Left-Definite Sturm-Liouville Problems

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Abstract

Left-definite regular self-adjoint Sturm-Liouville problems with separated and coupled boundary conditions are studied. A characterization is given in terms of right-definite problems, a concrete and "natural" indexing scheme for the eigenvalues is proposed. Pruefer transformation techniques can be used to establish the existence of and to give a characterization for the eigenvalues in the case of separated boundary conditions. Here we give an elementary proof of the existence of the eigenvalues for the coupled case. Furthermore we study the continuous and differentiable dependence of the eigenvalues on all parameters of the problem. For a fixed equation we find the range of each eigenvalue as a function of the boundary conditions and inequalities among the eigenvalues as the boundary conditions vary. These extend the classical inequalities among the Neumann, Dirichlet, Periodic, and Semi-Periodic eigenvalues and our recent generalizations for the right-definite case. Some of our results here yield an algorithm for the numerical computation of the eigenvalues of left-definite problems with coupled boundary conditions.

1 Introduction

In this paper we study boundary value problems associated with the regular Sturm-Liouville equation

$$-(py')' + qy = \lambda wy \quad \text{on } J = (a, b), \tag{1.1}$$

where p, q and w satisfy the following basic conditions

$$1/p,\ q,\ w \in L^1(J, \mathbb{R}), \quad p > 0 \ a.e. \text{ on } J, \quad w \text{ changes sign on } J. \tag{1.2}$$

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Here $\mathbb{R}$ denotes the set of real numbers, $L^1(J, \mathbb{R})$ the space of real valued Lebesgue integrable functions on J.

Richardson [17] showed that there may be non-real eigenvalues for boundary value problems associated with (1.1) even in the case of Dirichlet boundary conditions. But there is an important class of such problems, the so-called "left-definite" problems for which the eigenvalues are all real. This class is studied here. For other work on left-definite problems see Atkinson and Jabon [1], Atkinson and Mingarelli [2], Bennewitz and Everitt [3], Curgus and Langer [4], Daho and Langer [6], Haupt [9], Ince [10], Kwong [15], Mingarelli [16], Richardson [17] and the references cited there.

It is useful to consider the related right-definite equation:

$$-(py')' + qy = \lambda |w| y \quad \text{on} \quad J = (a, b). \tag{1.3}$$

According to the well known theory (see [19]) of right-definite Sturm-Liouville problems (SLP) the self-adjoint boundary conditions for (1.3) are well known and may be separated or coupled, see Section 2 below for details.

Our major results are:

1. A characterization of left-definite problems.

   The SLP consisting of the equation (1.1) together with either a separated boundary condition or a coupled boundary condition is left-definite if and only if the lowest eigenvalue of the (right-definite) problem (1.3) with the same boundary condition is positive. This characterization makes clear the dependence of left-definiteness on both the coefficients p and q and on the boundary conditions; note in particular that, although p is assumed positive, there is no sign restriction on q as is often found in the literature. Few results are known for left-definite problems with coupled boundary conditions compared with the separated case.

2. We propose a "natural" indexing scheme for the eigenvalues of left-definite problems based on the index set

   $$Z^* = \{\ldots, -2, -1, -0, 0, 1, 2, \ldots\}. \tag{1.4}$$

   By Prüefer transformation techniques [2] it can be shown that for left-definite problems with *separated* boundary conditions the eigenvalues can be indexed by Z^* such that

   $$\ldots < \lambda_{-2} < \lambda_{-1} < \lambda_{-0} < 0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots \tag{1.5}$$

   and the eigenfunctions (which are unique up to constant multiples) of λ_n have exactly $|n|$ zeros in the open interval (a, b) for $n \in Z^*$. We give an elementary proof to show that the eigenvalues of any left-definite problem with coupled boundary conditions are all real, there are countably infinitely many of them, they are unbounded above and below, they may be geometrically simple or double, and can be indexed to satisfy

$$\ldots \leq \lambda_{-2} \leq \lambda_{-1} \leq \lambda_{-0} < 0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \tag{1.6}$$

with only geometrically double eigenvalues appearing twice. (In the left-definite case $\lambda = 0$ is not an eigenvalue.)

3. The continuous and differentiable dependence of λ_n on all parameters of the problem is studied. Analogues of the recently established theorems in [13], [11], [12], are found. In fact, λ_n is not a continuous function of the problem in general, but does depend continuously on the equation, i.e. on each of $a, b, 1/p, q, w$ and is also a continuous function of the problem at all "interior" points in the "space of left-definite problems". Formulas for the derivatives, when they exist, of λ_n with respect to all parameters of the problem are found.

4. According to a well-known classical result [18], the following inequalities hold in the positive right-definite case

$$\begin{aligned} \lambda_0^N \quad & \leq \lambda_0^P < \lambda_0^S \leq \{\lambda_0^D, \lambda_1^N\} \\ & \leq \lambda_1^S < \lambda_1^P \leq \{\lambda_1^D, \lambda_2^N\} \\ & \leq \lambda_2^P < \lambda_2^S \leq \{\lambda_2^D, \lambda_3^N\} \\ & \leq \lambda_3^S < \lambda_3^D \leq \{\lambda_3^D, \lambda_4^N\} \leq \cdots, \end{aligned} \tag{1.7}$$

where λ_n^N, λ_n^P, λ_n^S and λ_n^D denote the Neumann, Periodic, Semi-periodic and Dirichlet eigenvalues, respectively, and the notation $\{\lambda_0^D, \lambda_1^N\}$ means each of λ_0^D and λ_1^N.

In the right-definite case these inequalities have been extended in [7] from $K = I$ to an arbitrary $K \in SL(2, \mathbb{R})$ for the coupled BC (2.7).

Recently, analogues of the inequalities (1.7) have been found by Constantin [5] for the left-definite case when $p = 1$ and both q and w are periodic functions with the same period. Here similar inequalities are established for general left-definite problems associated with equation (1.1) without any periodicity assumptions and for arbitrary coupled boundary conditions. These inequalities are comparable to those in [7] for right-definite problems and can be used to construct an algorithm to compute the eigenvalues of all left-definite problems.

Our proofs are elementary, based only on the basic theory of linear ordinary differential equations and on recent results for the right-definite case from the papers cited above. Motivated by [5], [16],

[10], etc., we use the two parameter equation

$$-(py')' + q - \lambda wy = \xi|w|y \quad \text{on } J. \tag{1.8}$$

as a key feature in our approach. With this equation the eigenvalues of left-definite and right-definite problems are connected with each other by "eigenvalue curves". This idea makes it possible for us to apply our recently established results for eigenvalue curves of right-definite problems to analyse the left-definite case.

The organization of the paper is as follows. Following the introduction in Section 1 we give the characterization of left-definite problems in Section 2. In Section 3 we review the results for the right-definite case which are needed in Section 4 to prove the new results for left-definite problems. In Section 5 we extend the left-definite results to a wider class of problems and comment on the indefinite case.

2 Characterization of Left-Definite Problems

We consider the self-adjoint SLP consisting of the equation

$$-(py')' + qy = \lambda w\, y \quad \text{on } J = (a, b), \tag{2.1}$$

and the BC

$$A\,Y(a) + B\,Y(b) = 0, \quad Y = \begin{pmatrix} y \\ y^{[1]} \end{pmatrix},\ y^{[1]} = py', \tag{2.2}$$

where

$$-\infty \leq a < b \leq \infty,\ 1/p,\ q,\ w \in L^1(J, \mathbb{R}), \quad p > 0 \text{ and } w \neq 0 \ a.e. \text{ on } J, \tag{2.3}$$

and A, B are 2×2 complex valued matrices such that

$$\text{the } 2 \times 4 \text{ matrix } (A|B) \text{ has full rank}, \tag{2.4}$$

and

$$AEA^* = BEB^*, \quad E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.5}$$

The BC's given by (2.2), (2.4), and (2.5) are classified into two disjoint classes: separated and coupled. The separated BC's have the canonical representation

$$\begin{aligned} \cos\alpha\, y(a) \,+\, \sin\alpha\, y^{[1]}(a) \,&=\, 0, \quad 0 \leq \alpha < \pi; \\ \cos\beta\, y(b) \,+\, \sin\beta\, y^{[1]}(b) \,&=\, 0, \quad 0 < \beta \leq \pi. \end{aligned} \tag{2.6}$$

And the coupled BC's have the canonical representation

$$Y(b) = e^{i\theta} K Y(a), \tag{2.7}$$

where

$$i = \sqrt{-1},\ -\pi < \theta \leq \pi, \quad \text{and } K \in SL(2,\mathbb{R}) := \left\{ K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2} : \det K = 1 \right\}. \tag{2.8}$$

Remark 2.1 *In (2.2), (2.6), (2.7) $y(a),\ (py')(a)$ are defined as limits:*

$$y(a) = \lim_{t\to a^+} y(t); \quad (py')(a) = lim_{t\to a^+}(py')(t).$$

It is shown in [19] that, given (2.3), these limits exist and are finite for any solution y. Similarly it can be shown that these limits exist and are finite for any maximal domain function f. Similar remarks apply to the endpoint b. Furthermore, it can be shown similarly that $f(a),\ f(b)$ and $(pf')(a),\ (pf'(b)$ exist as finite limits for any maximal domain function f.

Let (2.3)-(2.5) hold. We define a linear manifold $\mathcal{D} = \mathcal{D}(\mathcal{J}, \surd^{-\infty}, \amalg, |\sqsupseteq|, \mathcal{A}, \mathcal{B})$ in the Hilbert space $L^2(J, |w|)$ by:

$$\mathcal{D} = \left\{ \{ \in \mathcal{L}^{\in}(\mathcal{J}, |\sqsupseteq|) : \begin{array}{c} f, f^{[1]} \in AC_{loc}(J), |w|^{-1}(-(pf')' + qf) \in L^2(J,|w|), \\ \text{and } AF(a) + BF(b) = 0 \text{ for } F = \begin{pmatrix} f \\ f^{[1]} \end{pmatrix}. \end{array} \right\} \tag{2.9}$$

Then we define functionals $\mathcal{R}$ and $\mathcal{L}$ on $\mathcal{D}$ as follows:

$$\mathcal{R}\{ = \int_{\dashv}^{\lfloor} |\{|^{\in} \sqsupseteq, \tag{2.10}$$

$$\mathcal{L}\{ = \int_{\dashv}^{\lfloor} [-(\surd\{')'\overline{\{} + \amalg|\{|^{\in}]. \tag{2.11}$$

Definition 2.1 *The S-L problem (2.1), (2.2) is said to be right-definite (RD) if $\mathcal{R}$ is definite on $\mathcal{D}$, i.e., either $\mathcal{R}\{ > \imath$ for all $f \not\equiv 0$ in $\mathcal{D}$ or $\mathcal{R}\{ < \imath$ for all $f \not\equiv 0$ in $\mathcal{D}$.*

The S-L problem (2.1), (2.2) is said to be left-definite (LD) if $\mathcal{L}$ is definite on $\mathcal{D}$, i.e., either $\mathcal{L}\{ > \imath$ for all $f \not\equiv 0$ in $\mathcal{D}$ or $\mathcal{L}\{ < \imath$ for all $f \not\equiv 0$ in $\mathcal{D}$.

The S-L problem (2.1), (2.2) is said to be indefinite (IN) if it is neither right-definite nor left-definite.

It is clear that the S-L problem (2.1), (2.2) is RD if and only if $w > 0$ *a.e.* on J or $w < 0$ *a.e.* on J. To clarify the meaning of the left-definiteness we consider the RD problem consisting of the equation

$$-(py')' + qy = \xi |w| y \quad \text{on } J \tag{2.12}$$

and the same BC (2.2).

Theorem 2.1 *Let (2.3)-(2.5) hold. Then the following three statements are equivalent:*

(i) The S-L problem (2.1), (2.2) is left-definite;

(ii) $\mathcal{L}$ is positive definite on $\mathcal{D}$, i.e., $\mathcal{L}f > 0$ for all $f \not\equiv 0$ in $\mathcal{D}$;

(iii) the eigenvalues of the right-definite problem (2.12), (2.2) are all positive.

Remark 2.2 *It follows from Theorem 2.1 that, under the conditions (2.3), in particular $p > 0$, the SLP (2.3) - (2.5) cannot be left-negative-definite.*

Proof: (i)$\Longleftrightarrow$(ii). By definition, (ii) implies (i). Suppose that $\mathcal{L}f < 0$ for all $f \not\equiv 0$ in $\mathcal{D}$. Let ξ_n be the n-th eigenvalue of SLP (2.12), (2.2) and y_n an eigenfunction associated with ξ_n, $n \in \mathbb{N}_0$. Then $y_n \in \mathcal{D}$ and

$$\mathcal{L}y_n = \int_a^b [-(py_n')' + qy_n]\bar{y}_n = \xi_n \int_a^b |y_n|^2 |w| < 0.$$

Hence $\xi_n < 0$ for all $n \in \mathbb{N}_0$. This contradicts the well known fact that $\xi_n \to +\infty$ as $n \to +\infty$ in the right-definite case (2.12), (2.2). Thus the conclusion follows from the definition of left-definiteness.

(ii)$\Longleftrightarrow$(iii). Suppose that $\mathcal{L}f > 0$ for all $f \not\equiv 0$ in $\mathcal{D}$. Let ξ_0 be the least eigenvalue of SLP (2.12), (2.2) and y_0 an eigenfunction associated with ξ_0. Then as in the above, we have that $y_0 \in \mathcal{D}$, and

$$\mathcal{L}y_0 = \xi_0 \int_a^b |y_0|^2 |w| > 0.$$

Hence $\xi_0 > 0$.

Suppose $\xi_0 > 0$. From the variational characterization of the least eigenvalue we have that

$$0 < \xi_0 = \min \frac{\mathcal{L}f}{\int_a^b |f|^2 |w|}. \tag{2.13}$$

where the minimum is taken over all $f \not\equiv 0$ in $\mathcal{D}$. Hence $\mathcal{L}f > 0$ for all $f \not\equiv 0$ in $\mathcal{D}$. ∎

The next result exhibits some classes of left-definite SLP.

Corollary 2.1 *Let (2.3) hold and assume $q \geq 0$ a.e. on J. Then*

(i) The separated SLP (2.1), (2.6) is left-definite if $\pi/2 \le \alpha \le \pi$ and $0 \le \beta \le \pi/2$.

(ii) The coupled SLP (2.1), (2.7) is left-definite if $K = \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix}$ for some real number $c \neq 0$.

Proof: For both the two cases we let ξ_0 be the least eigenvalue of the SLP (2.12), (2.2), and y an eigenfunction associated with ξ_0. Then from (2.12) and by integration by parts we have that

$$\begin{aligned} \xi_0 \int_a^b |y|^2|w| &= \int_a^b [-(py')' + qy]\bar{y} \\ &= y^{[1]}(a)\bar{y}(a) - y^{[1]}(b)\bar{y}(b) + \int_a^b [p|y'|^2 + q|y|^2] \\ &> y^{[1]}(a)\bar{y}(a) - y^{[1]}(b)\bar{y}(b). \end{aligned} \tag{2.14}$$

(i) To prove the case for $\pi/2 < \alpha \le \pi$ and $0 \le \beta < \pi/2$ we may assume that y is real-valued and note that

$$y^{[1]}(a)\bar{y}(a) - y^{[1]}(b)\bar{y}(b) = -\tan\alpha|y^{[1]}(a)|^2 + \tan\beta|y^{[1]}(b)|^2 \ge 0. \tag{2.15}$$

The combination of (2.14) and (2.15) implies that $\xi_0 > 0$ and hence the SLP (2.1), (2.6) is LD by Theorem 2.1. The cases when $\alpha = \pi/2$ or $\beta = \pi/2$ can be proven similarly.

(ii) In this case we have that

$$y(b) = ce^{i\theta}y(a) \quad \text{and} \quad y^{[1]}(b) = \frac{1}{c}e^{i\theta}y^{[1]}(a).$$

Hence

$$y^{[1]}(a)\bar{y}(a) - y^{[1]}(b)\bar{y}(b) = 0.$$

This together with (2.14) implies that $\xi_0 > 0$. Therefore, the SLP (2.1), (2.7) is LD by Theorem 2.1. ∎

In Corollary 2.1 as in much of the literature on left-definite problems it is assumed that $q \ge 0$. The next result shows not only that this assumption is not needed but that q can be as negative as you want and even unbounded below. Also for any fixed self-adjoint boundary condition and for fixed p and w there is a potential q yielding a left-definite problem; Corollary 2.2 gives an explicit construction for such a q.

Corollary 2.2 *Let (2.3)-(2.5) hold, and let ξ_0 be the least eigenvalue of the SLP (2.12), (2.2). Then for any $\epsilon > 0$, the SLP consisting of the equation*

$$-(py')' + [q - (\xi_0 - \epsilon)|w|]\, y = \lambda w y \tag{2.16}$$

and the BC (2.2) is left-definite.

Proof: From Theorem 2.1, we see that the SLP (2.16), (2.2) is LD if and only if the SLP consisting of the equation

$$-(py')' + [q - (\xi_0 - \epsilon)|w|]\, y = \xi |w| y \tag{2.17}$$

and the BC (2.2) has no nonnegative eigenvalues. From the hypotheses it follows that ϵ is the least eigenvalue of (2.17), (2.2). Therefore, the SLP (2.16), (2.2) is LD. ∎

Remark 2.3 *When $w > 0$ a.e. on J, $L^2(J, w)$ is a Hilbert space with norm $(\mathcal{R}\{)^{\infty/\in}$ generated by the inner product $(f, g) = \int_a^b f\bar{g}w$. This is the right-definite case. When w changes sign on (a, b) but $|w| > 0$ a.e. on J, $L^2(J, w)$ is a Krein space and the theory of operators in Krein space can be applied to study such SLP, see [4], [6]. In the left-positive-definite case Hilbert space operator theory can also be applied where the Hilbert space H is constructed as follows: Since $\mathcal{L}\{$ is positive definite on the linear manifold D, the norm $(\mathcal{L}\{)^{\infty/\in}$ is generated by the inner product*

$$< f, g >= \int_a^b [(pf')' + q]\bar{g}$$

on D. The Hilbert space H is the space obtained from the completion of D with respect to this norm.

It is interesting to observe that in the right-definite case the Hilbert space used as the "natural" framework in which to study SLP depends only on the interval J and on the weight function w. On the other hand the Hilbert space H depends on D and therefore on $1/p, q, |w|$ and on the boundary conditions. Since we are studying how the eigenvalues change when the boundary conditions or the coefficients are changed our "elementary method" used here has the considerable advantage over using operator theory in H of not having to change the underlying Hilbert space every time a parameter of the problem, e.g. $p, q, |w|, a, b$ or the boundary condition, is changed.

3 Right-Definite Problems

There is a voluminous literature on right-definite SLP; here we summarize, for the convenience of the reader, the results needed to prove the theorems for the left-definite case in Section 4. Our discussion here is limited to the case when $w > 0$ *a.e.* on J. The case $w < 0$ *a.e.* on J can be transformed into the former by replacing w and λ by $-w$ and $-\lambda$, respectively. It is well-known that self-adjoint regular right-definite SLP (with $p > 0$ have an infinite, but countable, number of only real eigenvalues which can be ordered to form a nondecreasing sequence (with the double eigenvalues appearing twice). This sequence is bounded below, but not above, has no finite cluster point, and is denoted by $\{\lambda_n : n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}\}$.

In [7], inequalities are found among eigenvalues of coupled and related separated boundary conditions.

For a fixed $K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \in SL(2, \mathbb{R})$, we consider the separated BC's

$$y(a) = 0, \qquad k_{22}y(b) - k_{12}y^{[1]}(b) = 0, \tag{3.1}$$

and

$$y^{[1]}(a) = 0, \qquad k_{21}y(b) - k_{11}y^{[1]}(b) = 0. \tag{3.2}$$

Note that $(k_{22}, k_{12}) \neq (0, 0) \neq (k_{21}, k_{11})$ since $\det K = 1$. Now we know that Eq. (2.1) together with each of the BC's (2.7), (3.1), and (3.2) has an infinite number of eigenvalues. We denote by $\{\lambda_n(e^{i\theta}K) : n \in \mathbb{N}_0\}$ the eigenvalues of the coupled SLP (2.1), (2.7), and by $\{\mu_n : n \in \mathbb{N}_0\}$ and $\{\nu_n : n \in \mathbb{N}_0\}$ the eigenvalues of (2.1), (3.1) and (2.1), (3.2), respectively. The following theorem exhibits the relationships among these eigenvalues. Here we use the notation $\{a, b\}$ to denote either a or b.

Theorem 3.1 *Let (2.3) hold and assume $w > 0$ a.e. on J. Suppose $K \in SL(2, \mathbb{R})$.*

(a) If $k_{11} > 0$ and $k_{12} \leq 0$, then $\lambda_0(K)$ is simple, and for any $\theta \in (-\pi, \pi)$, $\theta \neq 0$, we have

$$\begin{aligned} \nu_0 \quad &\leq \lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) \leq \{\mu_0, \nu_1\} \\ &\leq \lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) \leq \{\mu_1, \nu_2\} \\ &\leq \lambda_2(K) < \lambda_2(e^{i\theta}K) < \lambda_2(-K) \leq \{\mu_2, \nu_3\} \\ &\leq \lambda_3(-K) < \lambda_3(e^{i\theta}K) < \lambda_3(K) \leq \{\mu_3, \nu_4\} \leq \cdots . \end{aligned} \tag{3.3}$$

(b) If $k_{11} \leq 0$ and $k_{12} < 0$, then $\lambda_0(K)$ is simple, and for any $\theta \in (-\pi, \pi)$, $\theta \neq 0$, we have

$$\begin{aligned} &\lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) \leq \{\mu_0, \nu_0\} \leq \\ &\lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) \leq \{\mu_1, \nu_1\} \leq \\ &\lambda_2(K) < \lambda_2(e^{i\theta}K) < \lambda_2(-K) \leq \{\mu_2, \nu_2\} \leq \\ &\lambda_3(-K) < \lambda_3(e^{i\theta}K) < \lambda_3(K) \leq \{\mu_3, \nu_3\} \leq \cdots . \end{aligned} \tag{3.4}$$

(c) If neither case (a) nor case (b) applies to K, then either case (a) or case (b) applies to $-K$.

Proof: See Theorem 3.2 in [7]. ∎

Corollary 3.1 *For any* $K \in SL(2,\mathbb{R})$*, either* $\lambda_0(K)$ *or* $\lambda_0(-K)$ *is simple.*

Corollary 3.2 *Let* $K \in SL(2,\mathbb{R})$ *with either* $k_{11} > 0$ *and* $k_{12} \leq 0$ *or* $k_{11} \leq 0$ *and* $k_{12} < 0$*. If for some* $n \in \mathbb{N}_0$*,* $\lambda_{2n+1}(K)$ *is simple, then so is* $\lambda_{2n+2}(K)$*. In particular, if* K *has a double eigenvalue, then the first double eigenvalue of* K *is preceded by an odd number of simple eigenvalues.*

Theorem 3.2 *Let (2.3)-(2.5) hold with* $w > 0$ *a.e. on* J*. Let* $\{\lambda_n^D : n \in \mathbb{N}_0\}$ *be the eigenvalues of the Dirichlet problem associated with Eq. (2.1). Then the range of the* n*-th eigenvalue* λ_n *of the SLP (2.1), (2.2) on the space of self-adjoint BC's* (A,B) *satisfying (2.6) is* $(-\infty, \lambda_n^D]$ *for* $n = 0, 1$*, and* $(-\lambda_{n-2}^D,\ \lambda_n^D]$ *for* $n \geq 2$*.*

Proof: See Theorem 4.1 in [12]. ∎

Assume that the SLP (2.1), (2.2) is RD and is represented by $\boldsymbol{\omega} = (a, b, A, B, 1/p, q, w)$. Then the space of all such problems can be written as

$$\boldsymbol{\Omega} = \{\boldsymbol{\omega} = (a, b, A, B, 1/p, q, w) : (2.3) - (2.5) \text{ hold}\}. \tag{3.5}$$

A natural topology on $\boldsymbol{\Omega}$ is the product topology induced from the usual topology on $\mathbb{R}^n$ and on L^1. More precisely, given $\epsilon > 0$ and $\boldsymbol{\omega}_0 = (a_0, b_0, A_0, B_0, 1/p_0, q_0, w_0) \in \boldsymbol{\Omega}$, the ϵ-neighborhood of $\boldsymbol{\omega}_0$ is defined to be the set of $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ satisfying

$$||A - A_0|| + ||B - B_0|| + \int_{-\infty}^{+\infty} \left(|\widetilde{1/p} - \widetilde{1/p_0}| + |\tilde{q} - \tilde{q}_0| + |\tilde{w} - \tilde{w}_0|\right) < \epsilon \tag{3.6}$$

where $||\cdot||$ is any matrix norm, and $\widetilde{f}$ is the extension of f to $\mathbb{R}$ which equals 0 on $\mathbb{R} \setminus J$. With this topology, we have the following result on the dependence of eigenvalues on the problem.

Theorem 3.3 *Let (2.3)-(2.5) hold, and assume the SLP (2.1), (2.2) is RD.*

(i) For $n \in \mathbb{N}_0$*, the* n*-th eigenvalue* λ_n *depends continuously on the equation (2.1), i.e., on* $a, b, 1/p, q, w$*.*

(ii) In general, $\lambda_n, n \in \mathbb{N}_0$*, does not depend continuously on the boundary conditions, not even when these are represented by their canonical forms (2.6) and (2.7). Nevertheless, each simple eigenvalue is on a locally unique continuous branch of simple eigenvalues; while each double eigenvalue is on two locally unique continuous branches of eigenvalues.*

(iii) Let $\lambda(\boldsymbol{\omega})$ *be a continuous eigenvalue branch on* $\boldsymbol{\Omega}$*. For any* $\boldsymbol{\omega}_0 \in \boldsymbol{\Omega}$ *and for each component of* $\boldsymbol{\omega}$*, assume that (a) either* $\lambda(\boldsymbol{\omega}_0)$ *is simple, or (b)* $\lambda(\boldsymbol{\omega})$ *is always double when this component*

varies on an open set of its domain. Then λ is continuously differentiable at $\boldsymbol{\omega}_0$ in this component of $\boldsymbol{\omega}$.

In particular, for $n \in \mathbb{N}_0$, assume that (a) and (b) of (iii) hold for $\lambda = \lambda_n$. Then λ_n is continuously differentiable with respect to each of the parameters $a, b, 1/p, q, w$.

Proof: See [12]. ∎

Remark 3.1 *Explicit expressions for the derivatives of a continuous eigenvalue branch $\lambda(\boldsymbol{\omega})$, and of $\lambda_n(\boldsymbol{\omega})$ at its continuous points, were established in [12]. For example, the Frechet derivative of λ_n with respect to q is given by the formula*

$$\lambda_n'(q)(h) = \int_a^b |u_n|^2 h, \quad h \in L^1(a, b), \ n \in \mathbb{N}_0. \tag{3.7}$$

where u_n is an eigenfunction associated with λ_n satisfying

$$\int_a^b |u_n|^2 w = 1.$$

For more details on the derivative formulas, see Theorems 4.1 and 4.2 in [13].

4 Eigenvalues of Left-Definite Problems

For left-definite problems with separated boundary conditions the existence of eigenvalues and their characterization in terms of the Pruefer transformation is known. Although this is not explicitly stated in Atkinson and Mingarelli [2] it is a consequence of their results. This characterization yields precise information on the number of zeros of the eigenfunctions. For a different approach based on the ricatti equation see M. K. Kwong, [15]. These results are summarized in the next theorem.

Theorem 4.1 *Let (2.3) hold. Assume the SLP (2.1), (2.6) is LD and not RD. Then all eigenvalues are real, there exist countably infinitely many positive and negative eigenvalues, they are unbounded below and above, have no finite cluster point, and can be indexed to satisfy the inequalities*

$$\cdots < \lambda_{-n} < \cdots < \lambda_{-1} < \lambda_{-0} < 0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots.$$

Furthermore, the eigenfunctions associated with λ_n have exactly $|n|$ zeros, $n \in \mathbb{Z}^* := \{\pm 0, \pm 1, \cdots\}$.

In this section, we establish the existence of eigenvalues of the LD problems with arbitray self-adjoint coupled boundary conditions and study their properties. For this purpose, we introduce the following equation with two parameters:

$$-(py')' + qy - \lambda wy = \xi|w|y. \tag{4.1}$$

Remark 4.1 *(i) For each fixed* $\lambda \in \mathbb{R}$*, the SLP (4.1), (2.2) with assumptions (2.3)-(2.5) is RD, and hence has a countably infinite number of eigenvalues* $\{\xi_n(\lambda) : n \in \mathbb{N}_0\}$*, which are all real, bounded below and unbounded above and can be indexed to satisfy*

$$-\infty \le \xi_0(\lambda) \le \xi_1(\lambda) \le \cdots, \quad \text{and } \ \xi_n(\lambda) \to \infty \text{ as } n \to \infty.$$

(ii) By Theorem 3.3 (i), for $n \in \mathbb{N}_0$*,* $\xi_n(\lambda)$ *is a continuous function of* λ *in* $\mathbb{R}$*.*

(iii) If the SLP (2.1), (2.2) is LD, then by Theorem 2.1, $\xi_0(0) > 0$*.*

(iv) If $\xi_n(\lambda^*) = 0$ *for some* $n \in \mathbb{N}_0$ *and* $\lambda^* \in \mathbb{R}$*, then* λ^* *is an eigenvalue of the LD problem (2.1), (2.2).*

Lemma 4.1 *Let* $n \in \mathbb{N}_0$ *and* $h \in \mathbb{R}$*. Assume* $\xi_n(h) < \xi_0(0)$ *and* y_n *is an eigenfunction of the SLP (4.1), (2.2) associated with* $\lambda = h$ *and* $\xi_n = \xi_n(h)$*. Then*

$$h \int_a^b |y_n|^2 w > 0. \tag{4.2}$$

Proof: Let $k = \xi_n(h)$. Then $k < \xi_0(0)$. Let $\xi = k + \tilde{\xi}$. Then (4.1) becomes

$$-(py')' + (q - k|w|)y - \lambda wy = \tilde{\xi}|w|y. \tag{4.3}$$

Hence, the n-th eigenvalue of (4.3), (2.2) is $\tilde{\xi}_n(\lambda) = \xi_n(\lambda) - k$, which satisfies $\tilde{\xi}_n(h) = 0$. Therefore, h is an eigenvalue of the equation

$$-(py')' + (q - k|w|)y = \lambda wy \tag{4.4}$$

together with the BC (2.2). By assumption,

$$\tilde{\xi}_0(0) = \xi_0(0) - k > 0.$$

For $f \in \mathcal{D}$ we define

$$\mathcal{L}(\|)\{ = \int_{\dashv}^{\lfloor} [-(\sqrt{}\{')'\{ + (\amalg - \||\sqsupseteq|)|\{|^{\in}].$$

Similar to (2.14) we have that

$$0 < \tilde{\xi}_0(0) = \min \frac{\mathcal{L}(\|)\{}{\int_a^b |f|^2|w|} \tag{4.5}$$

where the min is over all $f \not\equiv 0$ in $\mathcal{D}$. It is easy to see that $y_n \in \mathcal{D}$. Then (4.5) implies that $\mathcal{L}(\|)\dagger_\backslash > \prime$, i.e.,

$$\int_a^b [-(py_n')'\bar{y}_n + (q - k|w|)|y_n|^2 > 0.$$

Note that y_n is an eigenfunction of (4.4), (2.2) associated with $\lambda = h$, from (4.4) and the above inequality we obtain (4.2). ∎

Corollary 4.1 *Let (2.3)-(2.5) hold, and assume the SLP (2.1), (2.2) is LD. Then the eigenvalues of (2.1), (2.2) are all real.*

Proof: Let λ be an eigenvalue of (2.1), (2.2), and y an associated eigenfunction. From (2.1) we obtain that

$$(\lambda - \bar{\lambda})|y|^2 w = -(py')'\bar{y} + (p\bar{y}')'y = [y, \bar{y}]'$$

where $[u, v] := u(p\bar{v}') - \bar{v}(pu')$ is the usual Lagrange bracket. Thus,

$$(\lambda - \bar{\lambda}) \int_a^b |y|^2 w = [y, \bar{y}]_a^b = 0. \tag{4.6}$$

Since (2.1), (2.2) is LD, for $\xi_n(\lambda)$ defined in Remark 4.1, (i), we have that $\xi_0(0) > 0$, hence $0 = \xi_n(\lambda) < \xi_0(0)$. By Lemma 4.1, $\lambda \int_a^b |y|^2 w > 0$ and thus $\int_a^b |y|^2 w \neq 0$. Now, (4.6) implies that $\lambda = \bar{\lambda}$, i.e., λ is real. ∎

Remark 4.2 *Corollary 4.1 together with Remark 4.1 shows that each eigenvalue of the LD problem (2.1), (2.2) is a root of $\xi_n(\lambda) = 0$ for some $n \in \mathbb{N}_0$, where $\xi_n(\lambda)$ is defined in Remark 4.1. This leads to the following natural indexing of the eigenvalues of left-definite problems (2.1), (2.2).*

Definition 4.1 *Let $n \in \mathbb{N}_0$.*

- *If the curve $\xi = \xi_n(\lambda)$ defined in Remark 4.1 intersects the axis $\xi = 0$ at $(\lambda^*, 0)$ where $\lambda^* > 0$, then $\lambda_n = \lambda^*$.*
- *If the curve $\xi = \xi_n(\lambda)$ defined in Remark 4.1 intersects the axis $\xi = 0$ at $(\lambda_*, 0)$ where $\lambda_* < 0$, then $\lambda_{-n} = \lambda_*$.*

INSERT FIGURE HERE.

We adopt this indexing scheme below. It is with respect to this indexing scheme that we study the continuity of λ_n with respect to the parameters of the problem and also the relationships of the eigenvalues with respect to different boundary conditions.

To show the existence of eigenvalues of the LD problems with coupled BC's, we study the functions $\xi_n(\lambda),\ n \in \mathbb{N}_0$.

Lemma 4.2 *For $n \in \mathbb{N}_0$ and $h \in \mathbb{R}$, assume that either $\xi_n(h)$ is simple or $\xi_n(\lambda)$ is double for all λ in an open interval containing h. Then $\xi_n(\lambda)$ is continuously differentiable at h. Furthermore, if $\xi_n(h) < \xi_0(0)$, then $h\,\xi_n'(h) < 0$.*

Proof: The continuous differentiability follows from Theorem 3.3, (iii). To show the furthermore part, let y_n be defined as in Lemma 4.1, and let $u_n = cy_n$ for some $c > 0$ such that $\int_a^b |u_n|^2|w| = 1$. From Lemma 4.1 we have

$$h\int_a^b |u_n|^2|w| = c^2h\int_a^b |y_n|^2 w > 0. \tag{4.7}$$

Note that $\xi_n(\lambda)$ is an eigenvalue of (4.1), (2.2), by applying the derivative formula (3.7) and the chain rule for differentiation, we obtain that

$$\xi_n'(h) = -\int_a^b |u_n|^2 w. \tag{4.8}$$

Combining (4.7) and (4.8) we have that $h\,\xi_n'(h) < 0$. ∎

Corollary 4.2 *For each $n \in \mathbb{N}_0$, the function $\xi = \xi_n(\lambda)$ is strictly decreasing in the region*

$$E_1 = \{(\lambda, \xi) : \lambda > 0,\ \xi < \xi_0(0)\}$$

and strictly increasing in the region

$$E_2 = \{(\lambda, \xi) : \lambda < 0,\ \xi < \xi_0(0)\}.$$

Therefore, if $\xi_n(h) < \xi_0(0)$ for some $h > 0$, then $\xi_n(\lambda)$ is strictly decreasing for $\lambda > h$; if $\xi_n(h) < \xi_0(0)$ for some $h < 0$, then $\xi_n(\lambda)$ is strictly increasing for $\lambda < h$.

Proof: Corollary 4.2 is not a direct consequence of Lemma 4.2 since $\xi_n(\lambda)$ need not be differentiable everywhere. However, the Dini devivatives of $\xi_n(\lambda)$, say $D^*\xi_n(\lambda)$, exist everywhere. By an argument similar to that given in the proof of Theorem 6.1 in [11], one can show that $D^*\xi_n(\lambda) < 0$ on E_1 and $D^*\xi_n(\lambda) > 0$ on E_2. Then the conclusion of Corollary 4.2 follows. We omit the detail of the proof. ∎

In the following we assume (2.3) and (2.8) hold. We denote by $\xi_n(\lambda), \eta_n(\lambda), \zeta_n(\lambda),\ n \in \mathbb{N}_0$, the eigenvalues of (4.1) together with (2.7), (3.1), (3.2), respectively; and denote by $\lambda_n, \mu_n, \nu_n,\ n \in \mathbb{Z}^*$, the eigenvalues of (2.1) together with (2.7), (3.1), (3.2), respectively, whenever exist. If the SLP(2.1), (3.1) is LD and not RD, then by Theorem 4.1, μ_n exists for all $n \in \mathbb{Z}^*$, and $\eta_n(\lambda) = 0$ if and only if $\lambda = \mu_{\pm n}$. Similar results are true for $\zeta_n(\lambda)$ and ν_n. The next theorem is a new result on the existence of eigenvalues of LD problems with coupled BC's, with a simple elementary proof.

Theorem 4.2 *Assume (2.3), (2.8) hold, and the SLP (2.1), (2.7) is LD and not RD. Then there exists a countably infinite number of eigenvalues, they are all real, may be geometrically simple or double, are unbounded below and above with no finite cluster point, and when they are indexed according to Definition 4.1 they satisfy the inequalities*

$$\cdots \leq \lambda_{-n} \leq \cdots \leq \lambda_{-1} \leq \lambda_{-0} < 0 < \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots \tag{4.9}$$

with only geometrically double eigenvalues appearing twice.

Furthermore, $\xi_n(\lambda) = 0$ if and only if $\lambda = \lambda_{\pm n}$, $n \in \mathbb{N}_0$.

Proof: Without loss of generality we only consider the case that either $k_{11} > 0$ and $k_{12} \leq 0$ or $k_{11} \leq 0$ and $k_{12} < 0$. Since (2.1), (2.7) is LD, $\xi_0(0) > 0$. Applying Theorem 3.1 to (4.1), we see that $\eta_n(\lambda) \geq \xi_n(\lambda)$ for all $\lambda \in \mathbb{R}$, $n \in \mathbb{N}_0$, and hence $\eta_0(0) \geq \xi_0(0) > 0$. This means that the SLP (2.1), (3.1) is LD (and not RD). Thus, $\eta_n(\lambda) = 0$ if and only if $\lambda = \mu_{\pm n}$, $n \in \mathbb{N}_0$. Combining the facts that $\xi_n(0) > 0$ and $\xi_n(\mu_{\pm n}) \leq \eta_n(\mu_{\pm n}) = 0$, from the continuity of $\xi_n(\lambda)$, we see that there exist $\lambda_{-n} < 0 < \lambda_n$ such that $\xi_n(\lambda_{\pm n}) = 0$, i.e., $\lambda_{\pm n}$ are eigenvalues of the SLP (2.1), (2.7).

Next, we show that such $\lambda_{\pm n}$ are unique. If not, without loss of generality, assume that $0 < \lambda_* < \lambda^*$ are the first two consecutive positive numbers such that $\xi_n(\lambda_*) = \xi_n(\lambda^*) = 0$. Note that $\xi_n(\lambda_*) < \xi_0(0)$ since $\xi_0(0) > 0$. Thus by Corollary 4.2, ξ_n is strictly decreasing for $\lambda \geq \lambda_*$. This is impossible.

Finally, the order of $\{\lambda_n : n \in \mathbb{Z}^*\}$ in (4.9) is from the fact that $\xi_{n+1}(\lambda) \geq \xi_n(\lambda)$ for any $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}_0$. ∎

In the following, the notation $\{a, b\}$ means each of a and b.

Theorem 4.3 *Let (2.3) and (2.8) hold.*

(a) Assume $k_{11} > 0$ *and* $k_{12} \le 0$*, and the SLP (2.1),(3.2) is LD and not RD. Then both the SLP's (2.1), (2.7) and (2.1), (3.1) are LD and not RD. Furthermore,* $\lambda_{\pm 0}(K)$ *is geometrically simple, and for any* $\theta \in (-\pi, \pi)$, $\theta \ne 0$*, we have*

$$\begin{aligned} \nu_0 \quad &\le \lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) \le \{\mu_0, \nu_1\} \\ &\le \lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) \le \{\mu_1, \nu_2\} \\ &\le \lambda_2(K) < \lambda_2(e^{i\theta}K) < \lambda_2(-K) \le \{\mu_2, \nu_3\} \\ &\le \lambda_3(-K) < \lambda_3(e^{i\theta}K) < \lambda_3(K) \le \{\mu_3, \nu_4\} \le \cdots. \end{aligned} \tag{4.10}$$

and another half of the inequality is obtained by replacing λ_n, μ_n, ν_n *in (4.10) by* $\lambda_{-n}, \mu_{-n}, \nu_{-n}$*, and* $<, \le$ *by* $>, \ge$*, respectively.*

(b) Assume $k_{11} \le 0$ *and* $k_{12} < 0$*, and the SLP (2.1),(2.7) is LD and not RD. Then both the SLP's (2.1), (3.1) and (2.1), (3.2) are LD and not RD. Furthermore,* $\lambda_{\pm 0}(K)$ *is geometrically simple, and for any* $\theta \in (-\pi, \pi)$, $\theta \ne 0$*, we have*

$$\begin{aligned} &\lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) \le \{\mu_0, \nu_0\} \le \\ &\lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) \le \{\mu_1, \nu_1\} \le \\ &\lambda_2(K) < \lambda_2(e^{i\theta}K) < \lambda_2(-K) \le \{\mu_2, \nu_2\} \le \\ &\lambda_3(-K) < \lambda_3(e^{i\theta}K) < \lambda_3(K) \le \{\mu_3, \nu_3\} \le \cdots. \end{aligned} \tag{4.11}$$

and another half of the inequality is obtained by replacing λ_n, μ_n, ν_n *in (4.10) by* $\lambda_{-n}, \mu_{-n}, \nu_{-n}$*, and* $<, \le$ *by* $>, \ge$*, respectively.*

(c) If neither case (a) nor case (b) applies to K*, then either case (a) or case (b) applies to* $-K$*.*

Proof: (a) For each fixed $\lambda \in \mathbb{R}$, inequality (3.3) holds where λ_n, μ_n, ν_m are replaced by $\xi_n(\lambda), \eta_n(\lambda)$, $\zeta_n(\lambda)$, respectively. That is, the following inequality holds:

$$\begin{aligned} \zeta_0(\lambda) \quad &\le \xi_0(\lambda, K) < \xi_0(\lambda, e^{i\theta}K) < \xi_0(\lambda, -K) \le \{\eta_0(\lambda), \zeta_1(\lambda)\} \\ &\le \xi_1(\lambda, -K) < \xi_1(\lambda, e^{i\theta}K) < \xi_1(\lambda, K) \le \{\eta_1(\lambda), \zeta_2(\lambda)\} \\ &\le \xi_2(\lambda, K) < \xi_2(\lambda, e^{i\theta}K) < \xi_2(\lambda, -K) \le \{\eta_2(\lambda), \zeta_3(\lambda)\} \\ &\le \xi_3(\lambda, -K) < \xi_3(\lambda, e^{i\theta}K) < \xi_3(\lambda, K) \le \{\eta_4(\lambda), \zeta_4(\lambda)\} \le \cdots. \end{aligned} \tag{4.12}$$

In particular, $\zeta_n(0) > 0$ implies that $\xi_n(0) > 0$ and $\eta_n(0) > 0$. Hence by Theorem 2.1, we see that (2.1), (3.2) is LD (and not RD) implies that (2.1), (2.8) and (2.1), (3.1) are LD (and not RD). Note that for $n \in \mathbb{N}_0$, $\lambda_{\pm n}, \mu_{\pm n}$, and $\nu_{\pm n}$ are the positive and negative roots of $\xi_n(\lambda), \eta_n(\lambda)$, and $\zeta_n(\lambda)$, respectively, and each of $\xi_n(\lambda), \eta_n(\lambda)$, and $\zeta_n(\lambda)$ is continuous in λ and is increasing in n. Then (4.10) becomes clear.

Parts (b) and (c) can be proved in a similar way. ∎

In the following we denote by $\{\xi_n^D(\lambda) : n \in \mathbb{N}_0\}$ the eigenvalues of the SLP consisting of the equation (4.1) and the Dirichlet BC

$$y(a) = y(b) = 0. \tag{4.13}$$

If the SLP (2.1), (4.12) is LD and not RD, we denote its eigenvalue by $\{\lambda_n^D : n \in \mathbb{Z}^*\}$.

Theorem 4.4 *Let (2.3)-(2.5) hold, and assume the SLP (2.1), (2.2) is LD and not RD. Then the SLP (2.1), (4.12) is LD and not RD, and*

$$\lambda_n \in (0, \lambda_n^D] \text{ and } \lambda_{-n} \in [\lambda_{-n}^D, 0) \quad \text{for } n = 0, 1;$$
$$\lambda_n \in (\lambda_{n-2}^D, \lambda_n^D] \text{ and } \lambda_{-n} \in [\lambda_{-n}^D, \lambda_{-n+2}) \quad \text{for } n = 2, 3, \ldots.$$

Proof: Since (2.1), (2.2) is LD, $\xi_0(0) > 0$. Applying Theorem 3.2 to (4.1) we get that

$$\xi_0^D(\lambda) \geq \xi_0(\lambda) \quad \text{for } \lambda \in \mathbb{R}. \tag{4.14}$$

In particular, $\xi_0^D(0) \geq \xi_0(0) > 0$. Thus (2.1), (4.12) is LD. (4.13) also implies that $\xi_0(\lambda_0^D) \leq \xi_0^D(\lambda_0^D) = 0$. This shows that $\lambda_0 \in (0, \lambda_0^D]$ since λ_0 is the only root of $\xi_0(\lambda) = 0$. The rest can be proved similarly. ∎

Remark 4.3 *In the right-definite case λ_n is a continuous function of the equation i.e. of $a, b, 1/p, q, w$ but not, in general, of the boundary conditions, see [8]. In this case the discontinuities of λ_n are completely characterized in [12]. Corollary 4.3 below shows that also in the left-definite case, λ_n for $n \in \mathbb{Z}$ depends continuously on the equation. The example above Theorem 4.5 below shows that also in the left-definite case λ_n is not a continuous function of the boundary conditions, in general. However, Theorem 4.5 shows that λ_n is continuos at the left-definite problem $\boldsymbol{\omega}$ if there is a neighborhood of the $\boldsymbol{\omega}$ in the space of LD problems such that every problem in this neighborhood is LD and not RD; in other words $\lambda_n(\boldsymbol{\omega})$ is continuous at all the "interior points" of the space of LD problems.*

Example. The SLP consisting of the equation

$$-y'' = \lambda \operatorname{sgn} t\, y, \quad -1 < t < 1 \tag{4.15}$$

and the Dirichlet BC

$$y(-1) = 0 = y(1) \tag{4.16}$$

is LD (and is not RD) since the least eigenvalue of the SLP consisting of the Fourier equation

$$-y'' = \lambda y, \quad -1 < t < 1 \tag{4.17}$$

and the BC (4.16) is positive. Now, in any neighborhood of the SLP (4.15), (4.16), there is a SLP consisting of (4.15) and the BC

$$y(-1) - cy^{[1]}(-1) = 0, \quad y(1) = 0, \tag{4.18}$$

for some $c > 0$ sufficiently small. However, this SLP is not LD for all sufficiently small $c > 0$ since the lowest eigenvalue of the SLP (4.16), (4.17) approaches $-\infty$ as $c \to 0^+$, see [8].

Theorem 4.5 *Let (2.3)-(2.5) hold, and assume the SLP (2.1), (2.2) is LD and not RD in a neighborhood* $\mathcal{N}$ *of* $\boldsymbol{\omega}_0 = (a_0, b_0, A_0, B_0, 1/p_0, q_0, w_0) \in \boldsymbol{\Omega}$. *Then for any* $n \in \mathbb{Z}^*$, $\lambda_n(\boldsymbol{\omega})$ *is continuous at* $\boldsymbol{\omega}_0$.

Proof: Similar to the proof of Theorem 3.1 in [13] we can show that for any fixed $n \in \mathbb{Z}^*$, $\lambda_n(\boldsymbol{\omega}_0)$ can be imbedded into a continuous branch. We only need to show that this branch keeps the same index n. For $\lambda \in \mathbb{R}$ and $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ let $\{\xi_n(\lambda, \boldsymbol{\omega}) : n \in \mathbb{N}_0\}$ be the eigenvalues of (4.1), (2.2). From the assumption we know that $\xi_0(0, \boldsymbol{\omega}) > 0$ for all $\boldsymbol{\omega} \in \mathcal{N}$. This implies that $\xi_0(0, \boldsymbol{\omega})$ is continuous with respect to $\boldsymbol{\omega}$ in $\mathcal{N}$since otherwise, $\xi_0(0, \boldsymbol{\omega})$ would be unbounded below for $\boldsymbol{\omega} \in \mathcal{N}$, see Theorem 3.39 in [12]. Hence for $n \in \mathbb{N}_0$, $\xi_n(\lambda, \boldsymbol{\omega})$ is continuous in λ and $\boldsymbol{\omega}$ for $(\lambda, \boldsymbol{\omega}) \in \mathbb{R} \times \mathcal{N}$. From the fact that $\lambda_n(\boldsymbol{\omega})$ is the only root of $\xi_n(\lambda, \boldsymbol{\omega})$ we see that $\lambda_n(\boldsymbol{\omega}), \boldsymbol{\omega} \in \mathcal{N}$, is a continuous eigenvalue branch and hence $\lambda_n(\boldsymbol{\omega})$ is continuous at $\boldsymbol{\omega}_0$. ∎

Corollary 4.3 *Let (2.3)-(2.5) hold, and assume the SLP (2.1), (2.2) is LD and not RD at* $\boldsymbol{\omega}_0 = (a_0, b_0, A_0, B_0, 1/p_0, q_0, w_0) \in \boldsymbol{\Omega}$. *Then for any* $n \in \mathbb{Z}^*$, $\lambda_n(w)$ *is continuous with respect to the equation at* $\boldsymbol{\omega}_0$, *i.e.,* $\lambda_n(w)$ *is continuous with respect to a, b, 1/p, q, w at* $\boldsymbol{\omega}_0$.

Proof: From Theorem 3.3, (i) we see that for $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$, $\xi_n(\lambda, \boldsymbol{\omega})$ defined in the proof of Theorem 4.5 is continuous with respect to the equation. The rest of the proof is similar to that of Theorem 4.5 and is omitted. ∎

In the next theorem $u_{\pm n}$ denote normalized eigenfunctions of (2.1), (2.2) associated with $\lambda_{\pm n}$ satisfying

$$\int_a^b |u_{\pm n}|^2 w = \pm 1, \quad n \in \mathbb{N}_0. \tag{4.19}$$

Such eigenfunctions exist since from Lemma 4.1 we have for any eigenfunction y_n of λ_n

$$\lambda_n \int_a^b |y_n|^2 w > 0, \quad n \in \mathbb{Z}^*. \tag{4.20}$$

In Theorem 4.6 below the derivative λ_n' is the Frechet derivative in the appropriate Banach space. A map T from a Banach space X into a Banach space Y is differentiable at a point $x \in X$ if there exists a bounded linear transformation $T' : X \to Y$ such that for all $h \in X$

$$|T(x+h) - T(x) - T'(x)h| = o(h) \quad \text{as} \quad h \to 0 \text{ in } X.$$

Theorem 4.6 *Let (2.1)-(2.5) hold and let the boundary conditions have their canonical representation (2.6) or (2.7), (2.8). Assume the SLP (2.1), (2.2) is LD and not RD in a neighborhood $\mathcal{N}$ of the point $\boldsymbol{\omega}_0 = (a_0, b_0, A_0, B_0, 1/p_0, q_0, w_0) \in \boldsymbol{\Omega}$. Let $n \in \mathbb{Z}^*$.*

1. *Fix all components of $\boldsymbol{\omega} \in \mathbb{N}$ except a and consider λ_n as a function of a. Assume that $\lambda_n(a_0)$ is geometrically simple or $\lambda_n(a)$ is geometrically double in some neighborhood of a_0. Then $\lambda_n(a)$ is differentiable a.e. in some neighborhood $\mathbb{N}_a$ of a_0 and*

$$\lambda_{\pm n}'(a) = \pm\{\frac{1}{p}(a)|pu_{\pm n}'|^2(a) - |u_{\pm n}(a)|^2[q(a) - \lambda_n(a)w(a)]\} \quad \textit{a.e. in } \mathbb{N}_a. \tag{4.21}$$

   Furthermore, if p, q, w are continuous at a and $p(a) \neq 0$, then (4.21) holds at the point a.

2. *Fix all components of $\boldsymbol{\omega}$ except b and consider λ_n as a function of b. Assume that $\lambda_n(b_0)$ is geometrically simple or $\lambda_n(b)$ is geometrically double in some neighborhood of b_0. Then $\lambda_n(b)$ is differentiable a.e. in some neighborhood $\mathbb{N}_b$ of b_0 and*

$$\lambda_{\pm n}'(b) = \pm\{\frac{-1}{p}(b)|pu_{\pm n}'|^2(b) + |u_{\pm n}(b)|^2[q(b) - \lambda_n(b)w(b)]\} \quad \textit{a.e. in } \mathbb{N}_b. \tag{4.22}$$

   Furthermore, if p, q, w are continuous at b and $p(b) \neq 0$, then (4.22) holds at the point b.

3. *Assume the boundary conditions are separable and have the canonical form (2.6); in this case A, B in $\boldsymbol{\omega}$ are replaced by α, β. Fix all components of $\boldsymbol{\omega} \in \mathbb{N}$ except α and consider λ_n as a function of $\alpha \in [0, \pi)$. Then λ_n is differentiable and*

$$\lambda_{\pm n}'(\alpha) = \pm\{-u_{\pm n}^2(a) - (pu_{\pm n}')^2(a)\}, \quad 0 \leq \alpha < \pi. \tag{4.23}$$

4. *Assume the boundary conditions are separable and have the canonical form (2.6); in this case* A, B *in* $\boldsymbol{\omega}$ *are replaced by* α, β. *Fix all components of* $\boldsymbol{\omega} \in \mathbb{N}$ *except* β *and consider* λ_n *as a function of* $\beta \in (0, \pi]$. *Then* λ_n *is differentiable and*

$$\lambda'_{\pm n}(\beta) = \pm\{u^2_{\pm n}(b) + (pu'_{\pm n})^2(b)\}, \quad 0 < \beta \leq \pi. \tag{4.24}$$

5. *Let the boundary conditions be given by (2.7), (2.8); recall the in this case* A, B *are replaced by* θ, K *in* $\boldsymbol{\omega}$. *Fix all components of* $\boldsymbol{\omega} \in \mathbb{N}$ *except* θ *and consider* λ_n *as a function of* $\theta \in (-\pi, \pi]$. *Then* λ_n *is a differentiable function of* θ *and*

$$\lambda'_{\pm n}(\theta) = \pm\{-2Im[u_{\pm n}(b)(p\bar{u}'_{\pm n})(b)]\}, \tag{4.25}$$

where Im(z) denotes the imaginary part of z.

6. *Let the boundary conditions be given by (2.7), (2.8); recall the in this case* A, B *are replaced by* θ, K *in* $\boldsymbol{\omega}$. *Fix all components of* $\boldsymbol{\omega} \in \mathbb{N}$ *except* K *and consider* λ_n *as a function of* $K \in SL(2, \mathbb{R})$. *Assume that* $\lambda_n(K)$ *is simple. Then* λ_n *is simple in a neighborhood of* K, *is "differentiable within* $SL(2, \mathbb{R})$*" and*

$$\lambda'_{\pm n}(K)H = \pm\{[p\bar{u}'_{\pm n}(b), -\bar{u}_{\pm n}(b)]HK^{-1}\begin{pmatrix} u(b) \\ (pu')(b) \end{pmatrix}\}, H \in M_{2,2}(\mathbb{R}) such that K+H \in SL(2, \mathbb{R}). \tag{4.26}$$

The phrase "differentiable within $SL(2, \mathbb{R})$*" is used here to indicate that the definition of Frechet derivative needs to be modified here to take into account the fact that (4.26) does not hold for all* H *in the Banach space* $M_{2,2}$ *but only for those as indicated.*

7. *Fix all components of* $\boldsymbol{\omega} \in \mathbb{N}$ *except* $1/p$ *and consider* λ_n *as a function of* $1/p \in L^1(J, \mathbb{R})$. *Then* $\lambda_{\pm n}$ *is continuously differentiable with respect to* $1/p$ *and*

$$\lambda'_{\pm n}(1/p)h = \pm\{-\int_a^b |pu'_{\pm n}|^2 h\}, \quad h \in L^1(J, \mathbb{R}). \tag{4.27}$$

8. *Fix all components of* $\boldsymbol{\omega} \in \mathbb{N}$ *except* q *and consider* λ_n *as a function of* $q \in L^1(J, \mathbb{R})$. *Then* λ_n *is continuously differentiable with respect to* q *and*

$$\lambda'_{\pm n}(q)h = \pm\{\int_a^b |u_{\pm n}|^2 h\}, \quad h \in L^1(J, \mathbb{R}). \tag{4.28}$$

9. *Fix all components of* $\boldsymbol{\omega} \in \mathbb{N}$ *except* w *and consider* λ_n *as a function of* $w \in L^1(J, \mathbb{R})$*. Then* λ_n *is continuously differentiable with respect to* w *and*

$$\lambda'_{\pm n}(w)h = \pm\{-\lambda \int_a^b |u_{\pm n}|^2 h\}, \quad h \in L^1(J, \mathbb{R}). \tag{4.29}$$

Remark 4.4 *For right-definite problems it is known [8], [7] that* λ_n *is not a continuous function of the boundary conditions, in general. This is also true in the left-definite case. Note however that by Theorem 4.5 if the left-definite (but not right definite) SLP is left-definite in a neighborhood of the given problem, in other words if the problem is not an isolated point in the space of left-definite problems, then* $\lambda_{\pm n}$ *is a continuous function of the problem. These continuous* $\lambda_{\pm n}$ *are also differentiable. Thus the hypothesis that the problem is left-definite in a neighborhood implies continuity and differentiability of* $\lambda_{\pm n}$*, e.g. in Theorem 4.6, (6).*

Proof: The proof of Theorem 4.6 is similar to the proofs of the corresponding results for the right definite case [13], [12], [11], [14] and therefore omitted. ∎

5 Comments on Generalized Left-Definite Problems

In general, an indefinite S-L problem may have non-real eigenvalues, and the spectrum may exhibit complicated and strange behavior. However, among all the indefinite problems we now identify a class which has essentially the same properties as those in the LD case. This is the class of problems which can be transformed into the LD case by a translation. We adopt the notation T-LD for this class.

Definition 5.1 *Let* $T \in \mathbb{R}$*, and define a functional* $\mathcal{L}[\mathcal{T}]$ *on the linear manifold* $\mathcal{D}$ *defined by (2.10) as follows:*

$$\mathcal{L}[\mathcal{T}]\{ = \int_{\dashv}^{\lfloor} -(\sqrt{}\{')'\{ + (\amalg - \mathcal{T}\sqsupseteq)|\{|^{\in}.$$

Then the SLP (2.1), (2.2) is said to be T-left-definite (T-LD) if $\mathcal{L}[\mathcal{T}]\{ > \prime$ *for all* $f \not\equiv 0$ *in* $\mathcal{D}$*.*

From this definition it is easy to see the following:

(i) The SLP (2.1), (2.2) is LD if and only if it is 0-LD.

(ii) The SLP (2.1), (2.2) is T-LD for some $T \in \mathbb{R}$ if and only if the problem consisting of the equation

$$-(py')' + (q - Tw)y = \lambda wy \quad \text{on } (a, b) \tag{5.1}$$

together with the BC (2.2) is LD.

From the relation between (2.1) and (5.1), we see that all the results in Section 4 hold for the T-LD problem (5.1), (2.2) if we replace the condition LD by T-LD and replace 0 by T.

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