Dependence of the n-th Sturm-Liouville Eigenvalue on the Problem

Qingkai Kong, Hongyou Wu and Anton Zettl

Department of Mathematics, Northern Illinois University, DeKalb, IL 60115

Abstract. We consider the n-th eigenvalue as a function on the space of self-adjoint regular Sturm-Liouville problems with positive leading coefficient and weight functions. The discontinuity of the n-th eigenvalue is completely characterized.

Consider a self-adjoint regular Sturm-Liouville problem (SLP) with positive leading coefficient and weight functions, i.e.,

$$-(py')' + qy = \lambda wy \text{ on } (a,b), \tag{0.1}$$

$$(A \,|\, B) \begin{pmatrix} y(a) \\ (py')(a) \\ y(b) \\ (py')(b) \end{pmatrix} = 0, \tag{0.2}$$

where

$$-\infty \le a < b \le \infty, \qquad 1/p,\ q,\ w \in \mathrm{L}((a,b),\mathbb{R}), \qquad p, w > 0 \text{ a.e. on } (a,b), \tag{0.3}$$

$$(A \,|\, B) \in \mathrm{M}^*_{2\times 4}(\mathbb{C}), \qquad A \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A^* = B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B^*, \tag{0.4}$$

and $\lambda \in \mathbb{C}$ is the so called spectral parameter of (0.1). Here $\mathrm{L}((a,b),\mathbb{R})$ denotes the space of Lebesgue integrable real functions on (a,b), $\mathrm{M}^*_{2\times 4}(\mathbb{C})$ stands for the set of 2 by 4 matrices over $\mathbb{C}$ with rank 2, and A^* is the complex conjugate transpose of the complex matrix A. It is well-known that the eigenvalues of the problem can be ordered to form a non-decreasing sequence

---

1991 Mathematics Subject Classification: Primary 34B24; Secondary 34L05, 34L15.

Key words: regular Sturm-Liouville problems, n-th eigenvalue, continuity, discontinuity, range.

$$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \cdots \tag{0.5}$$

approaching $+\infty$ so that the number of times an eigenvalue appears in the sequence is equal to its multiplicity. Hence, for each $n \in \mathbb{N}_0 =: \{0, 1, 2, \cdots\}$, λ_n is a function defined on the space of such SLP's. Everitt, Möller and Zettl have shown in [5] that in general, λ_n does *not* depend on the problem continuously. They have shown, among other things, that λ_0 has an infinite jump when the differential equation (DE) in the SLP is fixed and the boundary condition (BC) in the problem approaches the Dirichlet BC in a certain way. Thus, a natural question arises: what is the discontinuity set of λ_n? This is the main question that we want to address in this paper.

Recently, a lot of progress has been made in computing the eigenvalues of SLP's and of higher order eigenvalue problems (see, for example, [2], [6] and [1]). In principle, the eigenvalues of such a problem can be computed as the zeros of the characteristic function of the problem with a root finder. For the SLP consisting of (0.1) and (0.2), one is interested in both the values and the indices of the eigenvalues. For example, after computing an eigenvalue, we want to know which one it is, i.e., whether it is λ_0 or λ_1 or λ_{283}. To figure out the indices of these eigenvalues is a rather difficult task, since sometimes the first few eigenvalues are not computable. From the theoretical point of view, these indices can be determined in terms of the Prüfer transformation for the case of separated BC's and with the inequalities recently established in [4] for the case of coupled BC's. In most of the algorithms for computing such eigenvalues, it is necessary to approximate the problem in question, i.e., to approximate the interval and coefficient functions of the DE in the problem and to approximate the coefficient matrix of the BC in the problem. So, in usual numerical computation, we only obtain the eigenvalues of an approximate problem which are hopefully close to the eigenvalues of the original problem. However, in general, these eigenvalues of the approximate problem and the desired eigenvalues of the original problem do not have the same set of indices. Thus, in order to have an algorithm for determining the indices in computing, one needs a way to approximate the given SLP so that the indices stay invariant, i.e., the indices do not jump. The jumps in indices correspond precisely to the discontinuity of λ_n considered as a function of the problem. Therefore, the question investigated in this paper not only is of obvious theoretical interest but also has strong computational motivation. The results of this paper provide parts of the theoretical foundation for codes, such as Sleign2 [1], for computing eigenvalues *together with their indices.*

For each $n \in \mathbb{N}_0$, we characterize the set of SLP's at which λ_n is discontinuous. The discontinuity is always a jump, i.e., in some directions, either λ_0 or both λ_0 and λ_1 jump to continuous eigenvalue branches coming from $-\infty$ and the other λ_n's jump to the other continuous eigenvalue branches accordingly. At any such problem, we also identify the directions (in the space of SLP's considered) in which the value of λ_n jumps. In particular, we show that λ_n depends *continuously* on the DE in the SLP. We then determine the range of λ_n on the space of self-adjoint BC's and use it to obtain the possibilities for the number of zeros of a corresponding eigenfunction in the case of coupled BC's. We also comment on the differentiability and analyticity of λ_n at an SLP where it is continuous and has multiplicity 1. We then give an example to demonstrate that a multiplicity assumption is necessary in general for the differentiability of λ_n with respect to any parameter of the SLP.

This paper is organized as follows. In Section 1, we recall some basic results, describe the space of SLP's considered, and prove a principle for the continuity of λ_n. Section 2 is devoted to proving the continuous dependence of λ_n on the DE in the SLP. In Section 3, we give a complete characterization of the discontinuity of λ_n. In Section 4, the range of λ_n on the space of self-adjoint BC's is obtained and then used to determine the number of zeros of an eigenfunction for λ_n. Finally, in Section 5, we comment on the differentiability and analyticity of λ_n and give a related example.

Throughout this paper, we always assume that the DE (0.1) satisfies (0.3) and the BC (0.2) satisfies (0.4).

§1. Notation and Prerequisite Results

In this section we recall some basic results, describe the space of self-adjoint regular SLP's with positive leading coefficient and weight functions, and prove a principle for the continuity of the n-th eigenvalue as a function on the space of SLP's considered.

For each $\lambda \in \mathbb{C}$, let $\phi_{11}(\cdot, \lambda)$ and $\phi_{12}(\cdot, \lambda)$ be the solutions to (0.1) determined by the initial conditions

$$(1.1) \qquad \phi_{11}(a,\lambda) = 1,\ (p\phi_{11}')(a,\lambda) = 0 \quad \text{and} \quad \phi_{12}(a,\lambda) = 0,\ (p\phi_{12}')(a,\lambda) = 1,$$

respectively. We will denote $p\phi_{11}'$ by ϕ_{21} and $p\phi_{12}'$ by ϕ_{22}. Set

$$(1.2) \qquad \Phi(t,\lambda) = \begin{pmatrix} \phi_{11}(t,\lambda) & \phi_{12}(t,\lambda) \\ \phi_{21}(t,\lambda) & \phi_{22}(t,\lambda) \end{pmatrix}, \quad t \in [a,b],\ \lambda \in \mathbb{C}.$$

Here the values of $\Phi(\cdot,\lambda)$ at a and b are defined by right and left limits, respectively. For each $t \in [a,b]$, $\Phi(t,\lambda)$ is an entire matrix function of λ. Moreover, $\Phi(t,\lambda) \in \mathrm{SL}(2,\mathbb{R})$ for $t \in [a,b]$ and $\lambda \in \mathbb{R}$. The following result is well-known, see [13] or [4].

Theorem 1.3. *The Sturm-Liouville problem consisting of (0.1) and (0.2) has an infinite number of eigenvalues, and they are all real and bounded from below. Moreover, the eigenvalues are the zeros of the* characteristic function

$$\Delta(\lambda) =: \det(A + B\Phi(b,\lambda)) \tag{1.4}$$

of the problem and hence do not have a finite accumulation point.

Thus, as mentioned in the introduction, the eigenvalues of the problem can be ordered to form a non-decreasing sequence

$$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \cdots \tag{1.5}$$

approaching $+\infty$ so that the number of times an eigenvalue appears in the sequence is equal to its multiplicity. Note that by Theorem 4.16 in [8], the algebraic and geometric multiplicities of each eigenvalue of the SLP consisting of (0.1) and (0.2) are equal. So, in this paper we are not going to distinguish these two concepts and the word multiplicity will be used for either of them. Moreover, when counting the number of eigenvalues in a given interval, we will always assume that the eigenvalues are counted according to their multiplicities.

In the work [4] on Sturm-Liouville eigenvalues, the following representations of the solutions to the DE (0.1) is of crucial importance. We will need these formulas to study the discontinuity of the n-th eigenvalue.

Theorem 1.6. *There exist $\lambda_* \in \mathbb{R}$, $k > 0$ and a continuous function*

$$\alpha : [a,b] \times (-\infty, \lambda_*] \to [0,\infty) \tag{1.7}$$

such that $\alpha(t,\lambda)$ is decreasing in λ for each $t \in (a,b]$, $\alpha_t(t,\lambda)$ exists a.e. on $[a,b]$ for each $\lambda \in (-\infty,\lambda_*]$, $(p\alpha')(t,\lambda) = p(t)\alpha_t(t,\lambda)$ is continuous on $[a,b]$ for each $\lambda \in (-\infty,\lambda_*]$,

$$\lim_{\lambda\to-\infty} \alpha(t,\lambda) = \infty, \qquad \lim_{\lambda\to-\infty} p(t)\alpha_t(t,\lambda) = \infty \tag{1.8}$$

for each $t \in (a, b]$, *and*

$$\phi_{11}(t,\lambda) = k\,\cosh(\alpha(t,\lambda)), \tag{1.9}$$

$$\phi_{12}(t,\lambda) = \frac{1}{k^2}\,\phi_{11}(t,\lambda)\int_a^t \frac{\operatorname{sech}^2(\alpha(s,\lambda))}{p(s)}\mathrm{d}s, \tag{1.10}$$

$$\phi_{21}(t,\lambda) = k\,(p\alpha')(t,\lambda)\,\sinh(\alpha(t,\lambda)), \tag{1.11}$$

$$\phi_{22}(t,\lambda) = \frac{1}{k^2}\,\phi_{21}(t,\lambda)\int_a^t \frac{\operatorname{sech}^2(\alpha(s,\lambda))}{p(s)}\mathrm{d}s + \frac{1}{k}\operatorname{sech}(\alpha(t,\lambda)) \tag{1.12}$$

on $[a, b] \times (-\infty, \lambda_*]$.

In order to discuss the dependence of the eigenvalues of a self-adjoint SLP on the problem, we need to know how to measure the closeness of two DE's of the form (0.1) and how to measure the closeness of two self-adjoint BC's. These two questions are discussed in the next several paragraphs.

If the DE (0.1) is abbreviated as $(a, b, 1/p, q, w)$, then the space of DE's used in self-adjoint regular SLP's with positive leading coefficient and weight functions can be written as

$$\Omega = \big\{(a, b, 1/p, q, w);\ (0.3) \text{ holds}\big\}. \tag{1.13}$$

Bold faced lower case Greek letters, such as $\boldsymbol{\omega}$, will be used to stand for elements of Ω. A natural topology on Ω is the product topology induced from the usual topologies on $\mathbb{R}$ and on $\mathrm{L}(\mathbb{R}, \mathbb{R})$. More precisely, given $\epsilon > 0$, each $(a_0, b_0, 1/p_0, q_0, w_0) \in \Omega$ with finite a_0 and b_0 has a neighborhood in Ω consisting of the elements $(a, b, 1/p, q, w)$ satisfying

$$|a - a_0| + |b - b_0| + \int_{-\infty}^{+\infty} \big(|\widetilde{1/p} - \widetilde{1/p_0}| + |\widetilde{q} - \widetilde{q_0}| + |\widetilde{w} - \widetilde{w_0}|\big) < \epsilon, \tag{1.14}$$

where $\widetilde{1/p}$ is the extension of $1/p$ to $\mathbb{R}$ that is equal to 0 on $\mathbb{R} \setminus (a, b)$ and $\widetilde{1/p_0}$, $\widetilde{q}$, $\widetilde{q_0}$, $\widetilde{w}$, $\widetilde{w_0}$ have similar meanings; each $(-\infty, b_0, 1/p_0, q_0, w_0) \in \Omega$ with finite b_0 has a neighborhood in Ω formed by the elements $(a, b, 1/p, q, w)$ satisfying

$$a < -\frac{1}{\delta}, \quad |b - b_0| + \int_{-\infty}^{+\infty} \big(|\widetilde{1/p} - \widetilde{1/p_0}| + |\widetilde{q} - \widetilde{q_0}| + |\widetilde{w} - \widetilde{w_0}|\big) < \delta; \tag{1.15}$$

and etc. This topology has already been used in [10] and [7]. We note that Ω is path connected.

It is convenient to have DE's defined on finite intervals. For this reason, we consider the substitution

$$(1.16) \qquad t = t(s) = \int_{-\infty}^{s} f(r)\,\mathrm{d}r,$$

where $f \in \mathrm{L}^+(\mathbb{R},\mathbb{R}) =: \{f \in \mathrm{L}(\mathbb{R},\mathbb{R});\ f > 0 \text{ a.e. on } \mathbb{R}\}$. After this substitution, the DE (0.1) is transformed to the DE

$$(1.17) \qquad -\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{p(t(s))}{f(s)}\frac{\mathrm{d}}{\mathrm{d}s}y(t(s))\Big) + f(s)q(t(s))y(t(s)) = \lambda f(s)w(t(s))y(t(s)) \text{ on } (c,d)$$

where

$$(1.18) \qquad c = \int_{-\infty}^{a} f(s)\,\mathrm{d}s \quad \text{and} \quad d = \int_{-\infty}^{b} f(s)\,\mathrm{d}s$$

are finite. This defines a transformation from Ω into itself, which will be called the *canonical transformation* corresponding to f.

Proposition 1.19. *Let $f \in \mathrm{L}^+(\mathbb{R},\mathbb{R})$. Then, the canonical transformation from Ω into itself corresponding to f is continuous, and all the transformed differential equations are on finite intervals.*

PROOF. The first claim can be verified directly using the definitions, while the second one has been mentioned above. ∎

REMARK 1.20. After the substitution (1.16), the linear system corresponding to (0.2) reads

$$(1.21) \qquad (A\,|\,B)\begin{pmatrix} y(t(c)) \\ \frac{p(t(c))}{f(c)}\frac{\mathrm{d}}{\mathrm{d}s}y(t(c)) \\ y(t(d)) \\ \frac{p(t(d))}{f(d)}\frac{\mathrm{d}}{\mathrm{d}s}y(t(d)) \end{pmatrix} = 0.$$

Thus, *the substitution* (1.16) *does not change the coefficient matrix of any BC.*

Following [8], we will take the quotient space

$$(1.22) \qquad \mathrm{GL}(2,\mathbb{C})\backslash \mathrm{M}^*_{2\times 4}(\mathbb{C})$$

as the space of BC's, i.e., *each BC is an equivalence class of coefficient matrices of linear systems such as* (0.2), and the BC represented by the linear system (0.2) will be denoted by $[A \,|\, B]$. Note here that square brackets, not parentheses, are used. Usual bold faced capital Latin letters, such as $\boldsymbol{A}$, will also be used for BC's. The space $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ of self-adjoint real BC's consists of the separated real BC's and the coupled real BC's of the form $[K \,|\, -I]$ with $K \in \mathrm{SL}(2,\mathbb{R})$. By Theorem 2.18 in [8], $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ is a connected and compact analytic 3-dimensional manifold. It can be obtained by "gluing" the open sets

$$\mathcal{O}_{1,\mathrm{S}}^{\mathbb{R}} = \mathcal{O}_{6,\mathrm{S}}^{\mathbb{R}} = \{[K \,|\, -I];\ K \in \mathrm{SL}(2,\mathbb{R})\}, \tag{1.23}$$

$$\mathcal{O}_{2,\mathrm{S}}^{\mathbb{R}} = \left\{ \begin{bmatrix} 1 & a_{12} & 0 & a_{22} \\ 0 & a_{22} & -1 & b_{22} \end{bmatrix};\ a_{12}, a_{22}, b_{22} \in \mathbb{R} \right\}, \tag{1.24}$$

$$\mathcal{O}_{3,\mathrm{S}}^{\mathbb{R}} = \left\{ \begin{bmatrix} 1 & a_{12} & -a_{22} & 0 \\ 0 & a_{22} & b_{21} & -1 \end{bmatrix};\ a_{12}, a_{22}, b_{21} \in \mathbb{R} \right\}, \tag{1.25}$$

$$\mathcal{O}_{4,\mathrm{S}}^{\mathbb{R}} = \left\{ \begin{bmatrix} a_{11} & 1 & 0 & -a_{21} \\ a_{21} & 0 & -1 & b_{22} \end{bmatrix};\ a_{11}, a_{21}, b_{22} \in \mathbb{R} \right\}, \tag{1.26}$$

$$\mathcal{O}_{5,\mathrm{S}}^{\mathbb{R}} = \left\{ \begin{bmatrix} a_{11} & 1 & a_{21} & 0 \\ a_{21} & 0 & b_{21} & -1 \end{bmatrix};\ a_{11}, a_{21}, b_{21} \in \mathbb{R} \right\} \tag{1.27}$$

via the coordinate transformations among these open sets. Note that the topology on $\mathrm{SL}(2,\mathbb{R})$ is the one induced from the usual topology on the set $\mathrm{M}_{2\times 2}(\mathbb{R})$ of 2×2 matrices over $\mathbb{R}$, and each of the four open sets in (1.24)–(1.27) can be identified with $\mathbb{R}^3$. A complex BC $[A \,|\, B]$ is self-adjoint if and only if either $[A \,|\, B]$ is real with $\det A = \det B$ or $[A \,|\, B] = [e^{i\theta}K \,|\, -I]$ with $\theta \in (0,\pi)$ and $K \in \mathrm{SL}(2,\mathbb{R})$. By Theorem 2.25 in [8], the space $\mathcal{B}_{\mathrm{S}}^{\mathbb{C}}$ of self-adjoint complex BC's is a connected and compact analytic 4-dimensional real manifold. It can be obtained by "gluing" the open sets

$$\mathcal{O}_{1,\mathrm{S}}^{\mathbb{C}} = \mathcal{O}_{6,\mathrm{S}}^{\mathbb{C}} = \left\{[e^{i\theta}K \,|\, -I];\ \theta \in [0,\pi),\ K \in \mathrm{SL}(2,\mathbb{R})\right\}, \tag{1.28}$$

$$\mathcal{O}_{2,\mathrm{S}}^{\mathbb{C}} = \left\{ \begin{bmatrix} 1 & a_{12} & 0 & \bar{z} \\ 0 & z & -1 & b_{22} \end{bmatrix};\ a_{12} \in \mathbb{R}, z \in \mathbb{C}, b_{22} \in \mathbb{R} \right\}, \tag{1.29}$$

$$\mathcal{O}_{3,\mathrm{S}}^{\mathbb{C}} = \left\{ \begin{bmatrix} 1 & a_{12} & -\bar{z} & 0 \\ 0 & z & b_{21} & -1 \end{bmatrix};\ a_{12} \in \mathbb{R}, z \in \mathbb{C}, b_{21} \in \mathbb{R} \right\}, \tag{1.30}$$

$$\mathcal{O}_{4,\mathrm{S}}^{\mathbb{C}} = \left\{ \begin{bmatrix} a_{11} & 1 & 0 & -\bar{z} \\ z & 0 & -1 & b_{22} \end{bmatrix};\ a_{11} \in \mathbb{R}, z \in \mathbb{C}, b_{22} \in \mathbb{R} \right\}, \tag{1.31}$$

$$\mathcal{O}_{5,\mathrm{S}}^{\mathbb{C}} = \left\{ \begin{bmatrix} a_{11} & 1 & \bar{z} & 0 \\ z & 0 & b_{21} & -1 \end{bmatrix};\ a_{11} \in \mathbb{R}, z \in \mathbb{C}, b_{21} \in \mathbb{R} \right\} \tag{1.32}$$

via the coordinate transformations among these open sets. Note that the topology on the open set in (1.28) is the one induced from the usual topology on $\mathrm{M}_{2\times 2}(\mathbb{C})$, and each of the four open sets in (1.29)–(1.32) can be identified with $\mathbb{R}^4$.

Therefore, $\Omega \times \mathcal{B}_{\mathrm{S}}^{\mathbb{C}}$ is the space of self-adjoint regular SLP's with positive leading coefficient and weight functions. For every $(\boldsymbol{\omega}, \boldsymbol{A}) \in \Omega \times \mathcal{B}_{\mathrm{S}}^{\mathbb{C}}$ and every $n \in \mathbb{N}_0$, $\lambda_n(\boldsymbol{\omega}, \boldsymbol{A})$ is well-defined. When the DE is fixed, we will also use $\lambda_n(\boldsymbol{A})$ for $\boldsymbol{A} \in \mathcal{B}_{\mathrm{S}}^{\mathbb{C}}$ and $\lambda_n(e^{i\theta}K)$ for $\theta \in (-\pi, \pi]$, $K \in \mathrm{SL}(2, \mathbb{R})$; when the BC is fixed, we will also use $\lambda_n(\boldsymbol{\omega})$ for $\boldsymbol{\omega} \in \Omega$, and etc.

Let $\boldsymbol{\omega} \in \Omega$. For $K \in \mathrm{SL}(2, \mathbb{R})$, we use $\{\mu_n(\boldsymbol{\omega}, K);\ n \in \mathbb{N}_0\}$ to denote the eigenvalues of the SLP consisting of $\boldsymbol{\omega}$ and the separated BC

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & k_{22} & -k_{12} \end{bmatrix}, \tag{1.33}$$

and $\{\nu_n(\boldsymbol{\omega}, K);\ n \in \mathbb{N}_0\}$ the eigenvalues of the SLP consisting of $\boldsymbol{\omega}$ and the separated BC

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -k_{21} & k_{11} \end{bmatrix}. \tag{1.34}$$

Here and in the rest of this paper, when a capital Latin letter stands for a matrix, the entrices of the matrix are denoted by the corresponding lower case letter with two indices. Note that $\mu_n(\boldsymbol{\omega}, K) = \mu_n(\boldsymbol{\omega}, -K)$ and $\nu_n(\boldsymbol{\omega}, K) = \nu_n(\boldsymbol{\omega}, -K)$ for any $n \in \mathbb{N}_0$. When $\boldsymbol{\omega}$ is fixed, we sometimes abbreviate $\mu_n(\boldsymbol{\omega}, K)$ and $\nu_n(\boldsymbol{\omega}, K)$ as $\mu_n(K)$ and $\nu_n(K)$, respectively. In the following result from [4] and the rest of this paper, for any integer $k \geq 2$ and any k numbers c_1, c_2, ..., c_k, the notation $\boldsymbol{\{}c_1, c_2, \cdots, c_k\boldsymbol{\}}$ with bold faced braces means each of c_1, c_2, ..., c_k.

Theorem 1.35. *Fix a differential equation in Ω, and let $K \in \mathrm{SL}(2, \mathbb{R})$.*

a) *If $k_{11} > 0$ and $k_{12} \leq 0$, then $\lambda_0(K)$ is simple, and for any $\theta \in (-\pi, \pi)$, $\theta \neq 0$, we have*

$$\begin{aligned} \nu_0(K) \leq \lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) &\leq \boldsymbol{\{}\mu_0(K), \nu_1(K)\boldsymbol{\}} \\ \leq \lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) &\leq \boldsymbol{\{}\mu_1(K), \nu_2(K)\boldsymbol{\}} \\ \leq \lambda_2(K) < \lambda_2(e^{i\theta}K) < \lambda_2(-K) &\leq \boldsymbol{\{}\mu_2(K), \nu_3(K)\boldsymbol{\}} \\ \leq \lambda_3(-K) < \lambda_3(e^{i\theta}K) < \lambda_3(K) &\leq \boldsymbol{\{}\mu_3(K), \nu_4(K)\boldsymbol{\}} \leq \cdots. \end{aligned} \tag{1.36}$$

b) *If $k_{11} \leq 0$ and $k_{12} < 0$, then $\lambda_0(K)$ is simple, and for any $\theta \in (-\pi, \pi)$, $\theta \neq 0$, we have*

$$\begin{aligned} &\lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) \leq \boldsymbol{\{}\mu_0(K), \nu_0(K)\boldsymbol{\}} \leq \\ &\lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) \leq \boldsymbol{\{}\mu_1(K), \nu_1(K)\boldsymbol{\}} \leq \\ &\lambda_2(K) < \lambda_2(e^{i\theta}K) < \lambda_2(-K) \leq \boldsymbol{\{}\mu_2(K), \nu_2(K)\boldsymbol{\}} \leq \\ &\lambda_3(-K) < \lambda_3(e^{i\theta}K) < \lambda_3(K) \leq \boldsymbol{\{}\mu_3(K), \nu_3(K)\boldsymbol{\}} \leq \cdots. \end{aligned} \tag{1.37}$$

c) *If neither Part a) nor Part b) applies to K, then either Part a) or Part b) applies to $-K$.*

For the proof of the principle on the continuity of λ_n we will need the following generalization, to the case of $\Omega \times \mathcal{B}_S^{\mathbb{C}}$, of Theorem 3.32 in [8], which is only for the case of $\mathcal{B}_S^{\mathbb{C}}$.

Theorem 1.38. *Let $(\boldsymbol{\omega}, \boldsymbol{A}) \in \Omega \times \mathcal{B}_S^{\mathbb{C}}$. Assume that r_1 and r_2, $r_1 < r_2$, are any two real numbers such that neither of them is an eigenvalue of $(\boldsymbol{\omega}, \boldsymbol{A})$, and $n \geq 0$ is the number of eigenvalues of $(\boldsymbol{\omega}, \boldsymbol{A})$ in the interval (r_1, r_2). Then there exists a neighborhood $\mathcal{O}$ of $(\boldsymbol{\omega}, \boldsymbol{A})$ in $\Omega \times \mathcal{B}_S^{\mathbb{C}}$ such that any $(\boldsymbol{\sigma}, \boldsymbol{B}) \in \mathcal{O}$ also has exactly n eigenvalues in (r_1, r_2).*

PROOF. The claim on each of the open sets in (1.28)–(1.32) is a direct consequence of the continuous dependence of the solution (as an analytic function of the parameter λ varying in a compact interval) to an initial value problem for a DE in Ω on the problem [13], Rouché's Theorem [3] and Theorem 1.3. ∎

REMARK 1.39. The conclusion of Theorem 1.38 also holds without the assumption that the leading coefficient is positive. Moreover, there are analogous results for general regular SLP's and similar eigenvalue problems of higher order.

To conclude this section, we prove the following theorem, which will be called the *Continuity Principle* in the rest of the paper.

Theorem 1.40. *Let $\mathcal{O}$ be a subset of $\Omega \times \mathcal{B}_S^{\mathbb{C}}$. If λ_0 is uniformly bounded from below on $\mathcal{O}$, then the restrictions of the eigenvalues to $\mathcal{O}$ are all continuous.*

PROOF. Let r_1+1 be a uniform lower bound for the eigenvalues on $\mathcal{O}$, $(\boldsymbol{\omega}, \boldsymbol{A}) \in \mathcal{O}$, and $n \geq 2$ an arbitrary integer such that $\lambda_n(\boldsymbol{\omega}, \boldsymbol{A}) \neq \lambda_{n+1}(\boldsymbol{\omega}, \boldsymbol{A})$. Fix $r_2 \in \big(\lambda_n(\boldsymbol{\omega}, \boldsymbol{A}), \lambda_{n+1}(\boldsymbol{\omega}, \boldsymbol{A})\big)$. By Theorem 1.38, when $(\boldsymbol{\sigma}, \boldsymbol{B}) \in \mathcal{O}$ is sufficiently close to $(\boldsymbol{\omega}, \boldsymbol{A})$, $(\boldsymbol{\sigma}, \boldsymbol{B})$ has exactly $n+1$ eigenvalues in (r_1, r_2). Since $\lambda_0(\boldsymbol{\sigma}, \boldsymbol{B}) > r_1$, these $n+1$ eigenvalues of $(\boldsymbol{\sigma}, \boldsymbol{B})$ are the first $n+1$. By separating the non-equal ones of $\lambda_0(\boldsymbol{\omega}, \boldsymbol{A})$, $\lambda_1(\boldsymbol{\omega}, \boldsymbol{A})$, ..., $\lambda_n(\boldsymbol{\omega}, \boldsymbol{A})$ using small open intervals in (r_1, r_2) and then applying Theorem 1.38 to these open intervals, we see that $\lambda_k(\boldsymbol{\sigma}, \boldsymbol{B})$ is close to $\lambda_k(\boldsymbol{\omega}, \boldsymbol{A})$ for $k = 0$, 1, ..., n when $(\boldsymbol{\sigma}, \boldsymbol{B}) \in \mathcal{O}$ is sufficiently close to $(\boldsymbol{\omega}, \boldsymbol{A})$. Thus, the restrictions of λ_0, λ_1, ..., λ_n to $\mathcal{O}$ are continuous at $(\omega, \boldsymbol{A})$. ∎

By the Continuity Principle, at any point of $\Omega \times \mathcal{B}_S^{\mathbb{C}}$ where one of the λ_n's is discontinuous, λ_0 must approach $-\infty$ in some way.

The proof of the following result is similar to that of Theorem 1.40 and hence is omitted.

Theorem 1.41. *If $\mathcal{O}$ is a subset of $\Omega \times \mathcal{B}_S^{\mathbb{C}}$, $(\boldsymbol{\omega}, \boldsymbol{A}) \notin \mathcal{O}$ is an accumulation point of $\mathcal{O}$,*

$$\lim_{\mathcal{O} \ni (\boldsymbol{\sigma}, \boldsymbol{B}) \to (\boldsymbol{\omega}, \boldsymbol{A})} \lambda_n(\boldsymbol{\sigma}, \boldsymbol{B}) = -\infty \tag{1.42}$$

for $n = 0, 1, \ldots, m$, where $m \in \mathbb{N}_0$, and λ_{m+1} is uniformly bounded from below on $\mathcal{O}$, then

$$\lim_{\mathcal{O} \ni (\boldsymbol{\sigma}, \boldsymbol{B}) \to (\boldsymbol{\omega}, \boldsymbol{A})} \lambda_n(\boldsymbol{\sigma}, \boldsymbol{B}) = \lambda_{n-m-1}(\boldsymbol{\omega}, \boldsymbol{A}) \tag{1.43}$$

for $n = m+1, m+2, m+3, \ldots$.

§2. Continuous Dependence of λ_n on the Differential Equation

In this section, we prove the continuous dependence of λ_n on the DE in the SLP. Our proof given here is based on the concept of continuous eigenvalue branch and the inequalities (1.36), (1.37).

Theorem 2.1. *For any $n \in \mathbb{N}$, the n-th eigenvalue of a regular Sturm-Liouville problem with positive leading coefficient and weight functions and a fixed self-adjoint boundary condition depends* continuously *on the differential equation in the problem.*

Proof. By Proposition 1.19 and Remark 1.20, we only need to show the continuity of λ_n at each DE with a finite interval for $n \in \mathbb{N}_0$.

First, we consider the case where the self-adjoint BC $\boldsymbol{A}$ is a separated one, i.e., $\boldsymbol{A}$ is a separated real BC. Let $\boldsymbol{\omega}_0 = (a_0, b_0, 1/p_0, q_0, w_0) \in \Omega$ with finite a_0 and b_0. Then, $\lambda_0(\boldsymbol{\omega}_0)$ is simple. Consider the continuous eigenvalue branch Λ through $\lambda_0(\boldsymbol{\omega}_0)$ defined on a neighborhood $\mathcal{O}$ of $\boldsymbol{\omega}_0$ in Ω. We can assume that the DE's in $\mathcal{O}$ have finite intervals. Let $\boldsymbol{\omega}_1 = (a_1, b_1, 1/p_1, q_1, w_1) \in \mathcal{O}$. For each $s \in (0,1)$, we define

$$a_s = (1-s)a_0 + s a_1, \quad q_s(t) = (1-s)\widetilde{q_0}(t) + s\widetilde{q_1}(t) \text{ for } t \in (a_s, b_s), \tag{2.2}$$

while b_s, $1/p_s$ and w_s are defined similarly. Then, $\boldsymbol{\omega}_s =: (a_s, b_s, 1/p_s, q_s, w_s) \in \Omega$. Moreover, $\boldsymbol{\omega}_s \in \mathcal{O}$ when $\boldsymbol{\omega}_1$ is sufficiently close to $\boldsymbol{\omega}_0$, which will be assumed. By Theorem 3.2

in [10], there is a normalized eigenfunction $u(\cdot, s)$ for $\Lambda(\boldsymbol{\omega}_s)$, $s \in [0,1]$, such that $u(t,s)$ and $p_s(t)u'(t,s)$ are continuous functions on

$$\{(t,s) \in \mathbb{R}^2;\ s \in [0,1],\, t \in [a_s, b_s]\}. \tag{2.3}$$

Note that $u(\cdot, 0)$ does not have a zero in (a_0, b_0). So, we may assume that $u(t,0) > 0$ on (a_0, b_0). Since the BC is fixed, there exists an $\epsilon > 0$ sufficiently small such that for any $s \in [0,1]$,

$$u(t,s) > 0 \quad \text{on } (a_s, a_s + \epsilon) \cup (b_s - \epsilon, b_s). \tag{2.4}$$

If $u(\cdot, s)$ has a zero in (a_s, b_s) for some $s \in (0,1]$, then the smallest value c of such s exists, we have $p_c u'(t_0, c) = 0$ at each zero t_0 of $u(\cdot, c)$ since $u(t,c) \ge 0$ on $[a_c, b_c]$, and hence $u(\cdot, c) = 0$, which is impossible. Thus, $u(\cdot, 1)$ does not have a zero in (a_1, b_1), i.e., $\Lambda(\boldsymbol{\omega}_1) = \lambda_0(\boldsymbol{\omega}_1)$. Therefore, λ_0 is continuous at $\boldsymbol{\omega}_0 \in \Omega$, and λ_1, λ_2, λ_3, ... are also continuous at $\boldsymbol{\omega}_0 \in \Omega$ by the Continuity Principle.

Next, assume that the self-adjoint BC is the coupled one $\boldsymbol{A} = [e^{i\theta}K \mid -I]$, where $\theta \in [0, \pi)$ and $K \in \mathrm{SL}(2, \mathbb{R})$ with either $k_{11} > 0$, $k_{12} \le 0$ or $k_{11} < 0$, $k_{12} \ge 0$. Then $\nu_0(\boldsymbol{\omega}, K)$ depends continuously on $\boldsymbol{\omega} \in \Omega$ by the proven case. On the other hand, by Part a) of Theorem 1.35,

$$\nu_0(\boldsymbol{\omega}, K) \le \lambda_n(\boldsymbol{\omega}, \boldsymbol{A}) \tag{2.5}$$

for any $\boldsymbol{\omega} \in \Omega$ and $n \in \mathbb{N}_0$. Hence, the eigenvalues for $\boldsymbol{A}$ are uniformly bounded from below near any fixed $\boldsymbol{\omega} \in \Omega$. Therefore, the Continuity Principle implies that for each $n \in \mathbb{N}_0$, $\lambda_n(\boldsymbol{\omega}, \boldsymbol{A})$ depends continuously on $\boldsymbol{\omega} \in \Omega$.

Finally, we consider the case where the self-adjoint BC is the coupled one $\boldsymbol{A} = [e^{i\theta}K \mid -I]$ or $\boldsymbol{B} = [-e^{i\theta}K \mid -I]$ with $\theta \in [0, \pi)$ and $K \in \mathrm{SL}(2, \mathbb{R})$ satisfying $k_{11} \le 0$, $k_{12} < 0$. By Part b) of Theorem 1.35, $\lambda_0(\boldsymbol{\omega}, \boldsymbol{A})$ is simple for any $\boldsymbol{\omega} \in \Omega$. Fix an $\boldsymbol{\omega}_0 \in \Omega$ and consider the continuous simple eigenvalue branch Λ through $\lambda_0(\boldsymbol{\omega}_0, \boldsymbol{A})$ defined on a connected neighborhood $\mathcal{O}$ of $\boldsymbol{\omega}_0$ in Ω. By Part b) of Theorem 1.35 again, $\Lambda(\boldsymbol{\omega}_0) = \lambda_0(\boldsymbol{\omega}_0, \boldsymbol{A}) < \nu_0(\boldsymbol{\omega}_0, K)$ and $\Lambda(\boldsymbol{\omega}) \ne \nu_0(\boldsymbol{\omega}, K)$ for any $\boldsymbol{\omega} \in \mathcal{O}$. Hence, we have $\Lambda(\boldsymbol{\omega}) < \nu_0(\boldsymbol{\omega}, K)$ for any $\boldsymbol{\omega} \in \mathcal{O}$, since both Λ and ν_0 are continuous functions on $\mathcal{O}$. Therefore, $\lambda_0(\cdot, \boldsymbol{A}) = \Lambda$ on $\mathcal{O}$ still by Part b) of Theorem 1.35, i.e., $\lambda_0(\cdot, \boldsymbol{A})$ is continuous on $\mathcal{O}$. Moreover, $\lambda_1(\cdot, \boldsymbol{A})$, $\lambda_2(\cdot, \boldsymbol{A})$, $\lambda_3(\cdot, \boldsymbol{A})$, ... and $\lambda_0(\cdot, \boldsymbol{B})$, $\lambda_1(\cdot, \boldsymbol{B})$, $\lambda_2(\cdot, \boldsymbol{B})$, ... are continuous at $\boldsymbol{\omega}_0$ by Part b) of Theorem 1.35 and the Continuity Principle. ∎

§3. Discontinuity of λ_n

In this section, we characterize the discontinuity set of λ_n as a function on $\Omega \times \mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ or $\Omega \times \mathcal{B}_{\mathrm{S}}^{\mathbb{C}}$ and determine the behavior of λ_n near each discontinuity point.

Firstly, let us fix a differential equation $(a, b, 1/p, q, w) \in \Omega$ and characterize the discontinuity of λ_n as a function on $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ or $\mathcal{B}_{\mathrm{S}}^{\mathbb{C}}$. The following result is a consequence of Theorem 1.6.

Lemma 3.1. *For any two positive constants c and ϵ, there exists a λ_* such that for any $\lambda \le \lambda_*$,*

$$\phi_{11}(b,\lambda) \ge c, \quad \phi_{12}(b,\lambda) > 0, \quad \phi_{21}(b,\lambda) \ge c, \quad \phi_{22}(b,\lambda) > 0, \tag{3.2}$$

$$\frac{\phi_{11}(b,\lambda)}{\phi_{21}(b,\lambda)} \le \epsilon, \quad \frac{\phi_{12}(b,\lambda)}{\phi_{11}(b,\lambda)} \le \epsilon, \quad \frac{\phi_{12}(b,\lambda)}{\phi_{21}(b,\lambda)} \le \epsilon, \quad \frac{\phi_{22}(b,\lambda)}{\phi_{21}(b,\lambda)} \le \epsilon. \tag{3.3}$$

Proof. This is a direct consequence of Theorem 1.6 together with the fact

$$\lim_{\lambda \to -\infty} \int_a^b \frac{\operatorname{sech}^2(\alpha(s,\lambda))}{p(s)} \mathrm{d}s = 0 \tag{3.4}$$

deduced from it and the Bounded Convergence Theorem. ∎

The following are some continuity results about λ_n on $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$. In this context, we will use the notation

$$\mathcal{F}_-^{\mathbb{R}} = \{[K \mid -I]; \quad K \in \mathrm{SL}(2,\mathbb{R}),\ k_{11}k_{12} \le 0\}, \tag{3.5}$$

$$\mathcal{G}_-^{\mathbb{R}} = \left\{ \begin{bmatrix} a_1 & 1 & 0 & -r \\ r & 0 & -1 & b_2 \end{bmatrix}; \quad b_2 \le 0,\ a_1, r \in \mathbb{R} \right\}, \tag{3.6}$$

$$\mathcal{H}_-^{\mathbb{R}} = \left\{ \begin{bmatrix} 1 & a_2 & -r & 0 \\ 0 & r & b_1 & -1 \end{bmatrix}; \quad a_2 \le 0,\ b_1, r \in \mathbb{R} \right\}, \tag{3.7}$$

$$\mathcal{I}_-^{\mathbb{R}} = \left\{ \begin{bmatrix} 1 & a_2 & 0 & r \\ 0 & r & -1 & b_2 \end{bmatrix}; \quad a_2, b_2 \le 0,\ r \in \mathbb{R},\ a_2 b_2 \ge r^2 \right\}, \tag{3.8}$$

$$\begin{aligned} \mathcal{J}^{\mathbb{R}} = & \{[K \mid -I]; \quad K \in \mathrm{SL}(2,\mathbb{R}),\ k_{12} = 0\} \\ & \bigcup \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \end{bmatrix} \in \mathcal{B}_{\mathrm{S}}^{\mathbb{R}}; \quad a_2 b_2 = 0 \right\}. \end{aligned} \tag{3.9}$$

Proposition 3.10. *Let $n \in \mathbb{N}_0$. Then, as a function on the space $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ of self-adjoint real boundary conditions, λ_n is continuous at each point not in $\mathcal{J}^{\mathbb{R}}$.*

PROOF. First, fix $\boldsymbol{K} = [K \mid -I] \in \mathcal{B}_{\mathrm{S}}^{\mathbb{R}} \setminus \mathcal{J}^{\mathbb{R}}$. Set

$$c = 2\max\big\{|k_{11}|, |k_{21}|, |k_{22}|\big\}, \qquad d = |k_{12}|. \tag{3.11}$$

Then, $c, d > 0$. By Lemma 3.1, there is a λ_* such that for any $\lambda \le \lambda_*$,

$$\phi_{21}(\lambda) \ge \frac{12}{d}, \quad \frac{|\phi_{11}(\lambda)|}{\phi_{21}(\lambda)} \le \frac{d}{12c}, \quad \frac{|\phi_{12}(\lambda)|}{\phi_{21}(\lambda)} \le \frac{d}{12c}, \quad \frac{|\phi_{22}(\lambda)|}{\phi_{21}(\lambda)} \le \frac{d}{12c}. \tag{3.12}$$

If $\boldsymbol{L} = [L \mid -I] \in \mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ satisfies

$$|l_{12}| > \frac{d}{2} \quad \text{and} \quad \big\{|l_{11}|, |l_{21}|, |l_{22}|\big\} < c, \tag{3.13}$$

then for any $\lambda \le \lambda_*$,

$$\begin{aligned} \big|\Delta_{\boldsymbol{L}}(\lambda)\big| &= \big|2 - l_{22}\phi_{11}(\lambda) + l_{21}\phi_{12}(\lambda) + l_{12}\phi_{21}(\lambda) - l_{11}\phi_{22}(\lambda)\big| \\ &\ge \Big(|l_{12}| - \frac{|l_{22}\phi_{11}(\lambda) + l_{21}\phi_{12}(\lambda) + l_{11}\phi_{22}(\lambda)|}{\phi_{21}(\lambda)}\Big)\phi_{21}(\lambda) - 2 \\ &\ge 1. \end{aligned} \tag{3.14}$$

Thus, by Theorem 1.3, $\lambda_0(\boldsymbol{L}) \ge \lambda_*$ for any $\boldsymbol{L} \in \mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ sufficiently close to $\boldsymbol{K}$. The Continuity Principle then assures that λ_n is continuous at $\boldsymbol{K}$.

Next, let us consider

$$\boldsymbol{A} = \begin{bmatrix} a_1 & 1 & 0 & 0 \\ 0 & 0 & b_1 & -1 \end{bmatrix} \in \mathcal{B}_{\mathrm{S}}^{\mathbb{R}} \setminus \mathcal{J}^{\mathbb{R}}. \tag{3.15}$$

Then $\boldsymbol{A}$ has a neighborhood in $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ given by (1.27) and for any

$$\boldsymbol{B} = \begin{bmatrix} c & 1 & r & 0 \\ r & 0 & d & -1 \end{bmatrix} \tag{3.16}$$

in that neighborhood,

$$\Delta_{\boldsymbol{B}}(\lambda) = -2r - d\phi_{11}(\lambda) + (cd - r^2)\phi_{12}(\lambda) + \phi_{21}(\lambda) - c\phi_{22}(\lambda). \tag{3.17}$$

Thus, as in the previous case, we see that λ_n is continuous at $\boldsymbol{A}$. ∎

Proposition 3.18. *For every $n \in \mathbb{N}_0$, the restriction of λ_n to each of $\mathcal{F}_-^{\mathbb{R}}$, $\mathcal{G}_-^{\mathbb{R}}$, $\mathcal{H}_-^{\mathbb{R}}$ and $\mathcal{I}_-^{\mathbb{R}}$ is continuous.*

PROOF. First, fix $\boldsymbol{K} = [K \mid -I] \in \mathcal{F}_-^{\mathbb{R}}$. If $k_{12} \neq 0$, then λ_n is continuous at $\boldsymbol{K}$ by Proposition 3.10. Assume that $k_{12} = 0$ and $k_{11} > 0$. Set

$$c = 2\max\big\{k_{11}, |k_{21}|, |k_{22}|\big\}. \tag{3.19}$$

Then, $c > 0$. By Lemma 3.1, there is a λ_* such that for any $\lambda \leq \lambda_*$,

$$\phi_{11}(\lambda) \geq 12c, \quad \frac{|\phi_{12}(\lambda)|}{\phi_{11}(\lambda)} \leq \frac{1}{4c^2}, \quad \phi_{21}(\lambda) > 0, \quad \phi_{22}(\lambda) > 0. \tag{3.20}$$

When $\boldsymbol{L} = [L \mid -I] \in \mathcal{F}_-^{\mathbb{R}}$ is sufficiently close to $\boldsymbol{K}$, we have

$$l_{11} > 0, \quad -\frac{1}{2c} < l_{12} \leq 0, \quad \big\{l_{11}, |l_{21}|, |l_{22}|\big\} < c, \tag{3.21}$$

and hence

$$l_{12}l_{21} \geq c\,l_{12} > -\frac{1}{2}, \qquad l_{11}l_{22} = 1 + l_{12}l_{21} > \frac{1}{2}, \qquad l_{22} > \frac{1}{2l_{11}} > \frac{1}{2c}. \tag{3.22}$$

Thus, for such an $\boldsymbol{L}$ and any $\lambda \leq \lambda_*$,

$$\begin{aligned} \Delta_{\boldsymbol{L}}(\lambda) &= 2 - l_{22}\phi_{11}(\lambda) + l_{21}\phi_{12}(\lambda) + l_{12}\phi_{21}(\lambda) - l_{11}\phi_{22}(\lambda) \\ &\leq 2 + \Big(\frac{|l_{21}\phi_{12}(\lambda)|}{\phi_{11}(\lambda)} - l_{22}\Big)\phi_{11}(\lambda) < -1, \end{aligned} \tag{3.23}$$

which and Theorem 1.3 imply that $\lambda_0(\boldsymbol{L}) > \lambda_*$. Therefore, the Continuity Principle assures that $\lambda_n\big|_{\mathcal{F}_-^{\mathbb{R}}}$ is continuous at $\boldsymbol{K}$. The case where $k_{12} = 0$ and $k_{11} < 0$ can be handled similarly.

Next, let us consider

$$\boldsymbol{A} = \begin{bmatrix} a_1 & 1 & 0 & -r \\ r & 0 & -1 & b_2 \end{bmatrix} \in \mathcal{G}_-^{\mathbb{R}}. \tag{3.24}$$

If $r \neq 0$ and $b_2 \neq 0$, then

$$\boldsymbol{A} = \begin{bmatrix} \frac{a_1 b_2 + r^2}{r} & \frac{b_2}{r} & -1 & 0 \\ \frac{a_1}{r} & \frac{1}{r} & 0 & -1 \end{bmatrix} \tag{3.25}$$

and λ_n is continuous at $\boldsymbol{A}$ by Proposition 3.10; if $r \neq 0$ and $b_2 = 0$, then $\lambda_n\big|_{\mathcal{G}_-^{\mathbb{R}}}$ is continuous at $\boldsymbol{A}$ by the proven case; if $r = 0$ and $b_2 \neq 0$, then λ_n is continuous at $\boldsymbol{A}$ also by Proposition

3.10. So, we now assume that $r = 0$ and $b_2 = 0$. By Lemma 3.1, there is a λ_* such that for any $\lambda \le \lambda_*$,

$$\phi_{11}(\lambda) > 6, \quad \phi_{21}(\lambda) > 0, \quad \phi_{22}(\lambda) > 0, \tag{3.26}$$

$$\frac{|\phi_{12}(\lambda)|}{\phi_{11}(\lambda)} \le \frac{1}{4\max\{|a_1|, 1\}}, \quad \frac{\phi_{22}(\lambda)}{\phi_{21}(\lambda)} \le \frac{1}{2\max\{|a_1|, 1\}}. \tag{3.27}$$

When

$$\boldsymbol{B} = \begin{bmatrix} c & 1 & 0 & -s \\ s & 0 & -1 & d \end{bmatrix} \in \mathcal{G}_-^{\mathbb{R}} \tag{3.28}$$

is sufficiently close to $\boldsymbol{A}$, we have

$$|c| < 2\max\{|a_1|, 1\}, \quad |s| < 1, \quad d \le 0. \tag{3.29}$$

Thus, for such a $\boldsymbol{B}$ and any $\lambda \le \lambda_*$,

$$\begin{aligned} \Delta_{\boldsymbol{B}}(\lambda) &= \phi_{11}(\lambda) - c\phi_{12}(\lambda) - d\phi_{21}(\lambda) + (s^2 + cd)\phi_{22}(\lambda) - 2s \\ &\ge \Big(1 - \frac{|c\phi_{12}(\lambda)|}{\phi_{11}(\lambda)}\Big)\phi_{11}(\lambda) - d\big(\phi_{21}(\lambda) - |c\phi_{22}(\lambda)|\big) - 2 \\ &\ge \frac{1}{2}\phi_{11}(\lambda) - 2 > 1, \end{aligned} \tag{3.30}$$

which and Theorem 1.3 imply that $\lambda_0(\boldsymbol{B}) > \lambda_*$. Therefore, the Continuity Principle assures that $\lambda_n\big|_{\mathcal{G}_-^{\mathbb{R}}}$ is continuous at $\boldsymbol{B}$.

Finally, the continuity of $\lambda_n\big|_{\mathcal{H}_-^{\mathbb{R}}}$ and $\lambda_n\big|_{\mathcal{I}_-^{\mathbb{R}}}$ can be proved by similar arguments. ∎

For each $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$, let

$$\boldsymbol{S}_{\alpha,\beta} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \end{bmatrix}. \tag{3.31}$$

Then, the set $\mathcal{T}$ of separated real BC's consists of these $\boldsymbol{S}_{\alpha,\beta}$'s and is topologically a torus. The following result is part of the theorem in [5], which is proved using some derivative formulas in [10] and the Prüfer transformation.

Lemma 3.32. *As a function of (α, β), $\lambda_n(\boldsymbol{S}_{\alpha,\beta})$ is continuous on $[0, \pi) \times (0, \pi]$, strictly decreasing in α, and strictly increasing in β. Moreover, for each $\alpha \in [0, \pi)$,*

$$\lim_{\beta \to 0^+} \lambda_0(\boldsymbol{S}_{\alpha,\beta}) = -\infty, \qquad \lim_{\beta \to 0^+} \lambda_n(\boldsymbol{S}_{\alpha,\beta}) = \lambda_{n-1}(\boldsymbol{S}_{\alpha,\pi}) \text{ for } n \in \mathbb{N}, \tag{3.33}$$

and for each $\beta \in (0, \pi]$,

$$\lim_{\alpha \to \pi^-} \lambda_0(\boldsymbol{S}_{\alpha,\beta}) = -\infty, \qquad \lim_{\alpha \to \pi^-} \lambda_n(\boldsymbol{S}_{\alpha,\beta}) = \lambda_{n-1}(\boldsymbol{S}_{0,\beta}) \textit{ for } n \in \mathbb{N}. \tag{3.34}$$

Note that the continuity claim in Lemma 3.32 is a consequence of Propositions 3.10 and 3.18.

In order to describe the discontinuity of λ_n on $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$, we let

$$\mathcal{F}_+^{\mathbb{R}} = \mathcal{O}_{6,\mathrm{S}}^{\mathbb{R}} \setminus \mathcal{F}_-^{\mathbb{R}}, \qquad \mathcal{G}_+^{\mathbb{R}} = \mathcal{O}_{4,\mathrm{S}}^{\mathbb{R}} \setminus \mathcal{G}_-^{\mathbb{R}}, \qquad \mathcal{H}_+^{\mathbb{R}} = \mathcal{O}_{3,\mathrm{S}}^{\mathbb{R}} \setminus \mathcal{H}_-^{\mathbb{R}}, \tag{3.35}$$

$$\mathcal{I}_+^{\mathbb{R}} = \left\{ \begin{bmatrix} 1 & a_2 & 0 & r \\ 0 & r & -1 & b_2 \end{bmatrix}; \quad a_2, b_2 > 0,\ r \in \mathbb{R},\ a_2 b_2 > r^2 \right\}, \tag{3.36}$$

$$\mathcal{I}_0^{\mathbb{R}} = \mathcal{O}_{2,\mathrm{S}}^{\mathbb{R}} \setminus (\mathcal{I}_-^{\mathbb{R}} \cup \mathcal{I}_+^{\mathbb{R}}). \tag{3.37}$$

Note that the coupled BC's in $\mathcal{J}^{\mathbb{R}}$ are all in $\mathcal{F}_-^{\mathbb{R}}$, and

$$\mathcal{J}^{\mathbb{R}} \cap \mathcal{T} = (\mathcal{J}^{\mathbb{R}} \cap \mathcal{G}_-^{\mathbb{R}}) \cup (\mathcal{J}^{\mathbb{R}} \cap \mathcal{H}_-^{\mathbb{R}}) \cup \{\boldsymbol{D}\}, \tag{3.38}$$

where $\boldsymbol{D}$ is the Dirichlet BC.

Theorem 3.39. *The function* λ_0 *on* $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ *is continuous on* $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}} \setminus \mathcal{J}^{\mathbb{R}}$ *and discontinuous at each point of* $\mathcal{J}^{\mathbb{R}}$. *For* $n \in \mathbb{N}$, *the function* λ_n *is continuous on* $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}} \setminus \mathcal{J}^{\mathbb{R}}$ *and at each coupled boundary condition in* $\mathcal{J}^{\mathbb{R}}$ *where* $\lambda_n = \lambda_{n-1}$ *and discontinuous at any other point of* $\mathcal{J}^{\mathbb{R}}$. *More precisely, for each coupled boundary condition* $\boldsymbol{A} \in \mathcal{J}^{\mathbb{R}}$, *the restriction of* λ_n *to* $\mathcal{F}_-^{\mathbb{R}}$ *is continuous at* $\boldsymbol{A}$ *for* $n \in \mathbb{N}_0$ *and*

$$\lim_{\mathcal{F}_+^{\mathbb{R}} \ni \boldsymbol{B} \to \boldsymbol{A}} \lambda_0(\boldsymbol{B}) = -\infty, \qquad \lim_{\mathcal{F}_+^{\mathbb{R}} \ni \boldsymbol{B} \to \boldsymbol{A}} \lambda_n(\boldsymbol{B}) = \lambda_{n-1}(\boldsymbol{A}) \textit{ for } n \in \mathbb{N}; \tag{3.40}$$

for each $\boldsymbol{A} \in \mathcal{J}^{\mathbb{R}} \cap \mathcal{G}_-^{\mathbb{R}}$, *the restriction of* λ_n *to* $\mathcal{G}_-^{\mathbb{R}}$ *is continuous at* $\boldsymbol{A}$ *for* $n \in \mathbb{N}_0$ *and*

$$\lim_{\mathcal{G}_+^{\mathbb{R}} \ni \boldsymbol{B} \to \boldsymbol{A}} \lambda_0(\boldsymbol{B}) = -\infty, \qquad \lim_{\mathcal{G}_+^{\mathbb{R}} \ni \boldsymbol{B} \to \boldsymbol{A}} \lambda_n(\boldsymbol{B}) = \lambda_{n-1}(\boldsymbol{A}) \textit{ for } n \in \mathbb{N}; \tag{3.41}$$

for each $\boldsymbol{A} \in \mathcal{J}^{\mathbb{R}} \cap \mathcal{H}_-^{\mathbb{R}}$, *the restriction of* λ_n *to* $\mathcal{H}_-^{\mathbb{R}}$ *is continuous at* $\boldsymbol{A}$ *for* $n \in \mathbb{N}_0$ *and*

$$\lim_{\mathcal{H}_+^{\mathbb{R}} \ni \boldsymbol{B} \to \boldsymbol{A}} \lambda_0(\boldsymbol{B}) = -\infty, \qquad \lim_{\mathcal{H}_+^{\mathbb{R}} \ni \boldsymbol{B} \to \boldsymbol{A}} \lambda_n(\boldsymbol{B}) = \lambda_{n-1}(\boldsymbol{A}) \textit{ for } n \in \mathbb{N}; \tag{3.42}$$

while the restriction of λ_n *to* $\mathcal{I}_-^{\mathbb{R}}$ *is continuous at the Dirichlet boundary condition* $\boldsymbol{D}$ *for* $n \in \mathbb{N}_0$ *and*

$$\lim_{\mathcal{I}_0^{\mathbb{R}} \cup \mathcal{I}_+^{\mathbb{R}} \ni \boldsymbol{B} \to \boldsymbol{D}} \lambda_0(\boldsymbol{B}) = \lim_{\mathcal{I}_+^{\mathbb{R}} \ni \boldsymbol{B} \to \boldsymbol{D}} \lambda_1(\boldsymbol{B}) = -\infty, \tag{3.43}$$

$$\lim_{\mathcal{I}_0^{\mathbb{R}} \ni \boldsymbol{B} \to \boldsymbol{D}} \lambda_n(\boldsymbol{B}) = \lambda_{n-1}(\boldsymbol{D}) \text{ for } n \in \mathbb{N}, \tag{3.44}$$

$$\lim_{\mathcal{I}_+^{\mathbb{R}} \ni \boldsymbol{B} \to \boldsymbol{D}} \lambda_n(\boldsymbol{B}) = \lambda_{n-2}(\boldsymbol{D}) \text{ for } n \geq 2. \tag{3.45}$$

PROOF. By Theorem 4.12 in [8], the eigenvalues for a separated real BC are all simple. Thus, by Propositions 3.10 and 3.18, we only need to prove (3.40)–(3.45).

Fix a $K \in \mathrm{SL}(2, \mathbb{R})$ with $k_{11} > 0$ and $k_{12} = 0$. When $\boldsymbol{L} = [L \mid -I] \in \mathcal{F}_+^{\mathbb{R}}$ is sufficiently close to $\boldsymbol{K} = [K \mid -I] \in \mathcal{J}^{\mathbb{R}}$, we have $l_{11} > 0$ and $l_{12} > 0$. Part b) of Theorem 1.35 implies

$$\lambda_0(L) \leq \{\mu_0(L), \nu_0(L)\} \leq \lambda_1(L), \tag{3.46}$$

where $\mu_0(L)$ and $\nu_0(L)$ are the first eigenvalues for the separated BC's

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & l_{22} & -l_{12} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -l_{21} & l_{11} \end{bmatrix}, \tag{3.47}$$

respectively. By Lemma 3.32, $\mu_0(L) \to -\infty$ and $\nu_0(L) \to \nu_0(K)$ as $\boldsymbol{L}$ in $\mathcal{F}_+^{\mathbb{R}}$ approaches $\boldsymbol{K}$, since then $l_{12} \to 0^+$, $l_{22} \to k_{22} > 0$ and $l_{11} \to k_{11} > 0$. Thus,

$$\lim_{\mathcal{F}_+^{\mathbb{R}} \ni \boldsymbol{L} \to \boldsymbol{K}} \lambda_0(\boldsymbol{L}) = -\infty, \qquad \lim_{\mathcal{F}_+^{\mathbb{R}} \ni \boldsymbol{L} \to \boldsymbol{K}} \lambda_n(\boldsymbol{L}) = \lambda_{n-1}(\boldsymbol{K}) \text{ for } n \in \mathbb{N} \tag{3.48}$$

by Theorem 1.41. Similarly, we prove (3.40) for $K \in \mathrm{SL}(2, \mathbb{R})$ with $k_{11} < 0$ and $k_{12} = 0$.

Next, let

$$\boldsymbol{A} = \begin{bmatrix} a_1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \in \mathcal{J}^{\mathbb{R}} \cap \mathcal{G}_-^{\mathbb{R}}. \tag{3.49}$$

By Theorem 4.12 in [8], the eigenvalues for $\boldsymbol{A}$ are all simple. Fix an integer $m \geq 2$. Then, there is a neighborhood $\mathcal{O}$ of $\boldsymbol{A}$ in $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ such that $\mathcal{O} \cap \mathcal{G}_-^{\mathbb{R}}$ and $\mathcal{O} \cap \mathcal{G}_+^{\mathbb{R}}$ are connected, and the continuous simple eigenvalue branches Λ_0, Λ_1, ..., Λ_m through $\lambda_0(\boldsymbol{A})$, $\lambda_1(\boldsymbol{A})$, ..., $\lambda_m(\boldsymbol{A})$ are defined on $\mathcal{O}$. By Proposition 3.18, the restriction of each λ_n to $\mathcal{G}_-^{\mathbb{R}}$ is continuous, which and the simplity of Λ_n imply that if $\boldsymbol{B} \in \mathcal{O} \cap \mathcal{G}_-^{\mathbb{R}}$, then

$$\Lambda_n(\boldsymbol{B}) = \lambda_n(\boldsymbol{B}) \tag{3.50}$$

for $n = 0, 1, \ldots, m$. By Proposition 3.10, each λ_n is continuous on $\mathcal{G}_+^{\mathbb{R}}$. There exist $r > 0$ and $d > 0$ such that

$$(3.51) \qquad a_1 s + r^2 > 0, \qquad \boldsymbol{B}_s =: \begin{bmatrix} a_1 & 1 & 0 & -r \\ r & 0 & -1 & s \end{bmatrix} \in \mathcal{O}$$

for $s \in [0, d]$. Since

$$(3.52) \qquad \boldsymbol{B}_s = \begin{bmatrix} \frac{a_1 s + r^2}{r} & \frac{s}{r} & -1 & 0 \\ \frac{a_1}{r} & \frac{1}{r} & 0 & -1 \end{bmatrix}$$

for $s \in [0, d]$, the element $\boldsymbol{B}_0$ of $\mathcal{J}^{\mathbb{R}} \cap \mathcal{G}_-^{\mathbb{R}}$ is also in $\mathcal{J}^{\mathbb{R}} \cap \mathcal{F}_-^{\mathbb{R}}$, and the element $\boldsymbol{B}_s$ of $\mathcal{G}_+^{\mathbb{R}}$ is also in $\mathcal{F}_+^{\mathbb{R}}$ for $s \in (0, d]$. Thus, from (3.50) with $\boldsymbol{B} = \boldsymbol{B}_0$, the proven case, the simplity of Λ_n and the continuity of λ_{n+1} on $\mathcal{G}_+^{\mathbb{R}}$ one deduces that $\Lambda_n(\boldsymbol{B}_s) = \lambda_{n+1}(\boldsymbol{B}_s)$ for $n = 0, 1, \ldots, m$ and $s \in (0, d]$. The simplity of Λ_n and the continuity of λ_{n+1} on $\mathcal{G}_+^{\mathbb{R}}$ then imply that

$$(3.53) \qquad \Lambda_n(\boldsymbol{B}) = \lambda_{n+1}(\boldsymbol{B})$$

for $n = 0, 1, \ldots, m$ and $\boldsymbol{B} \in \mathcal{O} \cap \mathcal{G}_+^{\mathbb{R}}$. Therefore, we have

$$(3.54) \qquad \lim_{\mathcal{G}_+^{\mathbb{R}} \ni \boldsymbol{B} \to \boldsymbol{A}} \lambda_n(\boldsymbol{B}) = \lambda_{n-1}(\boldsymbol{A})$$

for $n = 1, 2, \ldots, m + 1$, which and the simplicity of Λ_0 imply the other limit in (3.41). Similarly, one proves (3.42).

Finally, by Theorem 4.12 in [8], the eigenvalues for $\boldsymbol{D}$ are all simple. Fix an integer $m \geq 2$. Then, there is a neighborhood $\mathcal{O}$ of $\boldsymbol{D}$ in $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ such that $\mathcal{O} \cap \mathcal{I}_-^{\mathbb{R}}$, $\mathcal{O} \cap \mathcal{I}_0^{\mathbb{R}}$ and $\mathcal{O} \cap \mathcal{I}_+^{\mathbb{R}}$ are connected, and the continuous simple eigenvalue branches $\Lambda_0, \Lambda_1, \ldots, \Lambda_m$ through $\lambda_0(\boldsymbol{D})$, $\lambda_1(\boldsymbol{D}), \ldots, \lambda_m(\boldsymbol{D})$ are defined on $\mathcal{O}$. By Proposition 3.18, if $\boldsymbol{B} \in \mathcal{O} \cap \mathcal{I}_-^{\mathbb{R}}$, then

$$(3.55) \qquad \Lambda_n(\boldsymbol{B}) = \lambda_n(\boldsymbol{B})$$

for $n = 0, 1, \ldots, m$. For $n \in \mathbb{N}_0$, Proposition 3.10 implies that $\lambda_n\big|_{\mathcal{I}_0^{\mathbb{R}} \setminus \mathcal{J}^{\mathbb{R}}}$ and $\lambda_n\big|_{\mathcal{I}_+^{\mathbb{R}}}$ are continuous. By the definitions (3.8), (3.9), (3.36) and (3.37),

$$(3.56) \qquad \mathcal{I}_0^{\mathbb{R}} \setminus \mathcal{J}^{\mathbb{R}} = \left\{ \begin{bmatrix} 1 & a_2 & 0 & r \\ 0 & r & -1 & b_2 \end{bmatrix}; \quad a_2, b_2, r \in \mathbb{R}, \ a_2 b_2 < r^2 \right\},$$

$$(3.57) \qquad \mathcal{I}_0^{\mathbb{R}} \cap \mathcal{J}^{\mathbb{R}} = \left\{ \begin{bmatrix} 1 & a_2 & 0 & r \\ 0 & r & -1 & b_2 \end{bmatrix}; \quad \begin{matrix} a_2, b_2 \geq 0, \ a_2^2 + b_2^2 > 0 \\ r \in \mathbb{R}, \ a_2 b_2 = r^2 \end{matrix} \right\}.$$

If

$$\boldsymbol{B} = \begin{bmatrix} 1 & a_2 & 0 & r \\ 0 & r & -1 & b_2 \end{bmatrix} \in \mathcal{O}^{\mathbb{R}}_{2,\mathrm{S}} \tag{3.58}$$

satisfies $a_2 > 0$, then

$$\boldsymbol{B} = \begin{bmatrix} \frac{1}{a_2} & 1 & 0 & \frac{r}{a_2} \\ -\frac{r}{a_2} & 0 & -1 & \frac{a_2 b_2 - r^2}{a_2} \end{bmatrix}; \tag{3.59}$$

if $\boldsymbol{B}$ given by (3.58) satisfies $b_2 > 0$, then

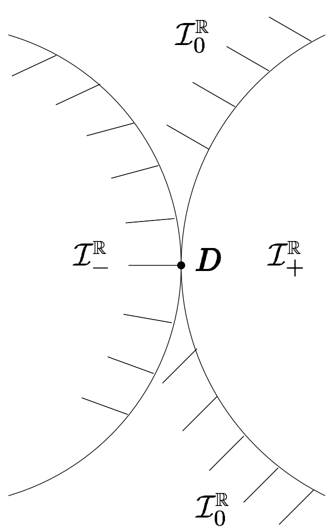
$$\boldsymbol{B} = \begin{bmatrix} 1 & \frac{a_2 b_2 - r^2}{b_2} & \frac{r}{b_2} & 0 \\ 0 & -\frac{r}{b_2} & \frac{1}{b_2} & -1 \end{bmatrix}. \tag{3.60}$$

Thus, for $n \in \mathbb{N}_0$, $\lambda_n\big|_{\mathcal{I}_0^{\mathbb{R}}}$ is continuous at each point of $\mathcal{I}_0^{\mathbb{R}} \cap \mathcal{J}^{\mathbb{R}}$ by Proposition 3.18. Hence, from the proven cases we deduce the following: if $\boldsymbol{B} \in \mathcal{O} \cap \mathcal{I}_0^{\mathbb{R}}$, then

$$\Lambda_n(\boldsymbol{B}) = \lambda_{n+1}(\boldsymbol{B}) \text{ for } n = 0, 1, ..., m; \tag{3.61}$$

if $\boldsymbol{B} \in \mathcal{O} \cap \mathcal{I}_+^{\mathbb{R}}$, then

$$\Lambda_n(\boldsymbol{B}) = \lambda_{n+2}(\boldsymbol{B}) \text{ for } n = 0, 1, ..., m. \tag{3.62}$$



Therefore, we have proven (3.44) for $n = 1, 2, ..., m+1$ and (3.45) for $n = 2, 3, ..., m+2$, which and the simplicity of Λ_0 imply (3.43). ∎

In order to to describe the discontinuity of λ_n as a function on $\mathcal{B}^{\mathbb{C}}_{\rm S}$, we set

$$\mathcal{F}^{\mathbb{C}}_{-} = \big\{[e^{i\theta}K \mid -I]; \quad K \in \mathrm{SL}(2,\mathbb{R}),\ k_{11}k_{12} \le 0,\ \theta \in [0,\pi)\big\}, \tag{3.63}$$

$$\mathcal{G}^{\mathbb{C}}_{-} = \left\{\begin{bmatrix} a_1 & 1 & 0 & -z \\ z & 0 & -1 & b_2 \end{bmatrix}; \quad b_2 \le 0,\ a_1 \in \mathbb{R},\ z \in \mathbb{C}\right\}, \tag{3.64}$$

$$\mathcal{H}^{\mathbb{C}}_{-} = \left\{\begin{bmatrix} 1 & a_2 & -r & 0 \\ 0 & r & b_1 & -1 \end{bmatrix}; \quad a_2 \le 0,\ b_1 \in \mathbb{R},\ z \in \mathbb{C}\right\}, \tag{3.65}$$

$$\mathcal{F}^{\mathbb{C}}_{+} = \mathcal{O}^{\mathbb{C}}_{6,\rm S} \setminus \mathcal{F}^{\mathbb{C}}_{-}, \qquad \mathcal{G}^{\mathbb{C}}_{+} = \mathcal{O}^{\mathbb{C}}_{4,\rm S} \setminus \mathcal{G}^{\mathbb{C}}_{-}, \qquad \mathcal{H}^{\mathbb{C}}_{+} = \mathcal{O}^{\mathbb{C}}_{3,\rm S} \setminus \mathcal{H}^{\mathbb{C}}_{-}, \tag{3.66}$$

$$\mathcal{I}^{\mathbb{C}}_{-} = \left\{\begin{bmatrix} 1 & a_2 & 0 & z \\ 0 & z & -1 & b_2 \end{bmatrix}; \quad a_2, b_2 \le 0,\ z \in \mathbb{C},\ a_2b_2 \ge z\bar{z}\right\}, \tag{3.67}$$

$$\mathcal{I}^{\mathbb{C}}_{+} = \left\{\begin{bmatrix} 1 & a_2 & 0 & z \\ 0 & z & -1 & b_2 \end{bmatrix}; \quad a_2, b_2 > 0,\ z \in \mathbb{C},\ a_2b_2 > z\bar{z}\right\}, \tag{3.68}$$

$$\mathcal{I}^{\mathbb{C}}_{0} = \mathcal{O}^{\mathbb{C}}_{2,\rm S} \setminus (\mathcal{I}^{\mathbb{C}}_{-} \cup \mathcal{I}^{\mathbb{C}}_{+}), \tag{3.69}$$

$$\begin{aligned} \mathcal{J}^{\mathbb{C}} = &\big\{[e^{i\theta}K \mid -I]; \quad K \in \mathrm{SL}(2,\mathbb{R}),\ k_{12} = 0,\ \theta \in [0,\pi)\big\} \\ &\bigcup \left\{\begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \end{bmatrix} \in \mathcal{B}^{\mathbb{R}}_{\rm S}; \quad a_2b_2 = 0\right\}. \end{aligned} \tag{3.70}$$

Note that the separated BC's in $\mathcal{J}^{\mathbb{C}}$ other than the Dirichlet BC are in $\mathcal{G}^{\mathbb{C}}_{-} \cup \mathcal{H}^{\mathbb{C}}_{-}$. The proofs of the following results are similar to those of Propositions 3.10, 3.18 and Theorem 3.39, so we omit them.

Proposition 3.71. *Let $n \in \mathbb{N}_0$. Then, as a function on the space $\mathcal{B}^{\mathbb{C}}_{\rm S}$ of self-adjoint complex boundary conditions, λ_n is continuous at each point not in $\mathcal{J}^{\mathbb{C}}$.*

Proposition 3.72. *For every $n \in \mathbb{N}_0$, the restriction of λ_n to each of $\mathcal{F}^{\mathbb{C}}_{-}$, $\mathcal{G}^{\mathbb{C}}_{-}$, $\mathcal{H}^{\mathbb{C}}_{-}$ and $\mathcal{I}^{\mathbb{C}}_{-}$ is continuous.*

Theorem 3.73. *The conclusions of Theorem 3.39 still hold when the super indices $\mathbb{R}$ in them are replaced by $\mathbb{C}$.*

REMARK 3.74. By Theorems 3.1 and 4.16 in [8], the complex self-adjoint BC's having double eigenvalues are

$$[\Phi(b,\lambda) \mid -I], \qquad \lambda \in \mathbb{R}. \tag{3.75}$$

All of them are real and coupled.

Finally, we give the characterization of the discontinuity of λ_n as a function on the space $\Omega \times \mathcal{B}^{\mathbb{R}}_{\rm S}$ or $\Omega \times \mathcal{B}^{\mathbb{C}}_{\rm S}$.

Theorem 3.76. a) *The function* λ_0 *on* $\Omega \times \mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ *is continuous on* $\Omega \times (\mathcal{B}_{\mathrm{S}}^{\mathbb{R}} \setminus \mathcal{J}^{\mathbb{R}})$ *and discontinuous at each point of* $\Omega \times \mathcal{J}^{\mathbb{R}}$. *For* $n \in \mathbb{N}$, *the function* λ_n *is continuous on* $\Omega \times (\mathcal{B}_{\mathrm{S}}^{\mathbb{R}} \setminus \mathcal{J}^{\mathbb{R}})$ *and at each problem with a coupled boundary condition in* $\mathcal{J}^{\mathbb{R}}$ *where* $\lambda_n = \lambda_{n-1}$ *and discontinuous at any other point of* $\Omega \times \mathcal{J}^{\mathbb{R}}$. *More precisely, for each problem* $(\boldsymbol{\omega}, \boldsymbol{A}) \in \Omega \times \mathcal{J}^{\mathbb{R}}$ *with a coupled boundary condition* $\boldsymbol{A}$, *the restriction of* λ_n *to* $\Omega \times \mathcal{F}_{-}^{\mathbb{R}}$ *is continuous at* $(\boldsymbol{\omega}, \boldsymbol{A})$ *for* $n \in \mathbb{N}_0$ *and*

$$\lim_{\Omega \times \mathcal{F}_{+}^{\mathbb{R}} \ni (\boldsymbol{\sigma}, \boldsymbol{B}) \to (\boldsymbol{\omega}, \boldsymbol{A})} \lambda_0(\boldsymbol{\sigma}, \boldsymbol{B}) = -\infty, \tag{3.77}$$

$$\lim_{\Omega \times \mathcal{F}_{+}^{\mathbb{R}} \ni (\boldsymbol{\sigma}, \boldsymbol{B}) \to (\boldsymbol{\omega}, \boldsymbol{A})} \lambda_n(\boldsymbol{\sigma}, \boldsymbol{B}) = \lambda_{n-1}(\boldsymbol{\omega}, \boldsymbol{A}) \text{ for } n \in \mathbb{N}; \tag{3.78}$$

for each problem $(\boldsymbol{\omega}, \boldsymbol{A}) \in \Omega \times \mathcal{J}^{\mathbb{R}}$ *with* $\boldsymbol{A} \in \mathcal{J}^{\mathbb{R}} \cap \mathcal{G}_{-}^{\mathbb{R}}$, *the restriction of* λ_n *to* $\Omega \times \mathcal{G}_{-}^{\mathbb{R}}$ *is continuous at* $(\boldsymbol{\omega}, \boldsymbol{A})$ *for* $n \in \mathbb{N}_0$ *and*

$$\lim_{\Omega \times \mathcal{G}_{+}^{\mathbb{R}} \ni (\boldsymbol{\sigma}, \boldsymbol{B}) \to (\boldsymbol{\omega}, \boldsymbol{A})} \lambda_0(\boldsymbol{\sigma}, \boldsymbol{B}) = -\infty, \tag{3.79}$$

$$\lim_{\Omega \times \mathcal{G}_{+}^{\mathbb{R}} \ni (\boldsymbol{\sigma}, \boldsymbol{B}) \to (\boldsymbol{\omega}, \boldsymbol{A})} \lambda_n(\boldsymbol{\sigma}, \boldsymbol{B}) = \lambda_{n-1}(\boldsymbol{\omega}, \boldsymbol{A}) \text{ for } n \in \mathbb{N}; \tag{3.80}$$

for each problem $(\boldsymbol{\omega}, \boldsymbol{A}) \in \Omega \times \mathcal{J}^{\mathbb{R}}$ *with* $\boldsymbol{A} \in \mathcal{J}^{\mathbb{R}} \cap \mathcal{H}_{-}^{\mathbb{R}}$, *the restriction of* λ_n *to* $\Omega \times \mathcal{H}_{-}^{\mathbb{R}}$ *is continuous at* $(\boldsymbol{\omega}, \boldsymbol{A})$ *for* $n \in \mathbb{N}_0$ *and*

$$\lim_{\Omega \times \mathcal{H}_{+}^{\mathbb{R}} \ni (\boldsymbol{\sigma}, \boldsymbol{B}) \to (\boldsymbol{\omega}, \boldsymbol{A})} \lambda_0(\boldsymbol{\sigma}, \boldsymbol{B}) = -\infty, \tag{3.81}$$

$$\lim_{\Omega \times \mathcal{H}_{+}^{\mathbb{R}} \ni (\boldsymbol{\sigma}, \boldsymbol{B}) \to (\boldsymbol{\omega}, \boldsymbol{A})} \lambda_n(\boldsymbol{\sigma}, \boldsymbol{B}) = \lambda_{n-1}(\boldsymbol{\omega}, \boldsymbol{A}) \text{ for } n \in \mathbb{N}; \tag{3.82}$$

while for each problem $(\boldsymbol{\omega}, \boldsymbol{D})$ *with the Dirichlet boundary condition* $\boldsymbol{D}$, *the restriction of* λ_n *to* $\Omega \times \mathcal{I}_{-}^{\mathbb{R}}$ *is continuous at* $(\boldsymbol{\omega}, \boldsymbol{D})$ *for* $n \in \mathbb{N}_0$ *and*

$$\lim_{\Omega \times (\mathcal{I}_{0}^{\mathbb{R}} \cup \mathcal{I}_{+}^{\mathbb{R}}) \ni (\boldsymbol{\sigma}, \boldsymbol{B}) \to (\boldsymbol{\omega}, \boldsymbol{D})} \lambda_0(\boldsymbol{\sigma}, \boldsymbol{B}) = \lim_{\Omega \times \mathcal{I}_{+}^{\mathbb{R}} \ni (\boldsymbol{\sigma}, \boldsymbol{B}) \to (\boldsymbol{\omega}, \boldsymbol{D})} \lambda_1(\boldsymbol{\sigma}, \boldsymbol{B}) = -\infty, \tag{3.83}$$

$$\lim_{\Omega \times \mathcal{I}_{0}^{\mathbb{R}} \ni (\boldsymbol{\sigma}, \boldsymbol{B}) \to (\boldsymbol{\omega}, \boldsymbol{D})} \lambda_n(\boldsymbol{\sigma}, \boldsymbol{B}) = \lambda_{n-1}(\boldsymbol{\omega}, \boldsymbol{D}) \text{ for } n \in \mathbb{N}, \tag{3.84}$$

$$\lim_{\mathcal{I}_{+}^{\mathbb{R}} \ni (\boldsymbol{\sigma}, \boldsymbol{B}) \to (\boldsymbol{\omega}, \boldsymbol{D})} \lambda_n(\boldsymbol{\sigma}, \boldsymbol{B}) = \lambda_{n-2}(\boldsymbol{\omega}, \boldsymbol{D}) \text{ for } n \geq 2. \tag{3.85}$$

b) *The conclusions of a) still hold when all the super indices* $\mathbb{R}$ *in them are replaced by* $\mathbb{C}$.

PROOF. Here we only prove the first claim of a), while the other claims of the theorem can be shown similarly.

Let $(\boldsymbol{\omega}, \boldsymbol{A}) \in \Omega \times \mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ be a problem with a coupled boundary condition $\boldsymbol{A}$ not in $\mathcal{J}^{\mathbb{R}}$. Since the case where $\lambda_0(\boldsymbol{\omega}, \boldsymbol{A})$ has multiplicity 1 is simpler, we will assume that the multiplicity of $\lambda_0(\boldsymbol{\omega}, \boldsymbol{A})$ is 2. Set

$$r_1 = \lambda_0(\boldsymbol{\omega}, \boldsymbol{A}) - 2, \qquad r_2 = \frac{1}{3}\lambda_1(\boldsymbol{\omega}, \boldsymbol{A}) + \frac{2}{3}\lambda_2(\boldsymbol{\omega}, \boldsymbol{A}), \tag{3.86}$$

$$r_3 = \lambda_0(\boldsymbol{\omega}, \boldsymbol{A}) - 1, \qquad r_4 = \frac{2}{3}\lambda_1(\boldsymbol{\omega}, \boldsymbol{A}) + \frac{1}{3}\lambda_2(\boldsymbol{\omega}, \boldsymbol{A}). \tag{3.87}$$

Then, $r_2 > r_4$. By Theorem 1.38, there are a connected neighborhood $\mathcal{N}_1$ of $\boldsymbol{\omega}$ in Ω and a connected neighborhood $\mathcal{N}_2$ of $\boldsymbol{A}$ in $\mathcal{O}_{6,\mathrm{S}}^{\mathbb{R}} \setminus \mathcal{J}^{\mathbb{R}}$ such that each $(\boldsymbol{\sigma}, \boldsymbol{B}) \in \mathcal{N} =: \mathcal{N}_1 \times \mathcal{N}_2$ has exactly two eigenvalues in (r_1, r_2) and they are in (r_3, r_4). Theorem 2.1 implies that $\lambda_0(\boldsymbol{\sigma}, \boldsymbol{A})$ and $\lambda_1(\boldsymbol{\sigma}, \boldsymbol{A})$ are continuous functions of $\boldsymbol{\sigma} \in \Omega$. Thus, for each $\boldsymbol{\sigma} \in \mathcal{N}_1$, the eigenvalues $\lambda_0(\boldsymbol{\sigma}, \boldsymbol{A})$ and $\lambda_1(\boldsymbol{\sigma}, \boldsymbol{A})$ of $(\boldsymbol{\sigma}, \boldsymbol{A})$ must be the two in (r_3, r_4). Fix a $\boldsymbol{\sigma} \in \mathcal{N}_1$. From Theorem 3.39 we see that $\lambda_0(\boldsymbol{\sigma}, \boldsymbol{B})$ and $\lambda_1(\boldsymbol{\sigma}, \boldsymbol{B})$ are continuous functions of $\boldsymbol{B} \in \mathcal{O}_{6,\mathrm{S}}^{\mathbb{R}} \setminus \mathcal{J}^{\mathbb{R}}$. So, for each $\boldsymbol{B} \in \mathcal{N}_2$, the eigenvalues $\lambda_0(\boldsymbol{\sigma}, \boldsymbol{B})$ and $\lambda_1(\boldsymbol{\sigma}, \boldsymbol{B})$ of $(\boldsymbol{\sigma}, \boldsymbol{B})$ must be the two in (r_3, r_4). Therefore, λ_0, λ_1, λ_2, ... take values in $(r_3, +\infty)$ on $\mathcal{N}$ and hence are continuous on $\mathcal{N}$ by the Continuity Principle. ∎

REMARK 3.88. In addition to Theorems 2.1 and 3.39, the above proof basically only uses the *local uniqueness of continuous eigenvalue branches* deduced from Theorem 1.38.

§4. Ranges of λ_n on $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ and $\mathcal{B}_{\mathrm{S}}^{\mathbb{C}}$

Fix a differential equation in Ω and consider λ_n as a function on $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ or $\mathcal{B}_{\mathrm{S}}^{\mathbb{C}}$. In this section, we first find the ranges of λ_n on $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ and $\mathcal{B}_{\mathrm{S}}^{\mathbb{C}}$, respectively, and then use these ranges to determine the possibilities for the number of zeros of an eigenfunction for λ_n.

Recall that $\lambda_n(e^{i\theta}K) = \lambda_n([e^{i\theta}K \mid -I])$ for any $[e^{i\theta}K \mid -I] \in \mathcal{B}_{\mathrm{S}}^{\mathbb{C}}$ and let λ_n^{D} be the value of λ_n at the Dirichlet BC.

Theorem 4.1. a) *The range of λ_n on the space $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$ of self-adjoint real boundary conditions is $(-\infty, \lambda_n^{\mathrm{D}}]$ if $n = 0$ or 1, and $(\lambda_{n-2}^{\mathrm{D}}, \lambda_n^{\mathrm{D}}]$ if $n \geq 2$.*

b) *For each $n \in \mathbb{N}_0$, the range of λ_n on the space $\mathcal{B}_{\mathrm{S}}^{\mathbb{C}}$ of self-adjoint complex boundary conditions is the same as that of λ_n on $\mathcal{B}_{\mathrm{S}}^{\mathbb{R}}$.*

PROOF. a) By Lemma 3.28, we have the following:

$$\sup_{\boldsymbol{A}\in\mathcal{T}} \lambda_n(\boldsymbol{A}) = \lambda_n(\boldsymbol{S}_{0,\pi}) = \lambda_n^{\rm D},\ n\in\mathbb{N}_0, \tag{4.2}$$

$$\begin{aligned}\lambda_n(\boldsymbol{S}_{0,\beta}) &> \lim_{\gamma\to 0^+} \lambda_n(\boldsymbol{S}_{0,\gamma}) = \lambda_{n-1}(\boldsymbol{S}_{0,\pi})\\ &= \lambda_{n-1}^{\rm D},\ \beta\in(0,\pi],\ n\in\mathbb{N},\end{aligned} \tag{4.3}$$

$$\begin{aligned}\inf_{\boldsymbol{A}\in\mathcal{T}} \lambda_1(\boldsymbol{A}) &= \inf_{0\le\alpha<\pi}\Big(\inf_{0<\beta\le\pi}\lambda_1(\boldsymbol{S}_{\alpha,\beta})\Big)\\ &= \inf_{0\le\alpha<\pi}\lambda_0(\boldsymbol{S}_{\alpha,\pi}) = -\infty,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\inf_{\boldsymbol{A}\in\mathcal{T}} \lambda_n(\boldsymbol{A}) &= \inf_{0\le\alpha<\pi}\Big(\inf_{0<\beta\le\pi}\lambda_n(\boldsymbol{S}_{\alpha,\beta})\Big)\\ &= \inf_{0\le\alpha<\pi}\lambda_{n-1}(\boldsymbol{S}_{\alpha,\pi}) = \lambda_{n-2}^{\rm D},\ n\ge 2,\end{aligned} \tag{4.5}$$

and the infimum in (4.5) is not achieved. By Theorem 1.35, for any $K\in{\rm SL}(2,\mathbb{R})$, there exists a $\beta\in(0,\pi]$ such that

$$\lambda_n(K)\le\lambda_n(\boldsymbol{S}_{0,\beta})\le\lambda_{n+1}(K),\quad n\in\mathbb{N}_0. \tag{4.6}$$

Clearly, (4.2)–(4.6) imply our claims.

b) By Theorem 1.35 again, if $\theta\in(0,\pi)$ and $K\in{\rm SL}(2,\mathbb{R})$, then for each $n\in\mathbb{N}_0$,

$$\lambda_n(K)\le\lambda_n(e^{i\theta}K)\le\lambda_n(-K)\qquad\text{or}\qquad\lambda_n(-K)\le\lambda_n(e^{i\theta}K)\le\lambda_n(K). \tag{4.7}$$

Therefore, the conclusions here are direct consequences of those in Part a). ∎

Note that the suprema of λ_n on $\mathcal{B}_{\rm S}^{\mathbb{R}}$ and $\mathcal{B}_{\rm S}^{\mathbb{C}}$ have been obtained in Corollary 3.1 of [4]. As an application of Theorem 4.1, we prove the following results.

Theorem 4.8. a) *Let $K\in{\rm SL}(2,\mathbb{R})$ and u_n be a real eigenfunction for $\lambda_n(K)$. Then the number of zeros of u_n on $[a,b)$ is 0 or 1 if $n=0$, and $n-1$ or n or $n+1$ if $n\ge 1$.*

b) *Let $\theta\in(0,\pi)$, $K\in{\rm SL}(2,\mathbb{R})$ and u_n be an eigenfunction for $\lambda_n(e^{i\theta}K)$. Then the number of zeros of ${\rm Re}\,u_n$ on $[a,b)$ is 0 or 1 if $n=0$, and $n-1$ or n or $n+1$ if $n\ge 1$. The same conclusion holds for ${\rm Im}\,u_n$. Moreover, u_n is never zero on $[a,b]$.*

PROOF. a) Let v_n be a real eigenfunction for $\lambda_n^{\rm D}$. Then, v_n has $n+2$ zeros on the interval $[a,b]$, and $v_n(a)=v_n(b)=0$. Thus, the conclusion follows from Part a) of Theorem 4.1 and the Sturm Comparison Theorem.

b) Note that $\operatorname{Re} u_n$ and $\operatorname{Im} u_n$ are nontrivial solutions to the fixed DE (0.1) with $\lambda = \lambda_n(e^{i\theta}K)$. Thus, the conclusions on them follow from Part b) of Theorem 4.1 and the Sturm Comparison Theorem. Since $\lambda_n(e^{i\theta}K)$ does not have a real eigenfunction, $\operatorname{Re} u_n$ and $\operatorname{Im} u_n$ are linearly independent on (a, b), and hence, $\operatorname{Re} u_n$ and $\operatorname{Im} u_n$ do not have a common zero on $[a, b]$. Therefore, u_n does not have a zero on $[a, b]$. ∎

We note that even though Part a) of Theorem 4.8 has been known, its existing proof involves operator theory (see, for example, [12]).

Remark 4.9. Each of the possibilities given by Theorem 4.8 is realized in some examples.

§5. Comments on Differentiability of λ_n

In this section, we first briefly discuss the differentiability or analyticity of λ_n and an important application of these properties, then give an example related to these properties.

Since we now know where λ_n is continuous, the derivative formulas in [9], [10] and [8] for continuous eigenvalue branches yield derivative formulas for λ_n with respect to the parameters defining the SLP. To give an example, let $n \in \mathbb{N}_0$, $\boldsymbol{\omega}_0 = (a_0, b_0, 1/p_0, q_0, w_0) \in \Omega$ and $\boldsymbol{A} \in \mathcal{B}_{\mathrm{S}}^{\mathbb{C}}$. Assume that λ_n is simple and continuous at $(\boldsymbol{\omega}_0, \boldsymbol{A})$. By Theorems 1.39 and 3.76, λ_n is simple and continuous on a neighborhood of $(\boldsymbol{\omega}_0, \boldsymbol{A})$ in $\Omega \times \mathcal{B}_{\mathrm{S}}^{\mathbb{C}}$. Consider

$$\lambda_n(b) = \lambda_n((a_0, b, 1/p_0, q_0, w_0), \boldsymbol{A}) \quad \text{for } b \in (a_0, b_0], \tag{5.1}$$

then $\lambda_n(b)$ is simple on $(b_0 - \delta, b_0]$ and differentiable a. e. on $(b_0 - \delta, b_0]$ for some $\delta > 0$, and

$$\lambda_n'(b) = -\frac{1}{p_0(b)}|(p_0 u_b')(b)|^2 + |u_b(b)|^2(q_0(b) - \lambda_n(b)w_0(b)) \tag{5.2}$$

a.e. on $(b_0 - \delta, b_0]$, where u_b is an eigenfunction for $\lambda_n(b)$ satisfying $\int_{a_0}^{b} |u_b|^2 w = 1$. By Theorem 4.1 in [8], when we change $\boldsymbol{A}$ only, λ_n depends on $\boldsymbol{A}$ analytically. Moreover, each of the derivative formulas also holds under the assumption that the multiplicity of λ_n is always 2 when the corresponding parameter varies on an open subset of its domain.

From the above derivative formulas one can deduce some monotone properties of λ_n with respect to the parameters $1/p$, q and w of the SLP. (When considering monotone properties of λ_n with respect to w, we need to take the sign of λ_n into account.) To give an example, let us fix a self-adjoint boundary condition. If $\boldsymbol{\omega}_1 = (a, b, 1/p_1, q, w) \in \Omega$ and $\boldsymbol{\omega}_2 = (a, b, 1/p_2, q, w) \in \Omega$ satisfy

$$p_1(t) \le p_2(t) \quad \text{a.e. on } (a, b), \tag{5.3}$$

then for each $n \in \mathbb{N}_0$, we have

$$\lambda_n(\boldsymbol{\omega}_1) \le \lambda_n(\boldsymbol{\omega}_2). \tag{5.4}$$

Next, we give an example to show that the multiplicity of the n-th eigenvalue can change when an end point of the interval in the DE varies and that in general, the n-th eigenvalue is not differentiable when its multiplicity changes. There are similar examples for the other parameters in the DE, and such examples can also be used in the discussion of the differentiability of continuous eigenvalue branches.

EXAMPLE 5.5. By Theorem 2.1, the n-th eigenvalue $\lambda_n(b)$ of the SLP

$$\begin{cases} -y'' = \lambda y \quad \text{on } (0, b) \\ \begin{pmatrix} y(b) \\ y'(b) \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{\pi} \\ -\frac{\pi}{2} & 0 \end{pmatrix} \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} \end{cases} \tag{5.6}$$

is a continuous function of $b > 0$. It is easy to see that

$$\lambda_0(1) = \lambda_1(1) = \left(\frac{\pi}{2}\right)^2, \qquad \lambda_2(1) > \pi^2. \tag{5.7}$$

Thus, when b is sufficiently close to 1, $\lambda_0(b)$ and $\lambda_1(b)$ are the zeros of

$$\Delta_b(\lambda) = 2 - \left(\frac{\pi}{2\sqrt{\lambda}} + \frac{2\sqrt{\lambda}}{\pi}\right) \sin(b\sqrt{\lambda}) \tag{5.8}$$

in $(0, \pi^2)$, i.e., the solutions to

$$\sin(b\sqrt{\lambda}) = \frac{4b\pi(b\sqrt{\lambda})}{b^2\pi^2 + 4(b\sqrt{\lambda})^2} \tag{5.9}$$

in $(0, \pi^2)$. When $b \neq 1$ is sufficiently close to 1,

$$b\sqrt{\lambda_1(b)} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right), \qquad b\sqrt{\lambda_2(b)} \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right), \tag{5.10}$$

and hence $\lambda_0(b)$, $\lambda_1(b)$ are simple. Then, by (5.2), $\lambda_0(\cdot)$ and $\lambda_1(\cdot)$ are strictly decreasing functions on $(1-\delta, 1+\delta)$ for some $\delta > 0$. In particular,

$$\lambda_1(b) > \lambda_0(b) > \left(\frac{\pi}{2}\right)^2 \text{ for } b \in (1-\delta, 1), \tag{5.11}$$

$$\lambda_0(b) < \lambda_1(b) < \left(\frac{\pi}{2}\right)^2 \text{ for } b \in (1, 1+\delta). \tag{5.12}$$

When $b < 1$ is sufficiently close to 1, (5.9) together with (5.11) and (5.12) yield

$$\cos\left(b\sqrt{\lambda_0(b)}\right) = 1 - \frac{2\pi^2}{\pi^2 + 4\lambda_0(b)}, \qquad \cos\left(b\sqrt{\lambda_1(b)}\right) = \frac{2\pi^2}{\pi^2 + 4\lambda_1(b)} - 1, \tag{5.13}$$

and hence

$$\lambda_0'(1^-) = -\frac{\pi^3}{2\pi + 4}, \qquad \lambda_1'(1^-) = -\frac{\pi^3}{2\pi - 4}. \tag{5.14}$$

Similarly,

$$\lambda_0'(1^+) = -\frac{\pi^3}{2\pi - 4}, \qquad \lambda_1'(1^+) = -\frac{\pi^3}{2\pi + 4}. \tag{5.15}$$

Therefore, $\lambda_0(\cdot)$ and $\lambda_1(\cdot)$ are not differentiable at 1.

In general, as a function on $\mathcal{B}_S^{\mathbb{C}}$, λ_n is also not differentiable at a self-adjoint BC where it is continuous and has multiplicity 2, see the example given in Section 7 of [7].

References

1. P. B. Bailey, W. N. Everitt, and A. Zettl: Home page of the code Sleign2. "`http://www.math.niu.edu/~zettl/SL2`"
2. P. B. Bailey, M. K. Gordon, and L. F. Shampine: *Automatic solution of the Sturm-Liouville problem.* ACM Trans. Math. Software 4 (1978), 193–208.
3. J. Conway: "Functions of One Complex Variable", second edition. Springer, 1986.
4. M. S. P. Eastham, Q. Kong, H. Wu & A. Zettl: *Inequalities among eigenvalues of Sturm-Liouville problems.* J. Inequalities & Appl., to appear.
5. W. N. Everitt, M. Möller & A. Zettl: *Discontinuous dependence of the n-th Sturm-Liouville eigenvalue.* In "General Inequalities" edited by C. Bandle, W. N. Everitt, L. Losonszi & W. Walter. Birkhauser, 1997.

6. C. T. Fulton and S. Pruess: *Mathematical software for Sturm-Liouville problems.* ACM Trans. Math. Software 19 (1993), 360–376.
7. Q. Kong, H. Wu & A. Zettl: *Dependence of eigenvalues on the problem.* Math. Nachr. 188 (1997), 173–201.
8. Q. Kong, H. Wu & A. Zettl: *Geometric aspects of Sturm-Liouville problems, I. Structures on spaces of boundary conditions.* Preprint.
9. Q. Kong & A. Zettl: *Dependence of eigenvalues of Sturm-Liouville problems on the boundary.* J. Differential Equations 126 (1996), 389–407.
10. Q. Kong & A. Zettl: *Eigenvalues of regular Sturm-Liouville problems.* J. Differential Equations 131 (1996), 1–19.
11. M. Möller & A. Zettl: *Differentiable dependence of eigenvalues of operators in Banach spaces.* J. Operator Theory 36 (1996), 335–355.
12. J. Weidmann: "Spectral Theory of Ordinary Differential Operators". Lecture Notes in Mathematics 1258. Springer, 1987.
13. A. Zettl: *Sturm-Liouville problems.* In "Spectral Theory and Computational Methods of Sturm-Liouville Problems" edited by Don Hinton & Philip W. Schaefer. Marcel Dekker, 1997.

kong@math.niu.edu, wu@math.niu.edu, zettl@math.niu.edu

1998-10-26