analytic subset of complex dimension 1. So, it is very rare for a complex BC to have an eigenvalue of geometric multiplicity 2. Moreover, since $\mathcal{B}_S^{\mathbb{C}}$ has dimension 4 (even $\mathcal{B}_S^{\mathbb{R}}$ has dimension 3) and $\mathcal{D}^{\mathbb{R}} \subset \mathcal{B}_S^{\mathbb{R}} \subset \mathcal{B}_S^{\mathbb{C}}$ is only a 1-dimensional analytic subset, it is also very rare for a self-adjoint complex BC (even a self-adjoint real BC) to have an eigenvalue of geometric multiplicity 2.

Next, we want to determine all the complex BC's that have a fixed $\lambda \in \mathbb{C}$ as an eigenvalue. Let $\mathcal{E}_\lambda^{\mathbb{C}}$ be the set of these BC's, i.e.,

$$\mathcal{E}_\lambda^{\mathbb{C}} = \big\{[A\,|\,B] \in \mathcal{B}^{\mathbb{C}};\ \det(A + B\Phi(b,\lambda)) = 0\big\}. \tag{4.4}$$

Then, $\mathcal{E}_\lambda^{\mathbb{R}}$ has its obvious meaning and when $\lambda \in \mathbb{R}$,

$$\begin{aligned} \mathcal{E}_\lambda^{\mathbb{R}} &= \mathcal{E}^{\mathbb{R}}{}_\bullet \Phi(b,\lambda) = \big\{[A\Phi(b,\lambda)\,|\,B];\ [A\,|\,B] \in \mathcal{E}^{\mathbb{R}}\big\} \\ &= \big\{[A\,|\,B\Phi(b,\lambda)^{-1}];\ [A\,|\,B] \in \mathcal{E}^{\mathbb{R}}\big\}, \end{aligned} \tag{4.5}$$

where

$$\mathcal{E}^{\mathbb{R}} = \big\{[A\,|\,B] \in \mathcal{B}^{\mathbb{R}};\ \det(A+B)=0\big\}. \tag{4.6}$$

Direct calculations yield

$$\begin{aligned} \mathcal{E}^{\mathbb{R}} &= \left\{\begin{bmatrix} \xi\cos\tau+1 & \xi\sin\tau & -1 & 0 \\ \eta\cos\tau & \eta\sin\tau+1 & 0 & -1 \end{bmatrix};\ \xi,\eta\in\mathbb{R},\ \tau\in\mathbb{R}/(\pi\mathbb{Z})\right\} \\ &\quad \bigcup\left\{\begin{bmatrix} 1 & 0 & \xi\cos\tau-1 & \xi\sin\tau \\ 0 & 1 & \eta\cos\tau & \eta\sin\tau-1 \end{bmatrix};\ \begin{matrix}\xi,\eta\in\mathbb{R},\ \tau\in\mathbb{R}/(\pi\mathbb{Z}) \\ 1-\xi\cos\tau-\eta\sin\tau=0\end{matrix}\right\} \\ &\quad \bigcup\left\{\begin{bmatrix} \cos\tau & \sin\tau & 0 & 0 \\ 0 & 0 & -\cos\tau & -\sin\tau \end{bmatrix};\ \tau\in\mathbb{R}/(\pi\mathbb{Z})\right\} \\ &= \left\{\begin{bmatrix} \xi\cos\tau+1 & \xi\sin\tau & -1 & 0 \\ \eta\cos\tau & \eta\sin\tau+1 & 0 & -1 \end{bmatrix};\ \begin{matrix}\xi,\eta\in\mathbb{R},\ \tau\in\mathbb{R}/(\pi\mathbb{Z}) \\ 1+\xi\cos\tau+\eta\sin\tau=0\end{matrix}\right\} \\ &\quad \bigcup\left\{\begin{bmatrix} 1 & 0 & \xi\cos\tau-1 & \xi\sin\tau \\ 0 & 1 & \eta\cos\tau & \eta\sin\tau-1 \end{bmatrix};\ \xi,\eta\in\mathbb{R},\ \tau\in\mathbb{R}/(\pi\mathbb{Z})\right\} \\ &\quad \bigcup\left\{\begin{bmatrix} \cos\tau & \sin\tau & 0 & 0 \\ 0 & 0 & -\cos\tau & -\sin\tau \end{bmatrix};\ \tau\in\mathbb{R}/(\pi\mathbb{Z})\right\}. \end{aligned} \tag{4.7}$$

Similarly, for $\lambda \in \mathbb{C}$,

$$\begin{aligned} \mathcal{E}_\lambda^{\mathbb{C}} &= \mathcal{E}^{\mathbb{C}}{}_\bullet \Phi(b,\lambda) = \big\{[A\Phi(b,\lambda)\,|\,B];\ [A\,|\,B] \in \mathcal{E}^{\mathbb{C}}\big\} \\ &= \big\{[A\,|\,B\Phi(b,\lambda)^{-1}];\ [A\,|\,B] \in \mathcal{E}^{\mathbb{C}}\big\}, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} \mathcal{E}^{\mathbb{C}} &= \big\{[A\,|\,B] \in \mathcal{B}^{\mathbb{C}};\ \det(A+B)=0\big\} \\ &= \left\{\begin{bmatrix} \xi z_1+1 & \xi z_2 & -1 & 0 \\ \eta z_1 & \eta z_2+1 & 0 & -1 \end{bmatrix};\ \xi,\eta\in\mathbb{C},\ (z_1,z_2)\in\mathbb{CP}^1\right\} \\ &\quad \bigcup\left\{\begin{bmatrix} 1 & 0 & \xi z_1-1 & \xi z_2 \\ 0 & 1 & \eta z_1 & \eta z_2-1 \end{bmatrix};\ \begin{matrix}\xi,\eta\in\mathbb{C},\ (z_1,z_2)\in\mathbb{CP}^1 \\ 1-\xi z_1-\eta z_2=0\end{matrix}\right\} \\ &\quad \bigcup\left\{\begin{bmatrix} z_1 & z_2 & 0 & 0 \\ 0 & 0 & -z_1 & -z_2 \end{bmatrix};\ (z_1,z_2)\in\mathbb{CP}^1\right\} \\ &= \left\{\begin{bmatrix} \xi z_1+1 & \xi z_2 & -1 & 0 \\ \eta z_1 & \eta z_2+1 & 0 & -1 \end{bmatrix};\ \begin{matrix}\xi,\eta\in\mathbb{C},\ (z_1,z_2)\in\mathbb{CP}^1 \\ 1+\xi z_1+\eta z_2=0\end{matrix}\right\} \\ &\quad \bigcup\left\{\begin{bmatrix} 1 & 0 & \xi z_1-1 & \xi z_2 \\ 0 & 1 & \eta z_1 & \eta z_2-1 \end{bmatrix};\ \xi,\eta\in\mathbb{C},\ (z_1,z_2)\in\mathbb{CP}^1\right\} \\ &\quad \bigcup\left\{\begin{bmatrix} z_1 & z_2 & 0 & 0 \\ 0 & 0 & -z_1 & -z_2 \end{bmatrix};\ (z_1,z_2)\in\mathbb{CP}^1\right\}, \end{aligned} \tag{4.9}$$

$\mathbb{C}\mathrm{P}^1 = (\mathbb{C}^2)^*/_\sim$ with $(\mathbb{C}^2)^* = \mathbb{C}^2 \setminus \{(0,0)\}$ and the equivalence relation $\sim$ being defined as follows: $(z_1, z_2) \sim (z_3, z_4)$ if $(z_1, z_2) = k(z_3, z_4)$ for some $k \in \mathbb{C}$. Therefore, we have proven the following result.

Theorem 4.3. i) *The characteristic surface determines all the eigenvalues for each complex boundary condition in the explicit manner given in* (4.8) *and* (4.9)*; the real characteristic surface determines all the real eigenvalues for each real boundary condition in the explicit manner given in* (4.5)–(4.7).

ii) *Each $\mathcal{E}_\lambda^{\mathbb{C}}$ is the image of $\mathcal{E}^{\mathbb{C}}$ under a diffeomorphism of $\mathcal{B}^{\mathbb{C}}$ given by a Lie group action, which sends $\mathcal{E}^{\mathbb{R}}$ to the corresponding $\mathcal{E}_\lambda^{\mathbb{R}}$ when λ is real.*

Remark 4.4. From the point of view of differential topology, the subsets $\mathcal{E}_\lambda^{\mathbb{C}}$, $\lambda \in \mathbb{C}$, of $\mathcal{B}^{\mathbb{C}}$ are the same as $\mathcal{E}^{\mathbb{C}}$, and the subsets $\mathcal{E}_\lambda^{\mathbb{R}}$, $\lambda \in \mathbb{R}$, of $\mathcal{B}^{\mathbb{R}}$ are the same as $\mathcal{E}^{\mathbb{R}}$. This means that the shapes of the sets $\mathcal{E}_\lambda^{\mathbb{C}}$ and $\mathcal{E}_\lambda^{\mathbb{R}}$ do not depend on the actual differential equation in question.

Remark 4.5. The subsets $\mathcal{E}_\lambda^{\mathbb{R}}$, $\lambda \in \mathbb{R}$, of $\mathcal{B}^{\mathbb{R}}$ and $\mathcal{E}_\lambda^{\mathbb{C}}$, $\lambda \in \mathbb{C}$, of $\mathcal{B}^{\mathbb{C}}$ are solely determined by $\Phi(b, \lambda)$, and no more information about the equation is needed. Moreover, the way in which $\Phi(b, \lambda)$ determines $\mathcal{E}_\lambda^{\mathbb{R}}$ or $\mathcal{E}_\lambda^{\mathbb{C}}$ is independent of the equation in question. In other words, the eigenvalues of the complex BC's are determined by the equation via an intermediate and geometric object–the characteristic surface $\mathcal{D}^{\mathbb{C}}$, and the real eigenvalues of the real BC's are determined by the equation also via an intermediate and geometric object–the real characteristic curve $\mathcal{D}^{\mathbb{R}}$. This observation implies the following result.

Corollary 4.6. i) *Let λ_* and $\lambda_\#$ be two complex numbers. If there is a complex boundary condition having λ_* and $\lambda_\#$ as eigenvalues of geometric multiplicity* 2*, then any complex boundary condition having one of λ_* and $\lambda_\#$ as an eigenvalue must have both of them as eigenvalues. Moreover, the converse holds: if every complex boundary condition having one of λ_* and $\lambda_\#$ as an eigenvalue actually has both of them as eigenvalues, then there is a complex boundary condition having both λ_* and $\lambda_\#$ as eigenvalues of geometric multiplicity* 2.

ii) *The results in* i) *still hold if only real boundary conditions and real eigenvalues are considered.*

For example, for the Fourier equation $-y'' = \lambda y$ on $[0, 1]$, any complex BC having one of $(2\pi)^2$, $(4\pi)^2$, $(6\pi)^2$, $\cdots$ as an eigenvalue must have all of them as eigenvalues, and any complex BC having one of π^2, $(3\pi)^2$, $(5\pi)^2$, $\cdots$ as an eigenvalue must have all of them as eigenvalues. This is because the BC $[I \mid -I]$ has $(2\pi)^2$, $(4\pi)^2$, $(6\pi)^2$, $\cdots$ as eigenvalues of geometric multiplicity 2 and the BC $[-I \mid -I]$ has π^2, $(3\pi)^2$, $(5\pi)^2$, $\cdots$ as eigenvalues of geometric multiplicity 2. Of course, the complex BC's having $(2\pi)^2$, $(4\pi)^2$, $(6\pi)^2$, $\cdots$ as eigenvalues are precisely the ones in $\mathcal{E}^{\mathbb{C}}$, while the BC's having π^2, $(3\pi)^2$, $(5\pi)^2$, $\cdots$ as eigenvalues are exactly the ones in $\mathcal{E}^{\mathbb{C}}{}_{\bullet}(-I)$.

Remark 4.7. Theorem 4.3 raises the following question: how can one determine the differential equation (1.1), i.e., its coefficient functions p, q and its weight function w, using the geometric properties of the real characteristic curve? We will come back to this topic in a later paper.

Remark 4.8. Since each $\mathcal{E}^{\mathbb{C}}_{\lambda}$ is an algebraic variety in $\mathcal{B}^{\mathbb{C}} = \mathrm{G}_{4,2}(\mathbb{C})$, the converse in Corollary 4.6 holds under a weaker assumption. Moreover, if G_1 and G_2 are in $\mathrm{SL}(2,\mathbb{R})$, then the intersection of $\mathcal{E}^{\mathbb{C}}{}_{\bullet}G_1$ and $\mathcal{E}^{\mathbb{C}}{}_{\bullet}G_2$ is generically of real dimension 5; ...; if G_1, ..., and G_8 are in $\mathrm{SL}(2,\mathbb{R})$, then the intersection of $\mathcal{E}^{\mathbb{C}}{}_{\bullet}G_1$, ..., and $\mathcal{E}^{\mathbb{C}}{}_{\bullet}G_8$ is generically empty. Thus, given an equation it is very "rare" for a fixed set of eight real numbers to be eigenvalues of a complex BC at the same time. Similarly, given an equation it is also very "rare" for a fixed set of five real numbers to be eigenvalues of a real BC at the same time. The following result clearly has a flavor along these lines.

Theorem 4.9. *Assume that $p, w > 0$ a.e. on (a, b). Then the eigenvalues of the separated real boundary conditions determine the real characteristic curve and, hence, the eigenvalues for every complex boundary condition.*

Proof. Assume that we know the eigenvalues of each BC in $\mathcal{T}$, i.e., know the circle $\mathcal{E}^{\mathbb{R}}_{\lambda} \cap \mathcal{T}$ in $\mathcal{T}$ for every $\lambda \in \mathbb{R}$. By (4.5) and (4.7), for each $\lambda \in \mathbb{R}$, $\Phi(b,\lambda)$ is among the elements Ψ of $\mathrm{SL}(2,\mathbb{R})$ such that

$$\mathcal{E}^{\mathbb{R}}_{\lambda} \cap \mathcal{T} = \mathcal{C}_{\bullet}\Psi. \tag{4.10}$$

Thus, Proposition 3.15 says that for any $\lambda \in \mathbb{R}$, the circle $\mathcal{E}^{\mathbb{R}}_{\lambda} \cap \mathcal{T}$ in $\mathcal{T}$ determines $\Phi(b,\lambda)$ up to a sign. Since the real characteristic curve $\mathcal{D}^{\mathbb{R}}$ is analytic, the family

$$\left\{\mathcal{E}^{\mathbb{R}}_{\lambda} \cap \mathcal{T}\right\}_{\lambda\in\mathbb{R}} \tag{4.11}$$

of circles in $\mathcal{T}$ determines the whole curve $\mathcal{D}^{\mathbb{R}}$ globally up to a sign. On the other hand, by Theorem 3.1 in [2], the entries of $\Phi(b,\lambda)$ are always positive when λ is sufficiently negative. Therefore, the family (4.11) actually determines the whole $\mathcal{D}^{\mathbb{R}}$ uniquely. Since $\Phi(b,\lambda)$ is an entire matrix function of λ, $\mathcal{D}^{\mathbb{R}}$ determines the characteristic surface $\mathcal{D}^{\mathbb{C}}$, and hence the eigenvalues of every complex boundary condition. ∎

Remark 4.10. By Theorem 4.9, there is a duality between the family (4.11) of circles in $\mathcal{T}$ and the real characteristic curve in $\mathcal{B}_S^{\mathbb{R}} \cap \mathcal{O}_6^{\mathbb{R}}$.

Now, let us look at some geometric aspects of the sets $\mathcal{E}_\lambda^{\mathbb{C}} \subset \mathcal{B}^{\mathbb{C}}$, $\lambda \in \mathbb{C}$, and $\mathcal{E}_\lambda^{\mathbb{R}} \subset \mathcal{B}^{\mathbb{R}}$, $\lambda \in \mathbb{R}$. For this purpose, we only need to look at $\mathcal{E}^{\mathbb{C}}$ and $\mathcal{E}^{\mathbb{R}}$, by (4.5) and (4.8). On the way to achieve this purpose, we will use the concept of *bottles*: for a manifold M, an M-bottle is a singular quotient space N that one obtains from $M \times [0,1]$ via modeling $M \times \{0\}$ by an equivalence relation on M to form a subset of N containing the singular points of N and modeling $M \times \{1\}$ by another equivalence relation on M to form a (smooth) submanifold of N.

Proposition 4.11. i) *The set $\mathcal{E}^{\mathbb{R}}$ is a singular submanifold of $\mathcal{B}^{\mathbb{R}}$ of dimension 3. Its only singular point is the boundary condition $[I \mid -I]$ and its tangent fan there is generated by the torus*

$$\left\{ \begin{pmatrix} \cos\sigma + \cos\tau & \sin\sigma - \sin\tau & 0 & 0 \\ \sin\sigma + \sin\tau & \cos\tau - \cos\sigma & 0 & 0 \end{pmatrix};\ \sigma, \tau \in \mathbb{R}/(2\pi\mathbb{Z}) \right\} \tag{4.12}$$

in $\mathrm{T}_{[I \mid -I]}\mathcal{O}_6^{\mathbb{R}}$. Moreover, $\mathcal{E}^{\mathbb{R}}$ is a torus-bottle with a point top and a torus bottom, while the map gluing its side to its bottom is the restriction to the torus

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4;\ x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1/2\} \subset \mathrm{S}^3 \tag{4.13}$$

of the natural projection from S^3 to $\mathbb{R}\mathrm{P}^3$ when this torus is regarded as the side torus.

ii) *The set $\mathcal{E}^{\mathbb{C}}$ is a singular complex submanifold of $\mathcal{B}^{\mathbb{C}}$ of complex dimension 3. Its only singular point is the boundary condition $[I \mid -I]$ and its tangent fan there is generated by the manifold*

$$\begin{aligned} \mathrm{S}^3 \times \mathrm{S}^2 = &\left\{ \begin{pmatrix} z_1 & z_2 & 0 & 0 \\ \eta z_1 & \eta z_2 & 0 & 0 \end{pmatrix};\ \begin{matrix} z_1, z_2, \eta \in \mathbb{C}, \quad |\eta| \le 1 \\ (1+|\eta|^2)(|z_1|^2 + |z_2|^2) = 1 \end{matrix} \right\} \\ &\bigcup \left\{ \begin{pmatrix} \zeta z_3 & \zeta z_4 & 0 & 0 \\ z_3 & z_4 & 0 & 0 \end{pmatrix};\ \begin{matrix} z_3, z_4, \zeta \in \mathbb{C}, \quad |\zeta| < 1 \\ (1+|\zeta|^2)(|z_3|^2 + |z_4|^2) = 1 \end{matrix} \right\} \end{aligned} \tag{4.14}$$

in $\mathrm{T}_{[I \mid -I]}\mathcal{O}_6^{\mathbb{C}}$. Moreover, $\mathcal{E}^{\mathbb{C}}$ is an $(\mathrm{S}^3 \times \mathrm{S}^2)$-bottle with a point top and an $\mathrm{S}^2 \times \mathrm{S}^2$ bottom, while the map gluing its side $\mathrm{S}^3 \times \mathrm{S}^2$ to its bottom $\mathrm{S}^2 \times \mathrm{S}^2$ is the Hopf fibration from S^3 to S^2 times the identity map from S^2 to S^2.

PROOF. Here we only prove i), while ii) can be proved similarly.

Define a function $f : \mathcal{O}_6^{\mathbb{R}} \longrightarrow \mathbb{R}$ by

$$f\big([A \,|\, -I]\big) = (a_{11}-1)(a_{22}-1) - a_{12}a_{21}. \tag{4.15}$$

Then $\mathcal{E}^{\mathbb{R}} \cap \mathcal{O}_6^{\mathbb{R}}$ is the zero set of f and the gradient ∇f of f has length

$$\big\|\nabla f\big\|\big([A \,|\, -I]\big) = \sqrt{(a_{22}-1)^2 + a_{21}^2 + a_{12}^2 + (a_{11}-1)^2}, \tag{4.16}$$

which is never zero away from $[I \,|\, -I]$. This proves the smoothness of $\mathcal{E}^{\mathbb{R}}$ at its points in $\mathcal{O}_6^{\mathbb{R}} \setminus \big\{[I \,|\, -I]\big\}$. Similarly, $\mathcal{E}^{\mathbb{R}}$ is smooth at its points in $\mathcal{O}_1^{\mathbb{R}} \setminus \big\{[I \,|\, -I]\big\}$. Since $\mathcal{E}^{\mathbb{R}} \setminus (\mathcal{O}_6^{\mathbb{R}} \cup \mathcal{O}_1^{\mathbb{R}}) = \mathcal{C} \subset \mathcal{O}_2^{\mathbb{R}} \cup \mathcal{O}_5^{\mathbb{R}}$, to see the smoothness of $\mathcal{E}^{\mathbb{R}}$ at these points we only need to notice that

$$\begin{aligned} \mathcal{E}^{\mathbb{R}} \cap \mathcal{O}_2^{\mathbb{R}} &= \left\{ \begin{bmatrix} 1 & a_{12} & 0 & b_{12} \\ 0 & a_{22} & -1 & b_{22} \end{bmatrix};\ \begin{matrix} a_{12}, a_{22}, b_{12}, b_{22} \in \mathbb{R} \\ a_{12} + a_{22} + b_{12} + b_{22} = 0 \end{matrix} \right\}, \\ \mathcal{E}^{\mathbb{R}} \cap \mathcal{O}_5^{\mathbb{R}} &= \left\{ \begin{bmatrix} a_{11} & 1 & b_{11} & 0 \\ a_{21} & 0 & b_{21} & -1 \end{bmatrix};\ \begin{matrix} a_{11}, a_{21}, b_{11}, b_{21} \in \mathbb{R} \\ a_{11} + a_{21} + b_{11} + b_{21} = 0 \end{matrix} \right\}. \end{aligned} \tag{4.17}$$

Hence, $\mathcal{E}^{\mathbb{R}} \setminus \big\{[I \,|\, -I]\big\}$ is a 3-dimensional submanifold of $\mathcal{B}^{\mathbb{R}}$. There are curves in $\mathcal{E}^{\mathbb{R}}$ through $[I \,|\, -I]$ yielding the four linearly independent tangent vectors

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathrm{T}_{[I \,|\, -I]}\mathcal{O}_6^{\mathbb{R}} \tag{4.18}$$

of $\mathcal{E}^{\mathbb{R}}$ at $[I \,|\, -I]$. Thus, $\mathcal{E}$ is singular at $[I \,|\, -I]$.

For each $\xi > 0$, the set

$$\Big\{[A \,|\, -I] \in \mathcal{E}^{\mathbb{R}};\ (a_{11}-1)^2 + a_{12}^2 + a_{21}^2 + (a_{22}-1)^2 = 4\xi^2\Big\} \tag{4.19}$$

of points in $\mathcal{E}^{\mathbb{R}} \cap \mathcal{O}_6^{\mathbb{R}}$ having distance 2ξ to $[I \,|\, -I]$ can be written as

$$\big\{[K(\xi,\sigma,\tau) \,|\, -I];\ \sigma, \tau \in \mathbb{R}/(2\pi\mathbb{Z})\big\}, \tag{4.20}$$

where

$$K(\xi,\sigma,\tau) = \begin{pmatrix} \xi(\cos\sigma + \cos\tau) + 1 & \xi(\sin\sigma - \sin\tau) \\ \xi(\sin\sigma + \sin\tau) & \xi(\cos\tau - \cos\sigma) + 1 \end{pmatrix}, \tag{4.21}$$

and hence is always a smooth torus. Thus, the tangent fan of $\mathcal{E}^{\mathbb{R}}$ at $[I \,|\, -I]$ is generated by the torus in (4.12).

Using the functions $g, h : \mathcal{O}_1^{\mathbb{R}} \longrightarrow \mathbb{R}$ defined by

$$g\big([I \,|\, B]\big) = (b_{11}+1)(b_{22}+1) - b_{12}b_{21}, \qquad h\big([I \,|\, B]\big) = b_{11}b_{22} - b_{12}b_{21} \tag{4.22}$$

for $B \in \mathrm{M}_{2\times 2}(\mathbb{R})$, we can show that $(\mathcal{E}^{\mathbb{R}} \setminus \mathcal{O}_6^{\mathbb{R}}) \cap \mathcal{O}_1^{\mathbb{R}}$ is a 2-dimensional submanifold of $\mathcal{B}^{\mathbb{R}}$ and, hence, of $\mathcal{E}^{\mathbb{R}}$. Since $\mathcal{E}^{\mathbb{R}} \setminus (\mathcal{O}_6^{\mathbb{R}} \cup \mathcal{O}_1^{\mathbb{R}}) = \mathcal{C} \subset \mathcal{O}_2^{\mathbb{R}} \cup \mathcal{O}_5^{\mathbb{R}}$ again, to see the smoothness of $\mathcal{E}^{\mathbb{R}} \setminus \mathcal{O}_6^{\mathbb{R}}$ at these points we only need to notice that

$$\begin{aligned}
(\mathcal{E}^{\mathbb{R}} \setminus \mathcal{O}_6^{\mathbb{R}}) \cap \mathcal{O}_2^{\mathbb{R}} &= \left\{ \begin{bmatrix} 1 & a_{12} & 0 & 0 \\ 0 & a_{22} & -1 & b_{22} \end{bmatrix} ; \; \begin{matrix} a_{12}, a_{22}, b_{22} \in \mathbb{R} \\ a_{12}+a_{22}+b_{22}=0 \end{matrix} \right\}, \\
(\mathcal{E}^{\mathbb{R}} \setminus \mathcal{O}_6^{\mathbb{R}}) \cap \mathcal{O}_5^{\mathbb{R}} &= \left\{ \begin{bmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & 0 & b_{21} & -1 \end{bmatrix} ; \; \begin{matrix} a_{11}, a_{21}, b_{21} \in \mathbb{R} \\ a_{11}+a_{21}+b_{21}=0 \end{matrix} \right\}.
\end{aligned} \tag{4.23}$$

Thus, $\mathcal{E}^{\mathbb{R}} \setminus \mathcal{O}_6^{\mathbb{R}}$ is a 2-dimensional submanifold of $\mathcal{E}^{\mathbb{R}}$ and, hence, is the limit set of the subset in (4.20) of $\mathcal{E}^{\mathbb{R}} \cap \mathcal{O}_6^{\mathbb{R}}$ as $\xi \longrightarrow +\infty$. Therefore, $\mathcal{E}^{\mathbb{R}} \setminus \mathcal{O}_6^{\mathbb{R}}$ is a quotient space of a torus and $\mathcal{E}^{\mathbb{R}}$ is a torus-bottle with $\mathcal{E}^{\mathbb{R}} \setminus \mathcal{O}_6^{\mathbb{R}}$ as its bottom.

If $\cos\tau \neq 0$, then $2\xi\cos\tau + 1 \neq 0$ for sufficiently large ξ and

$$\begin{aligned}
[K(\xi,\sigma,\tau) \,|\, -I] &= \begin{bmatrix} 1 & 0 & -\frac{\xi(\cos\tau-\cos\sigma)+1}{2\xi\cos\tau+1} & \frac{\xi(\sin\sigma-\sin\tau)}{2\xi\cos\tau+1} \\ 0 & 1 & \frac{\xi(\sin\sigma+\sin\tau)}{2\xi\cos\tau+1} & -\frac{\xi(\cos\sigma+\cos\tau)+1}{2\xi\cos\tau+1} \end{bmatrix} \\
&\longrightarrow \begin{bmatrix} 1 & 0 & \frac{\cos\sigma-\cos\tau}{2\cos\tau} & \frac{\sin\sigma-\sin\tau}{2\cos\tau} \\ 0 & 1 & \frac{\sin\sigma+\sin\tau}{2\cos\tau} & -\frac{\cos\sigma+\cos\tau}{2\cos\tau} \end{bmatrix} \quad \text{as } \xi \longrightarrow +\infty;
\end{aligned} \tag{4.24}$$

if $\cos\tau = 0$ and $\cos\sigma \neq 0$, then $\sin\tau = \pm 1$ and

$$\begin{aligned}
[K(\xi,\sigma,\tau) \,|\, -I] &= \begin{bmatrix} \xi\cos\sigma+1 & \xi(\sin\sigma\mp 1) & -1 & 0 \\ \xi(\sin\sigma\pm 1) & -\xi\cos\sigma+1 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\sigma+\frac{1}{\xi} & \sin\sigma\mp 1 & -\frac{1}{\xi} & 0 \\ 0 & \frac{1}{\xi} & \sin\sigma\pm 1 & -\cos\sigma-\frac{1}{\xi} \end{bmatrix} \\
&\longrightarrow \begin{bmatrix} \cos\sigma & \sin\sigma\mp 1 & 0 & 0 \\ 0 & 0 & \sin\sigma\pm 1 & -\cos\sigma \end{bmatrix} \\
&= \begin{bmatrix} \cos\sigma & \sin\sigma\mp 1 & 0 & 0 \\ 0 & 0 & -\cos\sigma & -(\sin\sigma\mp 1) \end{bmatrix} \quad \text{as } \xi \longrightarrow +\infty;
\end{aligned} \tag{4.25}$$

if $\cos\tau = \cos\sigma = 0$, then either

$$\begin{aligned}
[K(\xi,\sigma,\tau) \,|\, -I] &= \begin{bmatrix} 1 & 0 & -1 & 0 \\ \pm 2\xi & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & \pm\frac{1}{2\xi} & 0 & \mp\frac{1}{2\xi} \end{bmatrix} \\
&\longrightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{as } \xi \longrightarrow +\infty
\end{aligned} \tag{4.26}$$

or

$$[K(\xi,\sigma,\tau)\,|\,-I] = \begin{bmatrix} 1 & \mp 2\xi & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \mp\frac{1}{2\xi} & 1 & \pm\frac{1}{2\xi} & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{as } \xi \longrightarrow +\infty. \tag{4.27}$$

Hence, two distinct rays

$$\big\{[K(\xi,\sigma_1,\tau_1)\,|\,-I];\ \xi>0\big\} \quad \text{and} \quad \big\{[K(\xi,\sigma_2,\tau_2)\,|\,-I];\ \xi>0\big\} \tag{4.28}$$

go to the same limit as $\xi \longrightarrow +\infty$ if and only if

$$\frac{1}{\sqrt{2}}(\cos\sigma_1,\sin\sigma_1,\cos\tau_1,\sin\tau_1) = -\frac{1}{\sqrt{2}}(\cos\sigma_2,\sin\sigma_2,\cos\tau_2,\sin\tau_2). \tag{4.29}$$

Therefore, the bottom $\mathcal{E}^{\mathbb{R}} \setminus \mathcal{O}_6^{\mathbb{R}}$ is diffeomorphic to a torus in $\mathbb{R}\mathrm{P}^3$, and when it is identified with this torus, the map gluing $\mathcal{E}^{\mathbb{R}} \cap \mathcal{O}_6^{\mathbb{R}}$ to the bottom $\mathcal{E}^{\mathbb{R}} \setminus \mathcal{O}_6^{\mathbb{R}}$ is the restriction of the natural projection from S^3 to $\mathbb{R}\mathrm{P}^3$. ∎

Finally, let us look at the set $\mathcal{S}_\lambda^{\mathbb{C}}$ of all self-adjoint complex BC's that have a given λ as an eigenvalue and the set $\mathcal{S}_\lambda^{\mathbb{R}}$ of all self-adjoint real BC's that have λ as an eigenvalue. They will be called the λ-*solid* (in $\mathcal{B}_S^{\mathbb{C}}$) and λ-*surface* (in $\mathcal{B}_S^{\mathbb{R}}$), respectively. Note that

$$\mathcal{S}_\lambda^{\mathbb{C}} = \mathcal{E}_\lambda^{\mathbb{C}} \cap \mathcal{B}_S^{\mathbb{C}}, \qquad \mathcal{S}_\lambda^{\mathbb{R}} = \mathcal{E}_\lambda^{\mathbb{R}} \cap \mathcal{B}_S^{\mathbb{R}}, \tag{4.30}$$

and that if we set

$$\begin{aligned} \mathcal{S}^{\mathbb{C}} &= \big\{[A\,|\,B] \in \mathcal{B}_S^{\mathbb{C}};\ \det(A+B)=0\big\}, \\ \mathcal{S}^{\mathbb{R}} &= \big\{[A\,|\,B] \in \mathcal{B}_S^{\mathbb{R}};\ \det(A+B)=0\big\}, \end{aligned} \tag{4.31}$$

then

$$\begin{aligned} \mathcal{S}_\lambda^{\mathbb{C}} &= \mathcal{S}^{\mathbb{C}}{}_\bullet\Phi(b,\lambda) = \big\{[A\,|\,B\Phi(b,\lambda)^{-1}];\ [A\,|\,B] \in \mathcal{S}^{\mathbb{C}}\big\}, \\ \mathcal{S}_\lambda^{\mathbb{R}} &= \mathcal{S}^{\mathbb{R}}{}_\bullet\Phi(b,\lambda) = \big\{[A\,|\,B\Phi(b,\lambda)^{-1}];\ [A\,|\,B] \in \mathcal{S}^{\mathbb{R}}\big\}. \end{aligned} \tag{4.32}$$

Moreover, direct calculations yield

$$\begin{aligned} \mathcal{S}^{\mathbb{C}} &= \left\{ \begin{bmatrix} e^{i\theta}\hat{a}(\theta,a_{12},a_{21}) & e^{i\theta}a_{12} & -1 & 0 \\ e^{i\theta}a_{21} & e^{i\theta}\tilde{a}(\theta,a_{12},a_{21}) & 0 & -1 \end{bmatrix};\ \begin{matrix} \theta \in \mathbb{R}/(\pi\mathbb{Z}) \\ a_{12},a_{21} \in \mathbb{R} \\ a_{12}a_{21} \le -\sin^2\theta \end{matrix} \right\} \cup \mathcal{C}, \\ \mathcal{S}^{\mathbb{R}} &= \left\{ \begin{bmatrix} 1 \pm \sqrt{-a_{12}a_{21}} & a_{12} & -1 & 0 \\ a_{21} & 1 \mp \sqrt{-a_{12}a_{21}} & 0 & -1 \end{bmatrix};\ \begin{matrix} a_{12},a_{21} \in \mathbb{R} \\ a_{12}a_{21} \le 0 \end{matrix} \right\} \cup \mathcal{C}, \end{aligned} \tag{4.33}$$

where

$$\hat{a}(\theta, a_{12}, a_{21}) = \cos\theta \pm \sqrt{-\sin^2\theta - a_{12}a_{21}}, \tag{4.34}$$
$$\tilde{a}(\theta, a_{12}, a_{21}) = \cos\theta \mp \sqrt{-\sin^2\theta - a_{12}a_{21}}.$$

In the following proposition, by a *collapsed torus* we mean a singular surface obtained from a torus by shrinking exactly one position of the revolving circle of the torus to a point, the only singular point of the surface. A collapsed torus is also a sphere with two points glued together. Moreover, we also mention that for each point $[K \mid -I]$ of $\mathcal{B}_S^{\mathbb{R}} \cap \mathcal{O}_6^{\mathbb{R}}$, the tangent space $\mathrm{T}_{[K \mid -I]}\mathcal{B}_S^{\mathbb{R}} \subset \mathrm{T}_{[K \mid -I]}\mathcal{O}_6^{\mathbb{R}}$ of $\mathcal{B}_S^{\mathbb{R}}$ at $[K \mid -I]$ can be written as

$$\mathrm{T}_{[K \mid -I]}\mathcal{B}_S^{\mathbb{R}} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha & \beta+\gamma & 0 & 0 \\ \beta-\gamma & -\alpha & 0 & 0 \end{pmatrix} K;\ \alpha, \beta, \gamma \in \mathbb{R} \right\}, \tag{4.35}$$

and for each $[e^{i\theta}K \mid -I] \in \mathcal{B}_S^{\mathbb{C}} \cap \mathcal{O}_6^{\mathbb{C}}$,

$$\begin{aligned} \mathrm{T}_{[e^{i\theta}K \mid -I]}\mathcal{B}_S^{\mathbb{C}} &= \left\{ \frac{e^{i\theta}}{\sqrt{2}} \begin{pmatrix} \alpha+\mathrm{i}\delta & \beta+\gamma & 0 & 0 \\ \beta-\gamma & -\alpha+\mathrm{i}\delta & 0 & 0 \end{pmatrix} K;\ \alpha, \beta, \gamma, \delta \in \mathbb{R} \right\} \\ &\subset \mathrm{T}_{[e^{i\theta}K \mid -I]}\mathcal{O}_6^{\mathbb{C}}. \end{aligned} \tag{4.36}$$

Theorem 4.12. i) *For each $\lambda \in \mathbb{R}$, the λ-solid $\mathcal{S}_\lambda^{\mathbb{C}}$ is the image of $\mathcal{S}^{\mathbb{C}}$ under the left action of $\Phi(b,\lambda)$ on $\mathcal{B}_S^{\mathbb{C}}$, and the action sends $\mathcal{S}^{\mathbb{R}}$ to the λ-surface $\mathcal{S}_\lambda^{\mathbb{R}}$.*

ii) *For each $\lambda \in \mathbb{R}$, the λ-surface $\mathcal{S}_\lambda^{\mathbb{R}}$ in $\mathcal{B}_S^{\mathbb{R}}$ is a collapsed torus with the collapsed point being $[\Phi(b,\lambda) \mid -I]$ and the tangent cone there being the cone*

$$\left\{ \xi \begin{pmatrix} \cos\sigma & \sin\sigma+1 & 0 & 0 \\ \sin\sigma-1 & -\cos\sigma & 0 & 0 \end{pmatrix} \Phi(b,\lambda);\ \xi \in \mathbb{R},\ \sigma \in \mathbb{R}/(2\pi\mathbb{Z}) \right\} \tag{4.37}$$

in $\mathrm{T}_{[\Phi(b,\lambda) \mid -I]}\mathcal{B}_S^{\mathbb{R}} \subset \mathrm{T}_{[\Phi(b,\lambda) \mid -I]}\mathcal{O}_6^{\mathbb{R}}$.

iii) *For each $\lambda \in \mathbb{R}$, the λ-solid $\mathcal{S}_\lambda^{\mathbb{C}}$ is a 3-sphere with two points (on the 2-sphere corresponding to $\mathcal{S}_\lambda^{\mathbb{R}}$) glued together to become the point $[\Phi(b,\lambda) \mid -I]$ and its tangent cone there is the cone*

$$\left\{ \xi \begin{pmatrix} \cos\tau\cos\sigma + i\sin\tau & \cos\tau\sin\sigma+1 & 0 & 0 \\ \cos\tau\sin\sigma-1 & -\cos\tau\cos\sigma + i\sin\tau & 0 & 0 \end{pmatrix} \Phi(b,\lambda);\ \begin{matrix} \xi \in \mathbb{R} \\ \tau \in \mathbb{R}/(\pi\mathbb{Z}) \\ \sigma \in \mathbb{R}/(2\pi\mathbb{Z}) \end{matrix} \right\} \tag{4.38}$$

in $\mathrm{T}_{[\Phi(b,\lambda) \mid -I]}\mathcal{B}_S^{\mathbb{C}} \subset \mathrm{T}_{[\Phi(b,\lambda) \mid -I]}\mathcal{O}_6^{\mathbb{C}}$.

Proof. We only need to prove ii) and iii) for $\mathcal{S}^{\mathbb{R}}$ and $\mathcal{S}^{\mathbb{C}}$, respectively.

By (4.20), there holds

$$\mathcal{S}^{\mathbb{R}} = \{[I\,|\,-I]\} \cup \left\{[K(\xi,\sigma)\,|\,-I];\ \xi \in \mathbb{R},\, \xi \neq 0,\, \tau \in \mathbb{R}/_{(2\pi\mathbb{Z})}\right\} \cup \mathcal{C}, \tag{4.39}$$

where

$$K(\xi,\sigma) = \begin{pmatrix} 1+\xi\cos\sigma & \xi\sin\sigma+\xi \\ \xi\sin\sigma-\xi & 1-\xi\cos\sigma \end{pmatrix}. \tag{4.40}$$

So, $\mathcal{S}^{\mathbb{R}}$ is smooth away from $[I\,|\,-I]$ and $\mathcal{C}$, it is singular at $[I\,|\,-I]$, and its tangent cone at $[I\,|\,-I]$ is given by (4.37) with $\Phi(b,\lambda)$ removed. To see the smoothness of $\mathcal{S}^{\mathbb{R}}$ at the points in $\mathcal{C}$, we only need to notice that

$$\begin{aligned} \mathcal{S}^{\mathbb{R}} \cap \mathcal{O}_2^{\mathbb{R}} &= \left\{\begin{bmatrix} 1 & x_1-x_2 & 0 & x_2 \\ 0 & x_2 & -1 & -x_1-x_2 \end{bmatrix};\ x_1,x_2 \in \mathbb{R}\right\}, \\ \mathcal{S}^{\mathbb{R}} \cap \mathcal{O}_5^{\mathbb{R}} &= \left\{\begin{bmatrix} x_1+x_2 & 1 & -x_1 & 0 \\ -x_1 & 0 & x_1-x_2 & -1 \end{bmatrix};\ x_1,x_2 \in \mathbb{R}\right\}. \end{aligned} \tag{4.41}$$

Note that the circle $\mathcal{C}$ can be written as

$$\left\{\begin{bmatrix} 1-\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \sin\alpha-1 & -\cos\alpha \end{bmatrix};\ \alpha \in (\pi/2, 5\pi/2)\right\} \cup \{\boldsymbol{N}\}, \tag{4.42}$$

where $\boldsymbol{N}$ denotes the Neumann-Neumann BC, or as

$$\left\{\begin{bmatrix} \cos\alpha & 1+\sin\alpha & 0 & 0 \\ 0 & 0 & -\cos\alpha & -1-\sin\alpha \end{bmatrix};\ \alpha \in (-\pi/2, 3\pi/2)\right\} \cup \{\boldsymbol{D}\}, \tag{4.43}$$

where $\boldsymbol{D}$ stands for the Dirichlet-Dirichlet BC. The BC $[K(\xi,\sigma)\,|\,-I]$ is close to a separated BC if and only if $|\xi|$ is sufficiently large. When $\sigma \in (\pi/2, 5\pi/2)$,

$$\begin{aligned} [K(\xi,\sigma)\,|\,-I] &= \begin{bmatrix} 1-\sin\sigma & \cos\sigma-\frac{1}{\xi} & 0 & \frac{1}{\xi} \\ 0 & \frac{1}{\xi} & \sin\sigma-1 & -\cos\sigma-\frac{1}{\xi} \end{bmatrix} \\ &\longrightarrow \begin{bmatrix} 1-\sin\sigma & \cos\sigma & 0 & 0 \\ 0 & 0 & \sin\sigma-1 & -\cos\sigma \end{bmatrix} \end{aligned} \tag{4.44}$$

as $|\xi| \to +\infty$; and when $\sigma \in (-\pi/2, 3\pi/2)$,

$$\begin{aligned} [K(\xi,\sigma)\,|\,-I] &= \begin{bmatrix} \frac{1}{\xi}+\cos\sigma & \sin\sigma+1 & -\frac{1}{\xi} & 0 \\ -\frac{1}{\xi} & 0 & \frac{1}{\xi}-\cos\sigma & -\sin\sigma-1 \end{bmatrix} \\ &\longrightarrow \begin{bmatrix} \cos\sigma & 1+\sin\sigma & 0 & 0 \\ 0 & 0 & -\cos\sigma & -1-\sin\sigma \end{bmatrix} \end{aligned} \tag{4.45}$$

as $|\xi| \to +\infty$. Thus, the circle

$$\Big\{[K(\xi,\sigma)\,|\,-I];\ \sigma \in \mathbb{R}/(2\pi\mathbb{Z})\Big\}, \tag{4.46}$$

where $\xi \neq 0$, in $\mathcal{S}^{\mathbb{R}} \setminus \mathcal{C}$ uniformly approaches the circle $\mathcal{C}$ as $\xi \longrightarrow +\infty$ or $-\infty$, and hence $\mathcal{S}^{\mathbb{R}}$ is a collapsed torus.

Using the two functions f and g defined on $\mathcal{B}_S^{\mathbb{C}} \cap \mathcal{O}_6^{\mathbb{C}}$ by

$$f\big([e^{i\theta}K\,|\,-I]\big) = (k_{11}+k_{22})e^{-i\theta} - e^{-2i\theta} \tag{4.47}$$

if $\theta \in \mathbb{R}/(\pi\mathbb{Z})$ and $K \in \mathrm{SL}(2,\mathbb{R})$, one can prove that $\mathcal{S}^{\mathbb{C}}$ is smooth away from $[I\,|\,-I]$ and $\mathcal{C}$, since that part of $\mathcal{S}^{\mathbb{C}}$ is a level set of f and the gradient of f never vanishes there. To see the smoothness of $\mathcal{S}^{\mathbb{C}}$ at the points in $\mathcal{C}$, we only need to notice that

$$\begin{aligned}
\mathcal{S}^{\mathbb{C}} \cap \mathcal{O}_2^{\mathbb{C}} &= \left\{\begin{bmatrix} 1 & a_{12} & 0 & \bar{z} \\ 0 & z & -1 & b_{22}\end{bmatrix};\ \begin{matrix} a_{12}\in\mathbb{R}, z\in\mathbb{C}, b_{22}\in\mathbb{R} \\ a_{12}+z+\bar{z}+b_{22}=0\end{matrix}\right\},\\
\mathcal{S}^{\mathbb{C}} \cap \mathcal{O}_5^{\mathbb{C}} &= \left\{\begin{bmatrix} a_{11} & 1 & \bar{z} & 0 \\ z & 0 & b_{21} & -1\end{bmatrix};\ \begin{matrix} a_{11}\in\mathbb{R}, z\in\mathbb{C}, b_{21}\in\mathbb{R} \\ a_{11}+z+\bar{z}+b_{21}=0\end{matrix}\right\}.
\end{aligned} \tag{4.48}$$

For each $\xi > 0$, the set of points in $\mathcal{S}^{\mathbb{C}} \cap \mathcal{O}_6^{\mathbb{C}}$ having distance 2ξ to $[I\,|\,-I]$ can be written as

$$\begin{aligned}
&\Big\{[e^{i\theta}K_+(\xi,\sigma,\theta)\,|\,-I];\ \sigma \in \mathbb{R}/(2\pi\mathbb{Z}),\ \theta\in[0,\pi),\ \sin\theta \le \xi\Big\}\\
&\bigcup \Big\{[e^{i\theta}K_-(\xi,\sigma,\theta)\,|\,-I];\ \sigma \in \mathbb{R}/(2\pi\mathbb{Z}),\ \theta\in[0,\pi),\ \sin\theta \le \xi\Big\},
\end{aligned} \tag{4.49}$$

where

$$\begin{aligned}
K_+(\xi,\sigma,\theta) &= \begin{pmatrix} \cos\theta + \sqrt{\xi^2-\sin^2\theta}\,\cos\sigma & \sqrt{\xi^2-\sin^2\theta}\,\sin\sigma + \xi \\ \sqrt{\xi^2-\sin^2\theta}\,\sin\sigma - \xi & \cos\theta - \sqrt{\xi^2-\sin^2\theta}\,\cos\sigma \end{pmatrix},\\
K_-(\xi,\sigma,\theta) &= \begin{pmatrix} \cos\theta - \sqrt{\xi^2-\sin^2\theta}\,\cos\sigma & -\sqrt{\xi^2-\sin^2\theta}\,\sin\sigma - \xi \\ -\sqrt{\xi^2-\sin^2\theta}\,\sin\sigma + \xi & \cos\theta + \sqrt{\xi^2-\sin^2\theta}\,\cos\sigma \end{pmatrix}.
\end{aligned} \tag{4.50}$$

Thus, $\mathcal{S}^{\mathbb{C}}$ is singular at $[I\,|\,-I]$ and its tangent cone there is the one given in (4.38) with $\Phi(b,\lambda)$ removed. As in the last paragraph, for every $\theta \in [0,\pi)$,

$$\begin{aligned}
[e^{i\theta}K_+(\xi,\sigma,\theta)\,|\,-I] &\longrightarrow \begin{bmatrix} 1-\sin\sigma & \cos\sigma & 0 & 0 \\ 0 & 0 & \sin\sigma - 1 & -\cos\sigma \end{bmatrix},\\
[e^{i\theta}K_-(\xi,\sigma,\theta)\,|\,-I] &\longrightarrow \begin{bmatrix} 1-\sin\sigma & \cos\sigma & 0 & 0 \\ 0 & 0 & \sin\sigma - 1 & -\cos\sigma \end{bmatrix}
\end{aligned} \tag{4.51}$$

uniformly as $\xi \to +\infty$ if $\sigma \in (\pi/2, 5\pi/2)$, and

$$(4.52)\qquad \begin{aligned} [e^{i\theta}K_+(\xi,\sigma,\theta)\,|\,-I] &\longrightarrow \begin{bmatrix} \cos\sigma & 1+\sin\sigma & 0 & 0 \\ 0 & 0 & -\cos\sigma & -1-\sin\sigma \end{bmatrix}, \\ [e^{i\theta}K_-(\xi,\sigma,\theta)\,|\,-I] &\longrightarrow \begin{bmatrix} \cos\sigma & 1+\sin\sigma & 0 & 0 \\ 0 & 0 & -\cos\sigma & -1-\sin\sigma \end{bmatrix} \end{aligned}$$

uniformly as $\xi \to +\infty$ if $\sigma \in (-\pi/2, 3\pi/2)$. Hence, for each $\theta \in (0,\pi)$,

$$(4.53)\qquad \begin{aligned} \mathcal{C} \bigcup \Big\{ [e^{i\theta}K_+(\xi,\sigma,\theta)\,|\,-I];\ \xi \ge \sin\theta,\ \sigma \in \mathbb{R}/(2\pi\mathbb{Z}) \Big\} \\ \bigcup \Big\{ [e^{i\theta}K_-(\xi,\sigma,\theta)\,|\,-I];\ \xi \ge \sin\theta,\ \sigma \in \mathbb{R}/(2\pi\mathbb{Z}) \Big\} \end{aligned}$$

is a 2-sphere. Moreover, for any $\xi > 0$ and $\sigma \in \mathbb{R}/(2\pi\mathbb{Z})$,

$$(4.54)\qquad \begin{aligned} \lim_{\theta\to\pi^-} [e^{i\theta}K_+(\xi,\sigma,\theta)\,|\,-I] &= [K_-(\xi,\sigma,0)\,|\,-I], \\ \lim_{\theta\to\pi^-} [e^{i\theta}K_-(\xi,\sigma,\theta)\,|\,-I] &= [K_+(\xi,\sigma,0)\,|\,-I]. \end{aligned}$$

Therefore, $\mathcal{S}^{\mathbb{C}}$ is a 3-sphere with two two points glued together and its only singular point is on $\mathcal{S}^{\mathbb{R}}$. ∎

Remark 4.13. The surface $\mathcal{S}^{\mathbb{R}}$ is an algebraic variety in the Grassmann manifold $\mathrm{G}_2(\mathbb{R}^4)$, while $\mathcal{S}^{\mathbb{C}}$ is a real algebraic variety in the complex Grassmann manifold $\mathrm{G}_2(\mathbb{C}^4)$. Moreover, the subsets given in (4.49) of $\mathcal{S}^{\mathbb{C}}$ are spheres when $0 < \xi < 1$ and tori when $\xi \ge 1$.

§5. Analyticity of Continuous Eigenvalue Branches

In this section we investigate the smoothness of continuous eigenvalue branches under some assumptions on their multiplicities. As an application of the results obtained here and one of the main ideas used in their proofs we show that when $w > 0$ a.e. on (a,b), the algebraic and geometric multiplicities of an eigenvalue for a separated real BC are equal.

Theorem 5.1. *Let $\boldsymbol{A} = [A\,|\,B]$ be a complex boundary condition with a simple eigenvalue $\lambda_* \in \mathbb{C}$. Then,*

$$(5.1)\qquad \sum_{j,k=1}^{2} f_{jk}\partial_\lambda \phi_{jk}(b,\lambda_*) \ne 0,$$

the continuous simple eigenvalue branch Λ through λ_* is analytic, and its differential at $\boldsymbol{A}$ is given by

$$\mathrm{d}\Lambda\big|_{\boldsymbol{A}}((H\,|\,L)) = -\frac{\sum_{j,k=1}^{2}(c_{jk}h_{jk}+d_{jk}l_{jk})}{\sum_{j,k=1}^{2} f_{jk}\partial_\lambda\phi_{jk}(b,\lambda_*)} \tag{5.2}$$

for any $(H\,|\,L)\in \mathrm{T}_{\boldsymbol{A}}\mathcal{B}^{\mathbb{C}}$, where the coefficient matrices C, D and F are defined by

$$C = A^{\mathrm{a}} + B^{\mathrm{a}}\Phi(b,\lambda_*)^{\mathrm{a}}, \quad D = B^{\mathrm{a}} + A^{\mathrm{a}}\Phi(b,\lambda_*)^{\mathrm{t}}, \quad F = B^{\mathrm{t}}A^{\mathrm{a}} \tag{5.3}$$

with X^{a} being the accompanying matrix of a matrix X.

Proof. The continuous simple eigenvalue branch Λ through λ_* is the solution to the equation

$$\Delta_{\boldsymbol{X}}(\lambda) = \det(X + Z\Phi(b,\lambda)) = 0 \tag{5.4}$$

on λ for $\boldsymbol{X} = [X\,|\,Z]$ sufficiently close to $\boldsymbol{A}$. Since λ_* is simple, we have

$$\Delta'_{\boldsymbol{A}}(\lambda_*) \neq 0. \tag{5.5}$$

Direct calculations using (5.4) and (5.3) yield

$$\Delta'_{\boldsymbol{A}}(\lambda_*) = \sum_{j,k=1}^{2} f_{jk}\partial_\lambda\phi_{jk}(b,\lambda_*), \tag{5.6}$$

and hence (5.1) holds. Then, (5.4) and (5.5) together with the analyticity of Φ in λ, the analyticity of $\mathcal{B}^{\mathbb{C}}$ and the Implicit Function Theorem imply that the solution Λ to (5.4) is analytic. Moreover, for any $(H\,|\,L)\in \mathrm{T}_{\boldsymbol{A}}\mathcal{B}^{\mathbb{C}}$, from (5.4) and (5.3) one deduces

$$\Big(\sum_{j,k=1}^{2} f_{jk}\partial_\lambda\phi_{jk}(b,\lambda_*)\Big)\,\mathrm{d}\Lambda\big|_{\boldsymbol{A}}((H\,|\,L)) = \sum_{j,k=1}^{2}(c_{jk}h_{jk}+d_{jk}l_{jk}), \tag{5.7}$$

which together with (5.1) prove (5.2). ∎

We can restrict our attention to the space $\mathcal{B}^{\mathbb{C}}_S$ of self-adjoint complex BC's. There eigenvalues are all real and similar results in the real category hold. However, Kong and Zettl in [7] have proven the continuous differentiability of continuous eigenvalue branches over $\mathcal{B}^{\mathbb{C}}_S$ through an eigenvalue of geometric multiplicity 1 for a coupled BC in $\mathcal{B}^{\mathbb{C}}_S$ and of continuous eigenvalue branches over the space $\mathcal{T}$ of separated real BC's, where all eigenvalues are of geometric multiplicity 1. The following theorem unifies and generalizes their results.

Theorem 5.2. *Assume that $w > 0$ a.e. on (a,b) and let $\boldsymbol{A}$ be a self-adjoint complex boundary condition with an eigenvalue λ_* of geometric multiplicity 1. Then any continuous eigenvalue branch over $\mathcal{B}_S^{\mathbb{C}}$ through λ_* is differentiable at $\boldsymbol{A}$.*

Proof. The method used in [7] still applies to this general set-up. ∎

Now, we are ready to discuss the relations between the algebraic and geometric multiplicities of an eigenvalue. First, we have the following general result.

Proposition 5.3. *The multiplicity of an eigenvalue is greater than or equal to its geometric multiplicity.*

Proof. It suffices to prove that the multiplicity of any eigenvalue λ_* of geometric multiplicity 2 is at least 2. By Theorem 4.1 we only need to show that as an eigenvalue for $[\Phi(b,\lambda_*) \mid -I]$, λ_* has multiplicity ≥ 2. Now, the characteristic function is given by

$$\Delta(\lambda) = 2 - \phi_{22}(\lambda_*)\phi_{11}(\lambda) + \phi_{21}(\lambda_*)\phi_{12}(\lambda) + \phi_{12}(\lambda_*)\phi_{21}(\lambda) - \phi_{11}(\lambda_*)\phi_{22}(\lambda). \tag{5.8}$$

Here we have omitted b from the argument of each ϕ_{jk}. Using (2.7) one then directly verifies that $\Delta'(\lambda_*) = 0$. Thus, λ_* has multiplicity ≥ 2. ∎

Next, we establish the following result, whose proof uses Theorem 5.2 and the main idea in the proof of Theorem 5.1.

Theorem 5.4. *Assume that $w > 0$ a.e. on (a,b). Then the algebraic and geometric multiplicities of an eigenvalue for a separated real boundary condition are equal, i.e., the eigenvalue is (real and) simple.*

Proof. If λ_* is an eigenvalue for a separated real BC $\boldsymbol{A}$, say

$$\boldsymbol{A} = \begin{bmatrix} 1 & a_{12} & 0 & 0 \\ 0 & 0 & -1 & b_{22} \end{bmatrix}, \tag{5.9}$$

then a_{12}, $b_{22} \in \mathbb{R}$,

$$\vec{v} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix} \in \mathrm{T}_{\boldsymbol{A}}\mathcal{B}_S^{\mathbb{C}} \subset \mathrm{T}_{\boldsymbol{A}}\mathcal{O}_2^{\mathbb{C}}, \tag{5.10}$$

and λ_* has geometric multiplicity 1. Consider a smooth curve $s \mapsto \boldsymbol{A}(s) \in \mathcal{B}_S^{\mathbb{C}}$ such that $\boldsymbol{A}(0) = \boldsymbol{A}$ and $\boldsymbol{A}'(0) = \vec{v}$. Let Λ be a continuous eigenvalue branch over $\mathcal{B}_S^{\mathbb{C}}$ through λ_*. Then $\Lambda(\boldsymbol{A}(s))$ is differentiable at $s = 0$ by Theorem 5.2, and from $\Delta_{\boldsymbol{A}(s)}(\Lambda(\boldsymbol{A}(s))) \equiv 0$ one deduces

$$\Delta'_{\boldsymbol{A}}(\lambda_*)\,\mathrm{d}\Lambda\big|_{\boldsymbol{A}}(\vec{v}) = 2. \tag{5.11}$$

Thus, $\Delta'_{\boldsymbol{A}}(\lambda_*) \neq 0$, i.e., λ_* is simple. ∎

Combining Theorem 5.4 above and Theorem 4.2 in [2] yields the following result. Note that even though the whole paper [2] uses the assumptions $p, w > 0$ a.e. on (a, b), Theorem 4.2 there clearly holds without the condition $p > 0$ a.e. on (a, b).

Theorem 5.5. *Assume that $w > 0$ a.e. on (a, b). Then the algebraic and geometric multiplicities of an eigenvalue for an arbitrary self-adjoint boundary condition are equal.*

Remark 5.6. There is a proof of Theorem 5.5 which does not rely on any result from [2]. In other words, Theorem 5.5 can be regarded as a consequence of the differentiability of the continuous eigenvalue branches over $\mathcal{B}_S^{\mathbb{C}}$ when they have geometric multiplicity 1 and some geometric descriptions of $\mathcal{D}^{\mathbb{R}}$ and $\mathcal{S}_\lambda^{\mathbb{R}}$ in Section 4. Hence, Theorem 5.5 can be generalized to the case of eigenvalue problems for higher order ordinary differential equations, which will be addressed in a forthcoming publication. Moreover, we would like to mention that Theorem 5.5 can be proved without using the differentiability of eigenvalue branches over $\mathcal{B}_S^{\mathbb{C}}$ (actually, this part can be replaced by some arguments involving only the definitions of multiplicities and some descriptions about $\mathcal{D}^{\mathbb{R}}$, $\mathcal{E}_\lambda^{\mathbb{C}}$ and $\mathcal{S}_\lambda^{\mathbb{R}}$ in Section 4).

Remark 5.7. Assume that $w > 0$ a.e. on (a, b). If λ_* is a double eigenvalue for a BC $\boldsymbol{A} \in \mathcal{B}_S^{\mathbb{C}}$, then $\boldsymbol{A} = [\Phi(b, \lambda_*) \,|\, -I] \in \mathcal{D}^{\mathbb{R}}$ by Theorems 5.5 and 4.1. Each BC in $\mathcal{B}_S^{\mathbb{C}} \setminus \mathcal{D}^{\mathbb{R}}$ (or just in $\mathcal{B}_S^{\mathbb{R}} \setminus \mathcal{D}^{\mathbb{R}}$) sufficiently close to $\boldsymbol{A}$ has exactly two eigenvalues near λ_* and they are simple. So, λ_* is on two continuous eigenvalue branches over $\mathcal{B}_S^{\mathbb{C}}$ (or just over $\mathcal{B}_S^{\mathbb{R}}$) and they are locally unique. These continuous eigenvalue branches are in general not differentiable at $\boldsymbol{A}$, see Section 7 of [5].

Remark 5.8. Assume that $w > 0$ a.e. on (a, b). Let λ_* be an eigenvalue of geometric multiplicity 1 for $\boldsymbol{A} \in \mathcal{B}_S^{\mathbb{C}}$ and u a normalized eigenfunction for λ_*, i.e., an eigenfunction for λ_* satisfying

$$\int_a^b u(t)\overline{u}(t)w(t)\,\mathrm{d}t = 1. \tag{5.12}$$

Then, λ_* is simple by Theorem 5.5. Hence, there is a continuous eigenvalue branch Λ (over $\mathcal{B}^{\mathbb{C}}$, not just over $\mathcal{B}^{\mathbb{C}}_S$) through λ_* and, by Theorem 5.1, Λ is analytic. Moreover, the method used in the proof of the formulas (4.4)–(4.7) in [7] actually yields the following more general forms of the formulas: when $\boldsymbol{A}$ is coupled, i.e., $\boldsymbol{A} = [e^{i\theta}K \,|\, -I]$ for some $\theta \in {\mathbb{R}}/{(\pi\mathbb{Z})}$ and $K \in \mathrm{SL}(2,\mathbb{R})$, we have

$$\mathrm{T}_{\boldsymbol{A}}\mathcal{B}^{\mathbb{C}} = \mathrm{T}_{\boldsymbol{A}}\mathcal{O}^{\mathbb{C}}_6 = \{(e^{i\theta}KH \,|\, 0);\ H \in \mathrm{M}_{2\times 2}(\mathbb{C})\} \tag{5.13}$$

and for each $(e^{i\theta}KH \,|\, 0) \in \mathrm{T}_{\boldsymbol{A}}\mathcal{B}^{\mathbb{C}}$ (not necessarily tangent to $\mathcal{B}^{\mathbb{C}}_S$ at $\boldsymbol{A}$) there holds

$$\mathrm{d}\Lambda\big|_{\boldsymbol{A}}\big((e^{i\theta}KH \,|\, 0)\big) = (\, \overline{u}^{[1]}(a) \quad -\overline{u}(a)\,)\, H \begin{pmatrix} u(a) \\ u^{[1]}(a) \end{pmatrix}; \tag{5.14}$$

when $\boldsymbol{A}$ is given by (5.9), there holds

$$\mathrm{d}\Lambda\big|_{\boldsymbol{A}}((H \,|\, L)) = (\, u^{[1]}(a) \quad u^{[1]}(b)\,) \begin{pmatrix} h_{12} & l_{12} \\ h_{22} & l_{22} \end{pmatrix} \begin{pmatrix} u^{[1]}(a) \\ u^{[1]}(b) \end{pmatrix} \tag{5.15}$$

for any $(H \,|\, L)$ in

$$\mathrm{T}_{\boldsymbol{A}}\mathcal{B}^{\mathbb{C}} = \mathrm{T}_{\boldsymbol{A}}\mathcal{O}^{\mathbb{C}}_2 = \left\{ \begin{pmatrix} 0 & h_{12} & 0 & l_{12} \\ 0 & h_{22} & 0 & l_{22} \end{pmatrix};\ h_{12}, h_{22}, l_{12}, l_{22} \in \mathbb{C} \right\}; \tag{5.16}$$

when

$$\boldsymbol{A} = \begin{bmatrix} 1 & a_{12} & 0 & 0 \\ 0 & 0 & b_{21} & -1 \end{bmatrix}, \tag{5.17}$$

there holds

$$\mathrm{d}\Lambda\big|_{\boldsymbol{A}}((H \,|\, L)) = (\, u^{[1]}(a) \quad -u(b)\,) \begin{pmatrix} h_{12} & l_{11} \\ h_{22} & l_{21} \end{pmatrix} \begin{pmatrix} u^{[1]}(a) \\ u(b) \end{pmatrix} \tag{5.18}$$

for any $(H \,|\, L)$ in

$$\mathrm{T}_{\boldsymbol{A}}\mathcal{B}^{\mathbb{C}} = \mathrm{T}_{\boldsymbol{A}}\mathcal{O}^{\mathbb{C}}_3 = \left\{ \begin{pmatrix} 0 & h_{12} & l_{11} & 0 \\ 0 & h_{22} & l_{21} & 0 \end{pmatrix};\ h_{12}, h_{22}, l_{11}, l_{21} \in \mathbb{C} \right\}; \tag{5.19}$$

when

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & 1 & 0 & 0 \\ 0 & 0 & -1 & b_{22} \end{bmatrix}, \tag{5.20}$$

there holds

$$\mathrm{d}\Lambda\big|_{\boldsymbol{A}}((H \,|\, L)) = (\, -u(a) \quad u^{[1]}(b)\,) \begin{pmatrix} h_{11} & l_{12} \\ h_{21} & l_{22} \end{pmatrix} \begin{pmatrix} u(a) \\ u^{[1]}(b) \end{pmatrix} \tag{5.21}$$

for any $(H\,|\,L)$ in

$$\mathrm{T}_{\boldsymbol{A}}\mathcal{B}_S^{\mathbb{C}} = \mathrm{T}_{\boldsymbol{A}}\mathcal{O}_4^{\mathbb{C}} = \left\{ \begin{pmatrix} h_{11} & 0 & 0 & l_{12} \\ h_{21} & 0 & 0 & l_{22} \end{pmatrix};\ h_{11}, h_{21}, l_{12}, l_{22} \in \mathbb{C} \right\}; \tag{5.22}$$

when

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & 1 & 0 & 0 \\ 0 & 0 & b_{21} & -1 \end{bmatrix}, \tag{5.23}$$

there holds

$$\mathrm{d}\Lambda\big|_{\boldsymbol{A}}((H\,|\,L)) = -\,(\,u(a) \quad u(b)\,) \begin{pmatrix} h_{11} & l_{11} \\ h_{21} & l_{21} \end{pmatrix} \begin{pmatrix} u(a) \\ u(b) \end{pmatrix} \tag{5.24}$$

for any $(H\,|\,L)$ in

$$\mathrm{T}_{\boldsymbol{A}}\mathcal{B}_S^{\mathbb{C}} = \mathrm{T}_{\boldsymbol{A}}\mathcal{O}_5^{\mathbb{C}} = \left\{ \begin{pmatrix} h_{11} & 0 & l_{11} & 0 \\ h_{21} & 0 & l_{21} & 0 \end{pmatrix};\ h_{11}, h_{21}, l_{11}, l_{21} \in \mathbb{C} \right\}. \tag{5.25}$$

Therefore, at a self-adjoint complex BC $\boldsymbol{A}$, the derivative of the continuous eigenvalue branch through a simple eigenvalue for $\boldsymbol{A}$ is always a quadratic form in $u(a)$, $u^{[1]}(a)$, $u(b)$ and $u^{[1]}(b)$ if the canonical coordinate systems on $\mathcal{B}^{\mathbb{C}}$ are used. These formulas will be needed in [6].

The formulas (5.14), (5.15), (5.18), (5.21) and (5.24) are equivalent to special cases of (5.2). The equivalence can be established using (2.7), (2.8) and (5.12). It seems that the method used in the proof of the formulas (4.4)–(4.7) in [7] only works when the complex BC is self-adjoint, see also [5].

To end our discussion, we give an example to show that *the algebraic and geometric multiplicities of an eigenvalue are different in general.*

Example 5.9. Consider the Fourier equation $-y'' = \lambda y$ on the interval $[0, 1]$. Let $\lambda_* = (n\pi)^2$ with an integer $n > 0$. Then direct calculations using (2.8) and (2.7) yield that

$$\begin{gathered} \alpha_{11}(\lambda_*) = \frac{1}{2}, \qquad \alpha_{12}(\lambda_*) = 0, \qquad \alpha_{22}(\lambda_*) = \frac{1}{2\lambda_*}, \\ \Delta'_{\boldsymbol{A}}(\lambda_*) = \frac{\xi}{2\lambda_*}[(1-\lambda_*)\sin\sigma + (1+\lambda_*)\sin\tau], \end{gathered} \tag{5.26}$$

where $\boldsymbol{A} = [K(\xi,\sigma,\tau)\,|\,-\Phi(b,\lambda_*)^{-1}] \in \mathcal{E}^{\mathbb{R}}_{\lambda_*} \setminus \mathcal{D}^{\mathbb{R}}$ with $K(\xi,\sigma,\tau)$ defined by (4.21) for some $\xi > 0$ and σ, $\tau \in \mathbb{R}/(2\pi\mathbb{Z})$. Thus, when

$$\sin\tau = \frac{\lambda_* - 1}{\lambda_* + 1}\sin\sigma, \tag{5.27}$$

the eigenvalue λ_* for $\boldsymbol{A}$ has geometric multiplicity 1 and algebraic multiplicity at least 2.

Finally, we want to present some corollaries to Theorems 3.5, 3.8 and 5.4 and give an example to illustrate each of them. These corollaries relate the eigenvalues of separated BC's and those of coupled BC's, as mentioned in the introduction.

Corollary 5.10. *Assume that $w > 0$ a.e. on (a, b). Let $\boldsymbol{A}$ be a separated real boundary condition, $n > 0$ an integer, and $\mathcal{N} \subset \mathbb{C}$ a bounded domain containing n eigenvalues r_1, $r_2, \cdots, r_n$ for $\boldsymbol{A}$ such that its boundary does not contain any eigenvalue for $\boldsymbol{A}$. Then there exists a neighborhood $\mathcal{O}$ of $\boldsymbol{A}$ in $\mathcal{B}^{\mathbb{C}}$ such that each complex boundary condition in $\mathcal{O}$ has exactly n eigenvalues in $\mathcal{N}$ and they are simple. Moreover, these eigenvalues are given by the simple eigenvalue branches through $r_1, r_2, \cdots, r_n$, respectively.*

Proof. These conclusions are direct consequences of Theorem 3.5, Remark 3.6 and Theorem 5.4. ∎

To illustrate the above corollary, we have the following example, in which (and in the rest of this paper) λ_0^{DN}, λ_1^{DN}, λ_2^{DN}, $\cdots$ denote the eigenvalues for the Dirichlet-Neumann BC

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{5.28}$$

and λ_0^{ND}, λ_1^{ND}, λ_2^{ND}, $\cdots$ stand for the eigenvalues for the Neumann-Dirichlet BC

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{5.29}$$

Example 5.11. Assume that $p, w > 0$ a.e. on (a, b). Let $n \geq 0$ be an integer and set

$$\boldsymbol{A}(s) = \begin{bmatrix} c_1 e^s & 0 & -1 & 0 \\ 0 & c_2 e^{-s} & 0 & -1 \end{bmatrix} \tag{5.30}$$

for $s \in \mathbb{R}$, where c_1, $c_2 \in \mathbb{C}$ are non-zero constants. Then

$$\boldsymbol{A}(s) = \begin{bmatrix} c_1 & 0 & -e^{-s} & 0 \\ 0 & c_2 e^{-s} & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{5.31}$$

as $s \to +\infty$. Thus, for any real constants μ_1, μ_2, μ_3 and μ_4 satisfying $\mu_1 < \lambda_0^{DN}$, $\lambda_n^{DN} < \mu_2 < \lambda_{n+1}^{DN}$, $\mu_3 < 0$ and $\mu_4 > 0$, $\boldsymbol{A}(s)$ has exactly $n+1$ eigenvalues $z_0(s)$, $z_1(s)$, $\cdots$, $z_n(s)$ in the rectangle

$$\{z \in \mathbb{C};\ \mu_1 < \operatorname{Re} z < \mu_2,\ \mu_3 < \operatorname{Im} z < \mu_4\} \tag{5.32}$$

when s is sufficiently large, they are simple and continuous in s, and

$$\lim_{s\to+\infty} z_k(s) = \lambda_k^{DN} \tag{5.33}$$

for $k = 0, 1, \cdots, n$. Similarly, for any real constants ν_1, ν_2, ν_3 and ν_4 satisfying $\nu_1 < \lambda_0^{ND}$, $\lambda_n^{ND} < \nu_2 < \lambda_{n+1}^{ND}$, $\nu_3 < 0$ and $\nu_4 > 0$, $\boldsymbol{A}(s)$ has exactly $n+1$ eigenvalues $z_0(s)$, $z_1(s)$, $\cdots$, $z_n(s)$ in the rectangle

$$\{z \in \mathbb{C};\ \nu_1 < \operatorname{Re} z < \nu_2,\ \nu_3 < \operatorname{Im} z < \nu_4\} \tag{5.34}$$

when s is sufficiently negative, they are simple and continuous in s, and

$$\lim_{s\to-\infty} z_k(s) = \lambda_k^{ND} \tag{5.35}$$

for $k = 0, 1, \cdots, n$.

Corollary 5.12. *Assume that $w > 0$ a.e. on (a, b). Let $\boldsymbol{A}$ be a separated real boundary condition, $n > 0$ an integer, and $\mathcal{N} \subset \mathbb{C}$ a bounded domain containing n eigenvalues r_1, r_2, $\cdots$, r_n for $\boldsymbol{A}$ such that its boundary does not contain any eigenvalue for $\boldsymbol{A}$. Then there exists a neighborhood $\mathcal{O}$ of $\boldsymbol{A}$ in $\mathcal{B}^{\mathbb{R}}$ such that each boundary condition $\boldsymbol{C} \in \mathcal{O}$ has exactly n eigenvalues in $\mathcal{N}$ and they are real and simple. Moreover, these eigenvalues are given by the continuous simple eigenvalue branches over $\mathcal{B}^{\mathbb{R}}$ through r_1, r_2, $\cdots$, r_n, respectively.*

Proof. This refinement of the restriction to $\mathcal{B}^{\mathbb{R}}$ of an application of Corollary 5.10 is a direct consequence of Corollary 5.10 and Theorem 3.8. ∎

The following example is a refinement of a special case of Example 5.11.

Example 5.13. Assume that $p, w > 0$ a.e. on (a, b). Let $n \geq 0$ be an integer and set

$$\boldsymbol{A}(s) = \begin{bmatrix} re^s & 0 & -1 & 0 \\ 0 & e^{-s} & 0 & -1 \end{bmatrix} \tag{5.36}$$

for $s \in \mathbb{R}$, where $r \in \mathbb{R}$ is a non-zero constant. Then, the conclusions of Example 5.11 hold and, in addition, the eigenvalues $z_0(s)$, $z_1(s)$, $\cdots$, $z_n(s)$ are *real* for any sufficiently large or sufficiently negative s. Note that $\boldsymbol{A}(s)$ is not self-adjoint if $r \neq 1$.

One can also write down a result for the self-adjoint BC's that is similar to Corollary 5.10, which can be found in [6] and will be used there to give a proof [4] of the inequalities in [2] without referring to the periodic case.

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Department of Mathematics, Northern Illinois University, DeKalb, IL 60115USA

kong@math.niu.edu, wu@math.niu.edu, zettl@math.niu.edu

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