Inequalities among Eigenvalues of Singular Sturm-Liouville Problems

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Abstract

Inequalities among the eigenvalues of a regular Sturm-Liouville problem with an arbitrary coupled self-adjoint boundary condition and those for two related separated self-adjoint boundary conditions are established in [4]. In this paper, we extend these inequalities to the case of singular self-adjoint Sturm-Liouville problems whose endpoints are limit-circle non-oscillatory. The extension is achieved by reducing the singular problem to a regular one. These inequalities and their proofs directly yield the existence and boundedness from below of the eigenvalues of these problems. An algorithm based on these inequalities is employed by the Fortran code SLEIGN2 for numerically computing the eigenvalues. It is also shown that the geometric and algebraic multiplicities of any eigenvalue of such a problem are equal.

Key words: Singular Sturm-Liouville problems, eigenvalue inequalities, multiplicities of an eigenvalue.
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1 Introduction

We consider self-adjoint eigenvalue problems associated with the differential equation

$$-(py')' + qy = \lambda wy \text{ on } J = (a, b), \tag{1.1}$$

where $-\infty \le a < b \le \infty$ and p, q and w are real functions on J satisfying

$$p > 0,\ w > 0 \text{ a.e. on } J. \tag{1.2}$$

If we have

$$1/p,\ q,\ w \in L^1(J, \mathbb{R}), \tag{1.3}$$

then (1.1) is said to be *regular*. Here $L^1(J, \mathbb{R})$ denotes the space of real-valued Lebesgue integrable functions on J. In this case, we have regular Sturm-Liouville problems (SLP's) consisting of (1.1) and coupled boundary conditions (BC's) of the form

$$Y(b) = e^{i\theta} KY(a), \tag{1.4}$$

where

$$Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \qquad i = \sqrt{-1}, \qquad -\pi < \theta \leq \pi,$$

$$K \in \mathrm{SL}(2, \mathbb{R}) =: \left\{ K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}; \; k_{ij} \in \mathbb{R}, \det K = 1 \right\}. \tag{1.5}$$

Note that the conditions in (1.3) ensure that y and py' are well defined at both endpoints and, hence, the regular BC (1.4) is well defined. Set $\mathbb{N}_0 = \{0, 1, 2, ...\}$ and let

$$\{\lambda_n(e^{i\theta}K); \; n \in \mathbb{N}_0\}$$

be the eigenvalues, listed in non-decreasing order and with only the double ones appearing twice, of the regular SLP consisting of (1.1) and (1.4).

If (1.3) is not satisfied and we have

$$1/p, \; q, \; w \in L_{loc}(J, \mathbb{R}), \tag{1.6}$$

then (1.1) is said to be *singular*. Here $L_{loc}(J, \mathbb{R})$ denotes the space of real-valued functions on J which are Lebesgue integrable on all compact subintervals of J. In this case, one of the end points must be singular, and we can consider the singular SLP's consisting of (1.1) and singular BC's. The singular BC's are defined in Section 3.

According to a well-known classical result (see [3] and [2] for the case of smooth coefficients and [10] for the general case), we have the following inequalities for the regular SLP with $K = I$, the identity matrix:

$$\begin{aligned} \lambda_0^N \; & \leq \lambda_0(I) < \lambda_0(e^{i\theta}I) < \lambda_0(-I) \leq \{\lambda_0^D, \lambda_1^N\} \\ & \leq \lambda_1(-I) < \lambda_1(e^{i\theta}I) < \lambda_1(I) \leq \{\lambda_1^D, \lambda_2^N\} \\ & \leq \lambda_2(I) < \lambda_2(e^{i\theta}I) < \lambda_2(-I) \leq \{\lambda_2^D, \lambda_3^N\} \\ & \leq \lambda_3(-I) < \lambda_3(e^{i\theta}I) < \lambda_3(I) \leq \{\lambda_3^D, \lambda_4^N\} \leq \cdots, \end{aligned} \tag{1.7}$$

where $\theta \in (-\pi, \pi)$ and $\theta \neq 0$. In (1.7), λ_n^D and λ_n^N denote the n-th Dirichlet and Neumann eigenvalues, respectively, and the notation $\{\lambda_0^D, \lambda_1^N\}$ means each of λ_0^D and λ_1^N.

Inequality (1.7) has been extended in [10] from regular SLP's with $K = I$ to regular SLP's with

$$K = \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix}, \quad c > 0$$

and in [1] to regular SLP's with a larger class of K, including

$$K = \begin{pmatrix} c & h \\ 0 & 1/c \end{pmatrix}, \quad c > 0, \; h \leq 0.$$

For these two classes of coupling matrices K, the above inequalities were extended to the case of singular limit-circle non-oscillatory (LCNO) endpoints by Niessen and Zettl in [9] and by Bailey, Everitt and Zettl in [1].

Finally, in [4] a generalization of (1.7) was established for the regular SLP with an arbitrary $K \in \mathrm{SL}(2, \mathbb{R})$. A key feature of this result is the identification of two separated BC's whose eigenvalues play the role of the Dirichlet eigenvalues λ_n^D and the Neumann eigenvalues λ_n^N in (1.7). This feature plays an important role in the forthcoming update of the FORTRAN code SLEIGN2 where it is used not only to bracket the eigenvalues for the coupled BC but also to determine their indices.

As a consequence of these general inequalities, it is shown in [4] that for *any* $K \in \mathrm{SL}(2, \mathbb{R})$ either $\lambda_0(K)$ or $\lambda_0(-K)$ is simple; thus extending the classical result that the lowest periodic eigenvalue is simple, to the general case of arbitrary coupled self-adjoint BC's. Here simple refers to both the algebraic and geometric multiplicities, since these are equal, see [4] for a proof of the case of coupled BC's and [6] for the case of separated BC's.

The proof of the general inequalities analogous to (1.7) in [4] actually yields a new proof of the existence and boundedness from below of the eigenvalues of arbitrary coupled self-adjoint regular SLP's without using the theory of self-adjoint operators in Hilbert space. For separated BC's such a proof is provided by the Prüfer transformation.

In this paper, we show that these general inequalities established in [4] for regular end-points also hold for singular SLP's with arbitrary coupled singular BC's when each singular endpoint is LCNO. Our proof consists of reducing the singular LCNO case to the regular case by using a "regularizing" transformation employed by Niessen and Zettl in [9]. With this transformation, a regular problem is constructed from the singular one which has the same spectrum. Also, the eigenfunctions of the singular problem are obtained from the eigen-functions of the regular problem simply via being multiplied by the "regularizing" function. As a consequence, the basic properties such as the number of zeros of the eigenfunctions of the singular problem can be obtained from the known properties of the corresponding eigenfunctions of the regular problem.

In essence, this paper combines the results and methods of the two papers [4] by Eastham, Kong, Wu and Zettl and [9] by Niessen and Zettl.

Since our approach is to reduce the singular SLP in question to a regular SLP, we briefly review the related results about regular SLP's in Section 2, then discuss the nature of the singular BC's at a general limit-circle endpoint (i.e., not necessarily LCNO) in Section 3, and present our results on the inequalities for the eigenvalues of singular SLP's in Section 4.

Throughout this paper, we will always assume that (1.2) holds.

2 Regular Problems

In this section, we assume that (1.3) holds and consider the regular SLP consisting of (1.1) and (1.4). It follows from the well-known theory of regular self-adjoint SLP's that the problem has an infinite, but countable, number of only real eigenvalues which can be ordered to form a non-decreasing sequence (with only the double eigenvalues appearing twice). This sequence is bounded from below, but not from above. So, we have the notation $\{\lambda_n(e^{i\theta}K),\, n \in \mathbb{N}_0\}$ defined in the Introduction.

In general, the n-th eigenvalue does not depend on the BC continuously, see [5] and [7]. Nevertheless, each simple eigenvalue is on a locally unique continuous branch of simple eigenvalues; while each double eigenvalue is on two locally unique continuous branches of eigenvalues. See [6] for details.

For $\lambda \in \mathbb{C}$, let u and v be the solutions of (1.1) determined by the initial conditions

$$u(a,\lambda) = 0 = v^{[1]}(a,\lambda), \qquad v(a,\lambda) = 1 = u^{[1]}(a,\lambda), \tag{2.1}$$

respectively. Here and below $y^{[1]} = py'$ for any solution y of (1.1). For any fixed $K \in \mathrm{SL}(2,\mathbb{R})$ and all $\lambda \in \mathbb{C}$, we define

$$D(\lambda) = k_{11}u^{[1]}(b,\lambda) - k_{21}u(b,\lambda) + k_{22}v(b,\lambda) - k_{12}v^{[1]}(b,\lambda). \tag{2.2}$$

Let

$$\Phi(t,\lambda) = \begin{pmatrix} v(t,\lambda) & u(t,\lambda) \\ v^{[1]}(t,\lambda) & u^{[1]}(t,\lambda) \end{pmatrix}, \quad t \in J,\ \lambda \in \mathbb{C}. \tag{2.3}$$

Then, $\Phi(t,\lambda)$ is the fundamental matrix solution of

$$Y'(t) = [P(t) - \lambda W(t)]Y(t), \quad Y(a) = I, \tag{2.4}$$

where

$$P = \begin{pmatrix} 0 & 1/p \\ q & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}.$$

The equation usually used to characterize the eigenvalues is

$$\det(e^{i\theta}K - \Phi(b,\lambda)) = e^{i\theta}(2\cos\theta - D(\lambda)) = 0,$$

see Lemma 4.5 on p. 48 in [11]. This identity implies the following result.

Lemma 2.1 *A number λ is an eigenvalue of the Sturm-Liouville problem consisting of (1.1) and (1.4) if and only if*

$$D(\lambda) = 2\cos\theta. \tag{2.5}$$

Moreover, we have the following lemmas.

Lemma 2.2 *Let $\theta \in (-\pi, \pi]$ and $K \in \mathrm{SL}(2,\mathbb{R})$. Then, we have*

$$\lambda_n(e^{i\theta}K) = \lambda_n(e^{-i\theta}K) \tag{2.6}$$

for $n \in \mathbb{N}_0$. Furthermore, if f is an eigenfunction for $\lambda_n(e^{i\theta}K)$, then its complex conjugate $\bar{f}$ is an eigenfunction for $\lambda_n(e^{-i\theta}K)$.

Proof: See Lemma 2.3 in [4]. ∎

Lemma 2.3 *A number λ is an eigenvalue of the Sturm-Liouville problem consisting of (1.1) and (1.4) of geometric multiplicity two if and only if*

$$e^{i\theta}K = \Phi(b,\lambda). \tag{2.7}$$

In this case, $\theta = 0$ or $\theta = \pi$.

Proof: See Theorem 3.1 in [6]. ▮

For a fixed $K \in \mathrm{SL}(2,\mathbb{R})$, we consider the separated BC's

$$y(a) = 0, \qquad k_{22}y(b) - k_{12}y^{[1]}(b) = 0, \tag{2.8}$$

and

$$y^{[1]}(a) = 0, \qquad k_{21}y(b) - k_{11}y^{[1]}(b) = 0. \tag{2.9}$$

Note that $(k_{22}, k_{12}) \neq (0,0) \neq (k_{21}, k_{11})$ since $\det K = 1$. Therefore, there exist an infinite number of eigenvalues for each of the BC's (2.8) and (2.9). Let $\{\mu_n,\ n \in \mathbb{N}_0\}$ denote the eigenvalues for (2.8) and $\{\nu_n,\ n \in \mathbb{N}_0\}$ the eigenvalues for (2.9). Then, how are μ_n and ν_n related to $\lambda_n(K)$, $\lambda_n(-K)$ and $\lambda_n(e^{i\theta}K)$? This question is answered by the following result.

Theorem 2.1 *Let $K \in \mathrm{SL}(2,\mathbb{R})$.*

(a) If $k_{11} > 0$ and $k_{12} \leq 0$, then $\lambda_0(K)$ is simple, and for any $\theta \in (-\pi,\pi)$, $\theta \neq 0$, we have

$$\begin{aligned} \nu_0 \ &\leq \lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) \leq \{\mu_0, \nu_1\} \\ &\leq \lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) \leq \{\mu_1, \nu_2\} \\ &\leq \lambda_2(K) < \lambda_2(e^{i\theta}K) < \lambda_2(-K) \leq \{\mu_2, \nu_3\} \\ &\leq \lambda_3(-K) < \lambda_3(e^{i\theta}K) < \lambda_3(K) \leq \{\mu_3, \nu_4\} \leq \cdots. \end{aligned} \tag{2.10}$$

(b) If $k_{11} \leq 0$ and $k_{12} < 0$, then $\lambda_0(K)$ is simple, and for any $\theta \in (-\pi,\pi)$, $\theta \neq 0$, we have

$$\begin{aligned} &\lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) \leq \{\mu_0, \nu_0\} \leq \\ &\lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) \leq \{\mu_1, \nu_1\} \leq \\ &\lambda_2(K) < \lambda_2(e^{i\theta}K) < \lambda_2(-K) \leq \{\mu_2, \nu_2\} \leq \\ &\lambda_3(-K) < \lambda_3(e^{i\theta}K) < \lambda_3(K) \leq \{\mu_3, \nu_3\} \leq \cdots. \end{aligned} \tag{2.11}$$

(c) If neither case (a) nor case (b) applies to K, then either case (a) or case (b) applies to $-K$.

Proof: See Section 5 in [4]. ▮

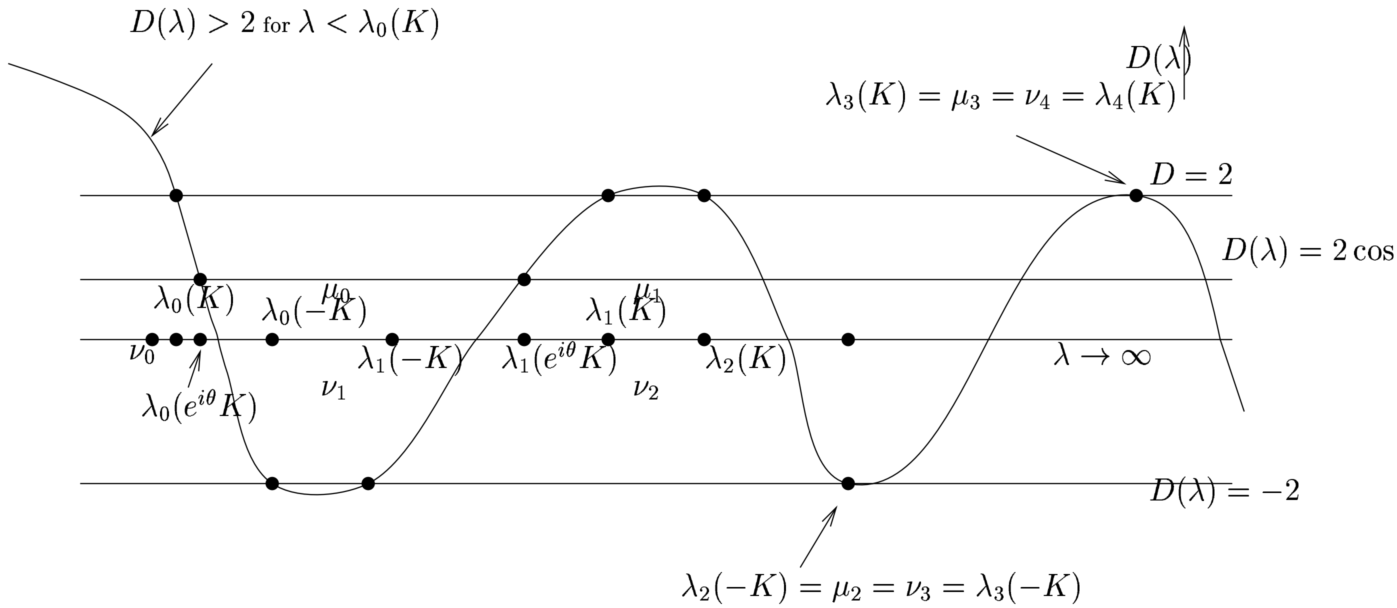
Corollary 2.1 *For any $K \in \mathrm{SL}(2,\mathbb{R})$, either $\lambda_0(K)$ or $\lambda_0(-K)$ is simple.*

Proof: This is a direct consequence of Theorem 2.1. ▮

Corollary 2.2 *Let $K \in \mathrm{SL}(2,\mathbb{R})$ with either $k_{11} > 0$ and $k_{12} \le 0$ or $k_{11} \le 0$ and $k_{12} < 0$. If $\lambda_{2n+1}(K)$ is simple, where $n \in \mathbb{N}_0$, then so is $\lambda_{2n+2}(K)$. In particular, if K has a double eigenvalue, then the first double eigenvalue of K is preceded by an odd number of simple eigenvalues.*

Proof: This is also a direct consequence of Theorem 2.1. ▮

The inequalities in (2.10) are illustrated with the following graph:



Remark 2.1. The existence and basic properties of the eigenvalues for separated self-adjoint BC's can be established using the Prüfer transformation. Theorem 2.1 together with its proof given in [4] yield a new and elementary proof of the existence and boundedness from below of the eigenvalues for an arbitrary coupled self-adjoint BC. In particular, the inequalities of Theorem 2.1 can be used to bound each eigenvalue for a coupled self-adjoint BC uniquely in an interval whose endpoints are given by eigenvalues for separated self-adjoint BC's. This also determines the index of the eigenvalue for the coupled BC. A variant of this scheme is used by SLEIGN2.

Theorem 2.2 *The algebraic and geometric multiplicities of an eigenvalue of a regular self-adjoint Sturm-Liouville problem are always equal.*

Proof: See Theorem 4.2 in [4] for coupled BC's and Theorem 4.12 in [6] for separated BC's. ▮

Theorem 2.3 *Suppose $\theta = 0$ or π. Let λ be an eigenvalue of the Sturm-Liouville problem consisting of (1.1) and (1.4). Then λ is double if and only if there exist $n, m \in \mathbb{N}_0$ such that $\lambda = \mu_n = \nu_m$. Here μ_n and ν_m are defined in front of Theorem 2.1.*

Proof: See Corollary 4.2 and Theorem 3.4 in [4]. ∎

Theorem 2.4 *a) The range of* λ_n *on the space of self-adjoint real boundary conditions is* $(-\infty, \lambda_n^D]$ *if* $n = 0$ *or 1, and* $(\lambda_{n-2}^D, \lambda_n^D]$ *if* $n \geq 2$.

b) For each $n \in \mathbb{N}_0$*, the range of* λ_n *on the space of self-adjoint complex boundary conditions is the same as that of* λ_n *on the space of self-adjoint real boundary conditions.*

Proof: See Theorem 4.1 in [7]. ∎

3 Singular Problems with LC Endpoints

In this section, we assume that (1.3) does not hold and instead (1.6) holds. Thus, either the endpoint a is singular or the endpoint b is singular, i.e., either there is no $c \in J$ such that $1/p,\ q,\ w \in L^1((a,c), \mathbb{R})$ or there is no $c \in J$ such that $1/p,\ q,\ w \in L^1((c,b), \mathbb{R})$. Note also that one of the endpoints may be regular.

If a is singular, then in general, $y(a)$ and $(py')(a)$ do not exit; and we have a similar statement for b. So, the regular BC (1.4) does not make sense in this case. What takes the place of (1.4) and also of the regular separated conditions? This depends on the endpoint classification, e.g., the limit-point (LP) limit-circle (LC) classification.

If the endpoint a for (1.1) is singular, then it is called an LC endpoint if all solutions of (1.1) are in $L^2((a,c), w)$ for some (and hence any) $c \in J$, and it is called an LP endpoint otherwise. Note that if a is regular, then all solutions of (1.1) are in $L^2((a,c), w)$ for any $c \in J$. The endpoint a is said to be oscillatory if every solution has an infinite number of zeros in the interval (a,c) for every $c \in J$. LCNO means LC and not oscillatory. Similar definitions are made at the endpoint b. At an LC endpoint, analogues of the regular BC's can be given in terms of maximal domain functions; at an LP endpoint, no BC is required or allowed to get a self-adjoint problem. For more details on the classification of endpoints, the reader is referred to the survey paper [11].

Although we establish in Section 4 the inequalities of Theorem 2.1 only for singular SLP's satisfying (1.2) whose endpoints are either LCNO or regular, the discussions of this section only require that each endpoint is either LC or regular.

An immediate consequence of this requirement and the well-known Lagrange identity is the fact that for any maximal domain functions y and z, the Lagrange bracket $[y,z]$ can be continuously extended to both endpoints. Here the maximal domain Δ is defined by

$$\Delta = \{f \in L^2(J,w) :\ f',\ pf' \in AC_{loc}(J),\ (-(pf')' + qf)/w \in L^2(J,w)\}$$

with $AC_{loc}(J)$ being the set of complex-valued functions which are absolutely continuous on all compact subintervals of J. Recall also that the Lagrange bracket is given by

$$[y,z] = y(p\bar{z}') - \bar{z}(py'), \quad y, z \in \Delta.$$

We can now give the self-adjoint BC's for singular differential equations (1.1) whose endpoints are either LC or regular.

Theorem 3.1 *Assume that the differential equation (1.1) is singular and its endpoints are either LC or regular. Then, we have the following results.*

a) There are real-valued maximal domain functions u and v satisfying

$$[u, v](a) = 1 = [u, v](b). \tag{3.1}$$

b) Fix such a pair u and v. Then, for any 2 by 2 complex matrices A and B satisfying

$$\mathrm{rank(A|B)} = 2, \qquad \mathrm{AEA}^* = \mathrm{BEB}^*, \ \textit{where } \mathrm{E} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.2}$$

the system

$$AY(a) + BY(b) = 0, \ \textit{where } Y = \begin{pmatrix} [u, y] \\ [v, y] \end{pmatrix}, \tag{3.3}$$

of linear equations defines a self-adjoint boundary condition; moreover, every self-adjoint boundary condition has this form.

c) The separated self-adjoint boundary conditions have the form

$$A_1\, [u, y](a) + A_2\, [v, y](a) = 0, \quad B_1\, [u, y](b) + B_2\, [v, y](b) = 0, \tag{3.4}$$

where a_1, a_2, b_1 and b_2 are real numbers such that one of a_1, a_2 and one of b_1, b_2 are non-zero; and the coupled self-adjoint boundary conditions have the canonical representation

$$Y(b) = e^{i\theta} K Y(a), \tag{3.5}$$

where Y is given in (3.3), $-\pi < \theta \le \pi$, and $K \in \mathrm{SL}(2, \mathbb{R})$.

Proof: Although this theorem is stated more generally than the corresponding result in [8], the proof given there can be easily adapted to yield a proof of our theorem here. ∎

Definition 3.1. Real-valued maximal domain functions u, v satisfying (3.1) are called *boundary condition functions.*

Note that there are more than one pair of BC functions. In order for (3.3) to define a given BC, we need to choose the coefficient matrices A and B according to the pair u, v of BC functions used. Since each pair of BC functions generates *all* self-adjoint BC's, by Theorem 3.1, it is natural to look for the correspondence between the two pairs of coefficient matrices in the same BC when two different pairs of BC functions are used. The next result gives this correspondence.

Theorem 3.2 *Let the hypotheses and notation of Theorem 3.1 be valid, and assume that* u_1, v_1 *is another pair of boundary condition functions. Then, the system*

$$A_1 Y_1(a) + B_1 Y_1(b) = 0, \text{ where } Y_1 = \begin{pmatrix} [u_1, y] \\ [v_1, y] \end{pmatrix}, \tag{3.6}$$

and (3.3) are the same self-adjoint boundary condition if and only if

$$A_1 = A \begin{pmatrix} [u, v_1](a) & [u_1, u](a) \\ [v, v_1](a) & [u_1, v](a) \end{pmatrix}, \quad B_1 = B \begin{pmatrix} [u, v_1](b) & [u_1, u](b) \\ [v, v_1](b) & [u_1, v](b) \end{pmatrix}. \tag{3.7}$$

Proof: The first identity in (3.7) can be obtained from

$$AY(t)[u_1, v_1](t) = A \begin{pmatrix} [u, v_1](t) & [u_1, u](t) \\ [v, v_1](t) & [u_1, v](t) \end{pmatrix} Y_1(t), \quad t \in J,$$

via taking limit $t \to a^+$. We can show the second identity in (3.7) in the same way. Since

$$\det \begin{pmatrix} [u, v_1] & [u_1, u] \\ [v, v_1] & [u_1, v] \end{pmatrix} (c) = ([u, v][u_1, v_1])(c) = 1 \tag{3.8}$$

for $c \in \{a, b\}$, we have

$$A_1 E A_1^* = AEA^*, \quad B_1 E B_1^* = BEB^*.$$

Thus, $A_1 E A_1^* = B_1 E B_1^*$. Note that (3.7), (3.8) and Part c) of Theorem 3.1 together imply that $\text{rank}(A_1|B_1) = 2$. Therefore, (3.6) is a self-adjoint BC. ∎

4 Singular Problems with LCNO Endpoints

In this section, we always assume that the differential equation (1.1) is singular and its endpoints are either LCNO or regular.

As a consequence of this assumption, we can always use BC functions u and v such that u is a principle solution at each endpoint and v is a non-principle solution at each endpoint, and v is positive on J. Principle and non-principle solutions exist at each endpoint, but the same solution may be principle at one endpoint and non-principle at the other. So, to get a BC function which is a principle solution at each endpoint, we choose a principle solution at each endpoint which is defined in an appropriate one-sided neighborhood of that endpoint and then use the "Patching Lemma" to patch them together to obtain a maximal domain function (defined through the whole interval J). Similarly, to construct a positive maximal domain function which is a non-principle solution at each endpoint, we use a strengthened version of the Patching Lemma due to Niessen and Zettl [9]. Specifically, we have the following results.

Theorem 4.1 *There exist real-valued functions u and v in the maximal domain satisfying the following conditions:*

a) $v > 0$ on $J = (a, b)$;

b) for some fixed real $\lambda = \lambda_a$, u is a principal solution at a and v is a non-principal solution at a;

c) for some fixed real $\lambda = \lambda_b$, u is a principal solution at b and v is a non-principal solution at b;

d) for any maximal domain function y,

$$\frac{y}{v}(a) = [u, y](a), \quad \frac{y}{v}(b) = [u, y](b). \tag{4.1}$$

Proof: See [9], in particular, pages 564 to 566 there. ∎

Note that the functions u and v of Theorem 4.1 need not be solutions throughout the whole interval (a,b) and, in case $\lambda_a = \lambda_b$, each of them needs not be the same solution near a and near b.

Since (4.1) implies that $[u, v](a) = 1$ and $[u, v](b) = 1$, u and v form a pair of BC functions. Let us fix such a pair u, v of BC functions and consider the singular self-adjoint SLP consisting of (1.1) and a self-adjoint BC.

Definition 4.1. The function v of Theorem 4.1 is called a *regularizing function* of the differential equation (1.1).

The reasons for this definition of regularizing functions will become clear from the following theorem.

Theorem 4.2 *Let v be a regularizing function of (1.1) and define*

$$P = pv^2, \quad Q = (-(pv')' + qv)v, \quad W = v^2 w \tag{4.2}$$

on J. Then,

$$P > 0, \ W > 0 \ \text{a.e. on } J, \quad 1/P, \ Q, \ W \ \in L^1(J, \mathbb{R})$$

and, consequently, the differential equation

$$-(Pz')' + Qz = \lambda W z \quad \text{on } J \tag{4.3}$$

is regular. Furthermore, we have the following results.

a) Let $\lambda \in \mathbb{C}$. If $y(\cdot, \lambda)$ is a solution of (1.1), then $z(\cdot, \lambda) = y(\cdot, \lambda)/v$ is a solution of (4.3); conversely, if $z(\cdot, \lambda)$ is a solution of (4.3), then $y(\cdot, \lambda) = vz(\cdot, \lambda)$ is a solution of (1.1).

b) All solutions z of (4.3) and their quasi-derivatives Pz' can be continuously extended to the endpoints of J.

Proof: See Lemmas 3.2 and 3.5 in [8]. ∎

Note that v is independent of λ, but does depend on the coefficients p, q, w and the endpoints a, b. Thus, the singular behavior of $y(\cdot, \lambda)$ is the same for all λ and is contained in v. Note also that at each endpoint c,

$$z(c) = \frac{y}{v}(c) = [u, y](c), \quad (Pz')(c) = (vpy' - ypv')(c) = [v, y](c) \tag{4.4}$$

exist and are finite for any maximal domain function y. However, in general, neither y, nor v, nor the individual terms in Pz' can be evaluated separately at a singular endpoint.

Remark 4.1. The first identity in (4.4) implies that we can use only a regularizing function v to express singular BC's, i.e., write them as restrictions on the values of y/v and $[v, y]$ at the endpoints.

Theorem 4.2 and Remark 4.1 allow us to extend the results of Section 2 to the case of singular SLP's whose endpoints are either LCNO or regular. The key point for doing this is the following result.

Theorem 4.3 *Assume that u, v are a pair of boundary condition functions given by Theorem 4.1 and let A, B be any 2 by 2 complex matrices satisfying*

$$\mathrm{rank(A|B)} = 2, \quad \mathrm{AEA^*} = \mathrm{BEB^*}. \tag{4.5}$$

Then, $\lambda \in \mathbb{C}$ is an eigenvalue for the singular Sturm-Liouville problem consisting of (1.1) and the boundary condition

$$AY(a) + BY(b) = 0, \;\; \textit{where } Y = \begin{pmatrix} y/v \\ [v, y] \end{pmatrix}, \tag{4.6}$$

if and only if it is an eigenvalue of the regular Sturm-Liouville problem consisting of (4.3) and the boundary condition

$$AZ(a) + BZ(b) = 0, \;\; \textit{where } Z = \begin{pmatrix} z \\ Pz' \end{pmatrix}.$$

Moreover, for each eigenvalue, its eigenfunctions for the singular problem can be obtained from its eigenfunctions for the regular problem via being multiplied by the regularizing function.

Proof: This is a direct consequence of Theorem 4.2 and the identities in (4.4). ∎

Remark 4.2. By Remark 2.1, there is an elementary proof of the existence and boundedness from below of the eigenvalues of a regular self-adjoint SLP satisfying (1.2). This proof and Theorem 4.3 together yield an elementary proof of the same result for the case of singular self-adjoint SLP's satisfying (1.2) such that their endpoints are either LCNO or regular.

For $\lambda \in \mathbb{C}$, let $\phi(\cdot,\lambda)$ and $\psi(\cdot,\lambda)$ be the solutions of (1.1) determined by the initial conditions

$$\Big(\frac{\phi(\cdot,\lambda)}{v}\Big)(a) = 1, \quad [v,\phi(\cdot,\lambda)](a) = 0$$

and

$$\Big(\frac{\psi(\cdot,\lambda)}{v}\Big)(a) = 0, \quad [v,\psi(\cdot,\lambda)](a) = 1,$$

respectively. Let

$$\Phi(t,\lambda) = \begin{pmatrix} (\phi(\cdot,\lambda)/v)(t) & (\psi(\cdot,\lambda)/v)(t) \\ [v,\phi(\cdot,\lambda)](t) & [v,\psi(\cdot,\lambda)](t) \end{pmatrix}, \quad t \in J,\ \lambda \in \mathbb{C}. \tag{4.7}$$

For any fixed $K \in \mathrm{SL}(2,\mathbb{R})$ and all $\lambda \in \mathbb{C}$, we set

$$D(\lambda) = k_{11}[v,\psi(\cdot,\lambda)](b) - k_{12}[v,\phi(\cdot,\lambda)](b) - k_{21}\Big(\frac{\psi(\cdot,\lambda)}{v}\Big)(b) + k_{22}\Big(\frac{\phi(\cdot,\lambda)}{v}\Big)(b). \tag{4.8}$$

Then, we have the following results derived from Lemma 2.1 and the general arguments in front of it.

Lemma 4.1 *a) Let A and B be any 2 by 2 complex matrices satisfying (4.5). Then, $\lambda \in \mathbb{C}$ is an eigenvalue of the singular Sturm-Liouville problem consisting of (1.1) and the boundary condition (4.6) if and only if*

$$\det(A + B\Phi(b,\lambda)) = 0. \tag{4.9}$$

b) Let $\theta \in (-\pi,\pi]$ and $K \in \mathrm{SL}(2,\mathbb{R})$. Then, $\lambda \in \mathbb{C}$ is an eigenvalue of the singular Sturm-Liouville problem consisting of (1.1) and the coupled boundary condition

$$Y(b) = e^{i\theta}KY(a), \ \text{where } Y = \begin{pmatrix} y/v \\ [v,y] \end{pmatrix}, \tag{4.10}$$

if and only if

$$D(\lambda) = 2\cos\theta.$$

Note that the left-hand side of (4.9) is an entire function of λ. The *algebraic multiplicity* of an eigenvalue of the singular SLP consisting of (1.1) and (4.6) will be defined to be its multiplicity as a zero of this entire function. Recall that the geometric multiplicity of an eigenvalue is the dimension of its eigenspace. By definition and Theorem 4.3, the algebraic multiplicity (resp. geometric multiplicity) of an eigenvalue of the singular problem is equal to its algebraic multiplicity as an eigenvalue of the regular problem. For any $\theta \in (-\pi,\pi]$ and $K \in \mathrm{SL}(2,\mathbb{R})$, let

$$\{\lambda_n(e^{i\theta}K);\ n \in \mathbb{N}_0\}$$

be the eigenvalues, listed in non-decreasing order and with their algebraic multiplicities taken into account, of the singular SLP consisting of (1.1) and (4.10). Then, the following two lemmas can be deduced from Lemmas 2.2 and 2.3, respectively.

Lemma 4.2 *Let $\theta \in (-\pi, \pi]$ and $K \in \mathrm{SL}(2, \mathbb{R})$. Then, we have*

$$\lambda_n(e^{i\theta} K) = \lambda_n(e^{-i\theta} K) \tag{4.11}$$

for $n \in \mathbb{N}_0$. Furthermore, if f is an eigenfunction for $\lambda_n(e^{i\theta} K)$, then its complex conjugate $\bar{f}$ is an eigenfunction for $\lambda_n(e^{-i\theta} K)$.

Lemma 4.3 *A number λ is an eigenvalue of the singular Sturm-Liouville problem consisting of (1.1) and (4.10) of geometric multiplicity two if and only if*

$$e^{i\theta} K = \Phi(b, \lambda).$$

In this case, $\theta = 0$ or $\theta = \pi$.

Moreover, from Theorem 2.2 we directly obtain the following result.

Theorem 4.4 *The algebraic and geometric multiplicities of an eigenvalue of the singular self-adjoint Sturm-Liouville problem consisting of (1.1) and (4.6) are always equal.*

For any $K \in \mathrm{SL}(2, \mathbb{R})$, let $\{\mu_n = \mu_n(K),\ n \in \mathbb{N}_0\}$ and $\{\nu_n = \nu_n(K),\ n \in \mathbb{N}_0\}$ denote the eigenvalues for the separated self-adjoint BC's

$$\left(\frac{y}{v}\right)(a) = 0, \qquad k_{22} \left(\frac{y}{v}\right)(b) - k_{12}\, [v, y](b) = 0 \tag{4.12}$$

and

$$[v, y](a) = 0, \qquad k_{21} \left(\frac{y}{v}\right)(b) - k_{11}\, [v, y](b) = 0, \tag{4.13}$$

respectively. Now, we can extend Theorem 2.1, Corollarys 2.1 and 2.2, and Theorem 2.3 to the case of singular SLP's whose endpoints are either LCNO or regular.

Theorem 4.5 *Let $K \in \mathrm{SL}(2, \mathbb{R})$.*

(a) If $k_{11} > 0$ and $k_{12} \le 0$, then $\lambda_0(K)$ is simple, and for any $\theta \in (-\pi, \pi)$, $\theta \neq 0$, we have

$$\begin{aligned} \nu_0 \quad &\le \lambda_0(K) < \lambda_0(e^{i\theta} K) < \lambda_0(-K) \le \{\mu_0, \nu_1\} \\ &\le \lambda_1(-K) < \lambda_1(e^{i\theta} K) < \lambda_1(K) \le \{\mu_1, \nu_2\} \\ &\le \lambda_2(K) < \lambda_2(e^{i\theta} K) < \lambda_2(-K) \le \{\mu_2, \nu_3\} \\ &\le \lambda_3(-K) < \lambda_3(e^{i\theta} K) < \lambda_3(K) \le \{\mu_3, \nu_4\} \le \cdots. \end{aligned} \tag{4.14}$$

(b) If $k_{11} \le 0$ and $k_{12} < 0$, then $\lambda_0(K)$ is simple, and for any $\theta \in (-\pi, \pi)$, $\theta \neq 0$, we have

$$\begin{aligned} &\lambda_0(K) < \lambda_0(e^{i\theta} K) < \lambda_0(-K) \le \{\mu_0, \nu_0\} \le \\ &\lambda_1(-K) < \lambda_1(e^{i\theta} K) < \lambda_1(K) \le \{\mu_1, \nu_1\} \le \\ &\lambda_2(K) < \lambda_2(e^{i\theta} K) < \lambda_2(-K) \le \{\mu_2, \nu_2\} \le \\ &\lambda_3(-K) < \lambda_3(e^{i\theta} K) < \lambda_3(K) \le \{\mu_3, \nu_3\} \le \cdots. \end{aligned} \tag{4.15}$$

(c) If neither case (a) nor case (b) applies to K, then either case (a) or case (b) applies to $-K$.

Corollary 4.1 *For any $K \in \mathrm{SL}(2,\mathbb{R})$, either $\lambda_0(K)$ or $\lambda_0(-K)$ is simple.*

Corollary 4.2 *Let $K \in \mathrm{SL}(2,\mathbb{R})$ with either $k_{11} > 0$ and $k_{12} \leq 0$ or $k_{11} \leq 0$ and $k_{12} < 0$. If $\lambda_{2n+1}(K)$ is simple, where $n \in \mathbb{N}_0$, then so is $\lambda_{2n+2}(K)$. In particular, if K has a double eigenvalue, then the first double eigenvalue of K is preceded by an odd number of simple eigenvalues.*

Remark 4.3. The inequalities of Theorem 4.5 can be used to bound each eigenvalue for a coupled self-adjoint BC uniquely in an interval whose endpoints are given by eigenvalues for separated self-adjoint BC's. This also determines the index of the eigenvalue for the coupled BC.

Theorem 4.6 *Suppose $\theta = 0$ or π. Let λ be an eigenvalue of the singular Sturm-Liouville problem consisting of (1.1) and (4.10). Then, λ is double if and only if there exist $n, m \in \mathbb{N}_0$ such that $\lambda = \mu_n = \nu_m$. Here μ_n and ν_m are defined in front of Theorem 4.5.*

In order to generalize Theorem 2.4, we denote by $\{\lambda_n^F,\ n \in \mathbb{N}_0\}$ the eigenvalues of the singular SLP consisting of (1.1) and the self-adjoint BC

$$\left(\frac{y}{v}\right)(a) = 0, \qquad \left(\frac{y}{v}\right)(b) = 0. \tag{4.16}$$

We use the notation λ_n^F since these are the eigenvalues of the Friedrichs extension, see [9]. They correspond to the Dirichlet eigenvalues of a regular SLP.

Theorem 4.7 *a) The range of λ_n on the space of self-adjoint real boundary conditions is $(-\infty, \lambda_n^F]$ if $n = 0$ or 1, and $(\lambda_{n-2}^F, \lambda_n^F]$ if $n \geq 2$.*

b) For each $n \in \mathbb{N}_0$, the range of λ_n on the space of self-adjoint complex boundary conditions is the same as that of λ_n on the space of self-adjoint real boundary conditions.

In a forthcoming publication, we will discuss the continuity and discontinuity of λ_n on the singular SLP in question.

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