Dependence of eigenvalues of Sturm-Liouville problems on the boundary

Q. Kong and A. Zettl*

Abstract

The eigenvalues of Sturm-Liouville (SL) problems depend not only continuously but smoothly on boundary points. The derivative of the nth eigenvalue as a function of an endpoint satisfies a first order differential equation. This for arbitrary (separated or coupled) self-adjoint regular boundary conditions. In addition, as the length of the interval shrinks to zero all higher eigenvalues march off to plus infinity. This is also true for the first (i.e. lowest) Dirichlet eigenvalue but not for the lowest Neumann eigenvalue. The latter has a finite limit.

1 Introduction

This paper was motivated by the work of Dauge and Helffer in [3] and [4]. These authors considered the Sturm-Liouville (SL) differential equation

$$-(py')' + qy = \lambda wy \tag{1.1}$$

with $p(t) \geq k > 0$ and $p, q, w \in C^\infty$ and showed that its Neumann eigenvalues, as functions of an endpoint, satisfy a differential equation of the form

$$\lambda' = u^2(q - \lambda w). \tag{1.2}$$

They also found the equation satisfied by the Dirichlet eigenvalues

$$\lambda' = -pu'^2 \tag{1.3}$$

and, more generally, the equation for the eigenvalues of any self-adjoint separated boundary condition at b parameterized by, say, β

$$\lambda' = u^2[-\beta/p + (q - \lambda w)]. \tag{1.4}$$

In addition, these authors showed that the lowest Neumann eigenvalue is, in general, not a decreasing function of the endpoints but, nevertheless, has a finite limit as the endpoints approach

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Key words and phrases: Sturm-Liouville problems, eigenvalues, dependence of eigenvalues on the boundary each other. On the other hand they showed that the lowest Dirichlet eigenvalue is a decreasing function of the endpoints and thus must have a finite or infinite limit as the endpoints approach each other, but these authors left open the question of whether this limit is finite or infinite.

Here we show that it is infinite. This is perhaps surprising since it implies that the difference between the Dirichlet and Neumann eigenvalues goes to infinity as the length of the interval shrinks to zero. This and the equations (1.2), (1.3), (1.4) are established without any smoothness assumptions on the coefficients and also for the case that the coefficient p is not assumed to be bounded away from zero and is even allowed to change sign. Our results - the extensions of the Dauge-Helffer theorems as well as the new theorems- are established for integrable coefficients with, in some cases, an additional mild technical condition. In most cases our proofs of the extensions of the results of [3] are simpler, more direct, and more complete.

We unify the eigenvalue differential equations (1.1), (1.3) and (1.4) into the form

$$\lambda' = -pu'^2 + u^2(q - \lambda w) \tag{1.5}$$

and show that it is also satisfied by the *simple and double* eigenvalues of arbitrary coupled self-adjoint regular boundary conditions. It is interesting to note that equation (1.5) has no explicit dependence on the boundary condition constants; of course, there is an implicit dependence since u is a normalized eigenfunction.

In Section 2 we summarize some of the basic results needed later and establish the notation. The results are given in Sections 3 and 4.

2 Notation and Basic Results

Consider the differential equation

$$-(py')' + qy = \lambda w\, y \;\text{ on } (A,B),\; -\infty \le A < B \le \infty \text{ with } \lambda \in \mathbb{R} \tag{2.1}$$

where

$$p, q, w : I = (A,B) \to \mathbb{R},\;\; 1/p,\; q,\; w \in L_{loc}(I),\;\; w > 0 \; a.e.\ on\ I\,. \tag{2.2}$$

Let

$$J = [a,b], \quad A < a < b < B, \tag{2.3}$$

and consider boundary conditions (BC)

$$C \begin{pmatrix} y(a) \\ (py')(a) \end{pmatrix} + D \begin{pmatrix} y(b) \\ (py')(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.4}$$

where the *complex* 2×2 matrices C and D satisfy:

$$\text{The } 2 \times 4 \text{ matrix } (C|D) \text{ has full rank,} \tag{2.5}$$

and

$$CEC^* = DED^*, \quad E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.6}$$

By a solution of (2.1) on I is meant a function $y \in AC_{loc}(I)$ such that $py' \in AC_{loc}(I)$ and the equation (2.1) is satisfied *a.e.* on I. Here $AC_{loc}(I)$ denotes the set of functions which are absolutely continuous on all compact subintervals of I. Clearly a solution of (2.1) on I is also a solution on any subinterval J of I.

A SL boundary value problem consists of equation (2.1) together with boundary conditions (BC) (2.4). With conditions (2.2), (2.3), (2.5) and (2.6) it is well known that problem (2.1), (2.4) is a regular self-adjoint SL problem which has an infinite but countable number of only real eigenvalues. In this paper we fix p, q, w and the boundary condition (constants) and one endpoint and study the dependence of the eigenvalues and eigenfunctions on the other endpoint.

For our purposes here it is convenient to divide these self-adjoint boundary conditions (2.4), (2.5), (2.6) into three disjoint subclasses:

1. **Separated self-adjoint BC.** These are

$$A_1 y(a) + A_2 (py')(a) = 0 \quad \text{where } A_1 \text{ and } A_2 \text{ are real and not both zero,} \tag{2.7}$$

$$B_1 y(b) + B_2 (py')(b) = 0 \quad \text{where } B_1 \text{ and } B_2 \text{ are real and not both zero.} \tag{2.8}$$

These separated conditions can be parameterized as follows:

$$\cos\alpha\, y(a) \; - \; \sin\alpha\, (py')(a) \; = \; 0, \quad 0 \le \alpha < \pi; \tag{2.9}$$

$$\cos\beta\, y(b) \; - \; \sin\beta\, (py')(b) \; = \; 0, \quad 0 < \beta \le \pi. \tag{2.10}$$

Note the different normalization in (2.10) for β than that used for α in (2.9). This is for convenience in stating some of the results below.

2. **All *real* coupled self-adjoint BC.** These can be formulated as follows:

$$\begin{pmatrix} y(b) \\ (py')(b) \end{pmatrix} = K \begin{pmatrix} y(a) \\ (py')(a) \end{pmatrix} \tag{2.11}$$

where $K \in SL_2(\mathbb{R})$ i.e. K satisfies

$$K = \begin{pmatrix} k_{11}, & k_{12} \\ k_{21}, & k_{22} \end{pmatrix}, \; k_{ij} \in \mathbb{R}, \; \det K = 1. \tag{2.12}$$

3. **All *complex* coupled self-adjoint BC.** These can be formulated as follows:

$$\begin{pmatrix} y(b) \\ (py')(b) \end{pmatrix} = \exp(i\,\theta)\, K \begin{pmatrix} y(a) \\ (py')(a) \end{pmatrix} \tag{2.13}$$

where K satisfies (2.12) and $-\pi < \theta < 0$, or $0 < \theta < \pi$.

Most of the following results are well-known. See [7] for some proofs with only integrable coefficients; see [6] for the case when p changes sign, and see [2] for the case of complex coupled BC.

Basic results and notation. Let (2.2) hold.

- (a) Assume that

$$p \geq 0 \text{ on } J = [a, b], \quad A < a < b < B. \tag{2.14}$$

  Then

  1. The BVP (2.1), (2.9) and (2.10) has only real and simple eigenvalues; there are an infinite but countable number of them; they are bounded below and can be ordered to satisfy

  $$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \ldots; \text{ and } \lambda_n \to +\infty \text{ as } n \to \infty. \tag{2.15}$$

     If u_n is an eigenfunction of λ_n, then u_n can be chosen real, is unique up to constant multiples and u_n has exactly n zeros in the open interval (a, b), $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

     Notation. Let

  $$\lambda_n = \lambda_n(\alpha, \beta; a, b); \quad u_n = u_n(\cdot, \alpha, \beta; a, b), \quad n \in \mathbb{N}_0, \tag{2.16}$$

     to highlight the dependence on these quantities. For the Dirichlet and Neumann eigenvalues we also use the special notation:

  $$\lambda_n^D = \lambda_n(0, \pi; a, b), \quad \lambda_n^N = \lambda_n(\pi/2, \pi/2; a, b), \ n \in \mathbb{N}_0. \tag{2.17}$$

     In addition to (2.15) we mention a couple of other properties of these eigenvalues which we need below.

     - (i) Fix a, b and omit these variables in (2.17). Let $n \in \mathbb{N}_0$. Then $\lambda_n(\alpha, \beta)$ is strictly decreasing in α for any fixed β and strictly increasing in β for any fixed α, $0 \leq \alpha < \pi$, $0 < \beta \leq \pi$.
     - (ii) For $0 < \alpha < \pi$ and $0 < \beta < \pi$ we have

  $$\begin{aligned} \lambda_n(\alpha,\beta) < \begin{pmatrix} \lambda_n(\alpha,\pi) \\ \lambda_n(0,\beta) \end{pmatrix} < \lambda_n(0,\pi) < \begin{pmatrix} \lambda_{n+1}(\alpha,\pi) \\ \lambda_{n+1}(0,\beta) \end{pmatrix} < \lambda_{n+2}(\alpha,\beta) \\ < \begin{pmatrix} \lambda_{n+2}(\alpha,\pi) \\ \lambda_{n+2}(0,\beta) \end{pmatrix} < \lambda_{n+2}(0,\pi) < \begin{pmatrix} \lambda_{n+3}(\alpha,\pi) \\ \lambda_{n+3}(0,\beta) \end{pmatrix} < \lambda_{n+4}(\alpha,\beta) < \ldots \end{aligned} \tag{2.18}$$

  2. The BVP (2.1), (2.11) and (2.12) has only real eigenvalues; each of these may be simple or double; there are an infinite but countable number of them and they can be ordered to satisfy

  $$-\infty < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots; \text{ and } \lambda_n \to +\infty \text{ as } n \to \infty. \tag{2.19}$$

Notation. Let

$$\lambda_n = \lambda_n(K; a, b); \quad u_n = u_n(\cdot, K; a, b), \quad n \in \mathbb{N}_0. \tag{2.20}$$

Note that there is some arbitrariness in the indexing of the eigenfunctions corresponding to a double eigenvalue. For the periodic and semi-periodic eigenvalues we also use the special notation:

$$\lambda_n^P = \lambda_n(I; a, b), \quad \lambda_n^S = \lambda_n(-I; a, b), \; n \in \mathbb{N}_0. \tag{2.21}$$

Here I denotes the 2×2 identity matrix.

3. The BVP (2.1), (2.12), (2.13) has only real eigenvalues; each of these is simple; there are an infinite but countable number of them and they can be ordered to satisfy

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \ldots; \text{ and } \lambda_n \to +\infty \text{ as } n \to \infty. \tag{2.22}$$

Notation. Denote these eigenvalues and eigenfunctions by

$$\lambda_n = \lambda_n(\exp(i\theta)K; a, b); \quad u_n = u_n(\cdot, \exp(i\theta)K; a, b), \quad n \in \mathbb{N}_0. \tag{2.23}$$

Then we have

$$\lambda_n(\exp(-i\,\theta)K; a, b) \; = \; \lambda_n(\exp(i\,\theta)K; a, b), \tag{2.24}$$

and the complex conjugate of an eigenfunction of $\lambda_n(\exp(i\,\theta)K; a, b)$ is an eigenfunction of $\lambda_n(\exp(-i\,\theta)K; a, b)$.

- (b) Assume that p changes sign in the interval (a, b), i.e., p is positive on a subset of (a, b) of positive Lebesgue measure and p is negative on a a subset of the interval (a, b) of positive Lebesgue measure. Then

  1. The BVP (2.1), (2.9) and (2.10) has only real and simple eigenvalues; there are an infinite but countable number of them; they are unbounded below and above and can be ordered to satisfy

  $$\ldots < \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < \ldots; \\ \lambda_n \to +\infty \text{ as } n \to \infty; \text{ and } \lambda_n \to -\infty \text{ as } n \to -\infty. \tag{2.25}$$

  2. The BVP (2.1), (2.11) and (2.12) has only real eigenvalues; each of these may be simple or double; there are an infinite but countable number of them; they are unbounded above and below and they can be ordered to satisfy

  $$\ldots \leq \lambda_{-2} \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots; \\ \lambda_n \to +\infty \text{ as } n \to \infty, \text{ and } \lambda_n \to -\infty \text{ as } n \to -\infty. \tag{2.26}$$

3. The BVP (2.1), (2.12), (2.13) has only real eigenvalues; each of these is simple; there are an infinite but countable number of them; they are unbounded above and below and they can be ordered to satisfy:

$$\ldots < \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < \ldots;$$
$$\lambda_n \to +\infty \text{ as } n \to \infty, \text{ and } \lambda_n \to -\infty \text{ as } n \to -\infty. \tag{2.27}$$

The notations for eigenvalues λ_n and eigenfunctions u_n, $n \in \mathbb{Z}$, for part (b) are the same as those introduced in part (a) for $n \in \mathbb{N}_0$.

3 Differential Equations for Eigenvalues

In this section we first show the continuity properties of eigenvalues and eigenfunctions, then obtain the differentiability of them and establish differential equations satisfied by them for every self-adjoint boundary condition.

By a normalized eigenfunction u of any self-adjoint SL problem we mean one that satisfies

$$\int_a^b |u|^2\, w = 1. \tag{3.1}$$

For fixed a and fixed boundary condition constants α, β (or A_1, A_2, B_1, B_2) or K, θ we abbreviate the notation of Section 2 to $\lambda_n(b)$ and study λ_n as a function of b for fixed $n \in \mathbb{N}_0$ or $n \in \mathbb{Z}$, as b varies in the interval (a, B).

Now we present a continuity result for the eigenvalues and eigenfunctions.

Theorem 3.1 *Let SL be a (regular) self-adjoint Sturm-Liouville problem on $J = [a, b]$ with separated or coupled boundary conditions as described in Section 2. Fix the BC and the endpoint a. Fix $n \in \mathbb{N}_0$ or $n \in \mathbb{Z}$ depending on whether p is positive or changes sign on J. Let $\lambda_n = \lambda_n(b)$ for $b \in (a, B)$. Then*

1. *$\lambda_n(b)$ is a continuous function of b for $b \in (a, B)$.*

2. *If $\lambda_n(b)$ is simple for some $b \in (a, B)$ then $\lambda_n(b)$ is simple for every $b \in (a, B)$.*

3. *There exists a normalized eigenfunction $u_n(\cdot, b)$ of $\lambda_n(b)$ for $b \in (a, B)$ such that $u_n(\cdot, b)$ and $(pu_n')(\cdot, b)$ are uniformly convergent in b on any compact subinterval of (a, B), i.e.,*

$$u_n(\cdot, b+h) \to u_n(\cdot, b), \quad (pu_n')(\cdot, b+h) \to (pu_n')(\cdot, b) \quad \text{as} \quad h \to 0, \tag{3.2}$$

and this convergence is uniform on any compact subinterval of (a, B).

Proof:

1. The continuity of $\lambda_n(b)$ as a function of b, although not explicitly given in [1], follows from Theorem 4.1 and its proof; see also Remark 2 on page 16 following this Theorem. Although this remark is given there specifically for Dirichlet eigenvalues only, it applies generally.

2. The fact that the multiplicity of $\lambda_n(b)$ is constant in b for $b \in (a, B)$ is a consequence of Theorem 4.1 of [1] and of the Spectral Theorem for self-adjoint operators in Hilbert space.

3. Firstly we show that there exist (not necessarily normalized) eigenfunctions $u_n(\cdot, b), u_n(\cdot, b+h)$ for h sufficiently small such that (3.2) holds uniformly on any compact subinterval of (a, B). For any solution y of (2.1) and any eigenfunction $u(\cdot, b)$ let

$$Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \quad U = \begin{pmatrix} u \\ pu' \end{pmatrix}.$$

Assume the boundary conditions are separated, i.e., (2.9) and (2.10) hold. Choose eigenfunctions $u = u_n(\cdot, b+h)$ for small h, all satisfying the same initial condition at a. Then the uniform convergence $U(\cdot, b+h) \to U(\cdot, b)$ on compact subintervals follows from part 1 and from the continuous dependence of solutions y and their quasi-derivatives py' on the parameter λ, see [7] and [5].

Assume the boundary conditions are coupled, i.e. (2.12) and (2.13) hold with $-\pi < \theta \leq \pi$. By part 2 either $\lambda_n(b)$ is simple for all $b \in (a, B)$ or it is double for all such b.

Suppose $\lambda_n(b)$ is double. Then we can argue as before by choosing eigenfunctions $u_n(\cdot, b+h)$ of $\lambda_n(b+h)$ all of which satisfy the same initial condition at a since a linear combination of two independent eigenfunctions can be chosen to satisfy an arbitrary initial conditions.

Suppose $\lambda_n(b)$ is simple for all $b \in (a, B)$. Let $u = u_n(\cdot, b+h)$ be an eigenfunction satisfying

$$\|U(a, b+h)\| = 1$$

for all $b + h \in (a, B)$. Here $\| \cdot \|$ denotes any fixed norm in $\mathbb{C}^2$. It suffices to show that

$$U(a, b+h) \to U(a, b) \quad \text{as} \quad h \to 0 \tag{3.3}$$

since the uniform convergence on compact subintervals then follows from the continuous dependence of solutions y and their quasi-derivatives py' on initial conditions and on the parameter λ. If (3.3) does not hold, then there exists a sequence $h_k \to 0$ such that

$$U(a, b) - U(a, b+h_k) := U_k \to U_0 \neq 0, \quad \text{as} \quad h_k \to 0. \tag{3.4}$$

Let Y_k, Z_k, Y be the solution vectors of (2.1) with $\lambda = \lambda_n(b)$ determined by the initial conditions

$$Y_k(a) = U_k, \; Z_k(a) = U(a, b+h_k), \; Y(a) = U_0, \quad k \in \mathbb{N},$$

respectively. Then by the uniqueness of solutions to initial value problems we have

$$Y_k = U(\cdot, b) - Z_k$$

in (a, B). Using (2.13) in (3.4) we get

$$\begin{aligned} Y_k(b) &= U(b,b) - Z_k(b) = U(b,b) - U(b+h_k, b+h_k) + U(b+h_k, b+h_k) - Z_k(b) \\ &= \exp(i\theta)K[U(a,b) - U(a, b+h_k)] + U(b+h_k, b+h_k) - Z_k(b) \\ &= \exp(i\theta)KY_k(a) + U(b+h_k, b+h_k) - Z_k(b). \end{aligned} \tag{3.5}$$

Letting $k \to \infty$ in (3.5) and using the continuous dependence of the solution vectors on initial conditions and parameters we conclude that

$$Y(b) = \exp(i\theta)KY(a).$$

Hence Y is a nontrivial eigenfunction solution vector corresponding to the eigenvalue $\lambda_n(b)$. Since $\lambda_n(b)$ is simple, there is a constant $c \neq 0$ such that $Y = cU(\cdot, b)$. In particular, $U_0 = Y(a) = cU(a, b)$. Letting $k \to \infty$ in (3.4) we obtain that

$$U(a,b) - \lim_{k\to\infty} U(a, b+h_k) = U_0 = cU(a,b),$$

i.e.

$$\lim_{k\to\infty} U(a, b+h_k) = (1-c)U(a,b),$$

and hence

$$\lim_{k\to\infty} \|U(a, b+h_k)\| = |1-c| \, \|U(a,b)\|$$

which contradicts

$$\|U(a, b+h_k)\| = \|U(a,b)\| = 1.$$

The above discussion shows that for every self-adjoint boundary condition and every fixed index n the eigenfunction $u_n(\cdot, b)$ and its quasi-derivative $(pu_n')(\cdot, b)$ are uniformly convergent in b on any compact subinterval of (a, B). By normalizing the eigenfunctions we complete the proof. ∎

It turns out that the eigenvalues are differentiable functions of the endpoints satisfying first order differential equations. The following lemmas are used to obtain these differential equations.

Lemma 3.1 *Assume u and v are solutions of (2.1) with $\lambda = \mu$ and $\lambda = \nu$, respectively. Then*

$$[u,v]_a^b := [u,v](b) - [u,v](a) := [u(p\bar{v}') - \bar{v}(pu')](b) - [u(p\bar{v}') - \bar{v}(pu')](a) = (\mu - \nu)\int_a^b u\bar{v}w. \tag{3.6}$$

Proof: This follows from integration by parts. ∎

Lemma 3.2 *Assume a real valued function* $f \in L_{loc}(A,B)$. *Then*

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} f \;=\; f(t) \;\; a.e. \; in \; (A,B). \tag{3.7}$$

Proof: Let $a \in (A,B)$ and define $F(t) = \int_a^t f$. Then $F \in AC_{loc}(A,B)$ and $F'(t) = f(t)$ *a.e.* in (A,B). Thus

$$\frac{1}{h}\int_t^{t+h} f = \frac{1}{h}[F(t+h) - F(t)] \to F'(t) = f(t) \;\; \text{as} \;\; h \to 0 \; a.e. \text{ in } \; (A,B).$$

This completes the proof. ∎

Theorem 3.2 (Dirichlet Eigenvalue-Eigenfunction Differential Equation) *Let (2.2) hold. Consider the BVP (2.1), (2.9) and (2.10) with* $0 \le \alpha < \pi$ *and* $\beta = \pi$, *i.e., an arbitrary separated condition at a and the Dirichlet condition at b. Using the notation of Section 2 and letting* $\lambda = \lambda_n$, $u = u_n$ *we have the following differential equation:*

$$(p\,\lambda')(b) \;=\; -(pu')^2(b,b), \;\; a.e. \; in \; (a,B). \tag{3.8}$$

In particular, if p is continuous at $b \in [a,B)$ *and* $p(b) \neq 0$, *then (3.8) holds at b.*

Proof: For small h, in (3.6) choose $\mu = \lambda(b), \nu = \lambda(b+h)$, and $u = u(\cdot,b), v = u(\cdot,b+h)$. From (3.6) and the boundary conditions, noting that $[u,v](a) = 0$ and $u(b,b) = 0$, we have

$$-u(b,b+h)(pu')(b,b) = [\lambda(b) - \lambda(b+h)] \int_a^b u(s,b)u(s,b+h)w(s)\,ds. \tag{3.9}$$

By Theorem 3.1 and the normalization (3.1) we have

$$\int_a^b u(s,b)u(s,b+h)w(s)\,ds \to \int_a^b u^2(s,b)w(s)\,ds = 1, \;\; \text{as} \;\; h \to 0. \tag{3.10}$$

Hence

$$u(b,b+h) = u(b,b+h) - u(b+h,b+h) = -\int_b^{b+h} u'(s,b+h)\,ds = -\int_b^{b+h} \frac{1}{p(s)}(pu')(s,b+h)\,ds$$

$$= -\int_b^{b+h} \frac{1}{p(s)}(pu')(s,b)\,ds + \int_b^{b+h} \frac{1}{p(s)}[(pu')(s,b) - (pu')(s,b+h)]\,ds. \tag{3.11}$$

Noting that $(pu')(s,b) - (pu')(s,b+h) \to 0$ uniformly on any compact subinterval of $[a,B)$ as $h \to 0$, by Lemma 3.2 and (3.11)

$$\lim_{h\to 0} \frac{u(b,b+h)}{h} = -\frac{1}{p(b)}(pu')(b,b) \quad a.e. \text{ in } (a,B).$$

Dividing (3.9) by h and taking the limit as $h \to 0$ we get (3.8). The second part of the theorem follows from the above. ∎

Theorem 3.3 (Neumann Eigenvalue-Eigenfunction Differential Equation) *Let (2.2) hold. Consider the BVP (2.1), (2.9) and (2.10) with* $0 \le \alpha < \pi$ *and* $\beta = \pi/2$*, i.e., an arbitrary separated condition at a and the Neumann condition at b. Using the notation of Section 2 and letting* $\lambda = \lambda_n,\ u = u_n$ *we have the following differential equation:*

$$\lambda'(b) \;=\; u^2(b,b)(q(b) - \lambda\, w(b)) \quad a.e.\ in\ (a,B). \tag{3.12}$$

In particular, if q and w are continuous at b then (3.12) holds at b.

Proof: The proof is similar to that of Theorem 3.2. For small h, in (3.6) choose $\mu = \lambda(b), \nu = \lambda(b+h)$, and $u = u(\cdot,b),\ v = u(\cdot,b+h)$. From (3.6) and the boundary conditions, noting that $[u,v](a) = 0$ and $(pu')(b,b) = 0$, we have

$$-u(b,b)(pu')(b,b+h) = [\lambda(b) - \lambda(b+h)] \int_a^b u(s,b)u(s,b+h)w(s)ds. \tag{3.13}$$

In place of equation (3.11) we get

$$\begin{aligned}
&(pu')(b,b+h)(b) = [(pu')(b,b+h) - (pu')(b+h,b+h)] = -\int_b^{b+h} (pu')'(s,b+h) \\
&= -\int_b^{b+h} [q(s)u(s,b+h) - \lambda(b+h)u(s,b+h)w(s)]\, ds \\
&= -\int_b^{b+h} q(s)u(s,b)\, ds + \int_b^{b+h} q(s)[u(s,b) - u(s,b+h)]\, ds \\
&\quad +\lambda(b+h)\int_b^{b+h} u(s,b)w(s)\, ds - \lambda(b+h)\int_b^{b+h} [u(s,b) - u(s,b+h)]w(s)\, ds.
\end{aligned} \tag{3.14}$$

Now dividing (3.14) by h and taking the limit as $h \to 0$, using the continuity of λ at b, the uniform convergence of $u(\cdot,b+h)$ to $u(\cdot,b)$, and Lemma 3.2 we obtain (3.12). The second part of the theorem follows from the above. ∎

Theorem 3.4 (Eigenvalue-Eigenfunction Differential Equation for Separated BVP's) *Let (2.2) hold. Consider the BVP (2.1), (2.9) and (2.10) with* $0 \le \alpha < \pi$ *and* $0 < \beta \le \pi$ *i.e. arbitrary separated conditions at a and b. Using the notation of Section 2 and letting* $\lambda = \lambda_n,\ u = u_n$ *we have the following differential equations:*

$$\lambda'(b) \;=\; -\frac{1}{p(b)}(pu')^2(b,b) + \; u^2(b,b)(q(b) - \lambda(b)\, w(b)) \ a.e.\ in\ \ (a,B). \tag{3.15}$$

Furthermore, if $\beta \ne \pi$*, then*

$$\lambda'(b) \;=\; u^2(b,b)\left(-\frac{\cot^2\beta}{p(b)} + q(b) - \lambda(b)\, w(b)\right) \ a.e.\ in\ (a,B); \tag{3.16}$$

if $\beta \ne \pi/2$*, then*

$$\lambda'(b) \;=\; (pu')^2(b,b)\left(-\frac{1}{p(b)} + \tan^2\beta(q(b) - \lambda(b)\, w(b))\right) \ a.e.\ in\ (a,B). \tag{3.17}$$

In particular, if p, q and w are continuous at b and $p(b) \neq 0$, then equations (3.15)-(3.17) hold at b.

It is easy to see that Theorem 3.4 includes Theorems 3.2 and 3.3. The proof is more complicated, but consists basically of combining the techniques in the proofs of the previous theorems and is therefore omitted.

Theorem 3.5 (Eigenvalue-Eigenfunction Differential Equation for Coupled BVP's) *Let (2.2) hold. Consider the coupled BVP (2.1) with (2.13), (2.12) where $-\pi < \theta \leq \pi$. Using the notation of Section 2 and letting $\lambda = \lambda_n$, $u = u_n$ we have the following differential equation:*

$$\lambda'(b) = -\frac{1}{p(b)}|pu'|^2(b,b) + |u|^2(b,b)(q(b) - \lambda(b)\,w(b)), \text{ a.e. in } (a,B). \tag{3.18}$$

In particular, if p, q and w are continuous at b in (a,B) and $p(b) \neq 0$, then equation (3.18) holds at b.

Proof: For small h, in (3.6) we choose μ, ν and u, v as in the proof of Theorem 3.2. Noting that u, v are complex functions, from Lemma 3.1 we get

$$\begin{aligned}
&[\lambda(b+h) - \lambda(b)] \int_a^b u\bar{v}w = -[u,v]_a^b = -[u(p\bar{v})' - \bar{v}(pu')]_a^b \\
=& -\left[(p\bar{v}', -\bar{v})\begin{pmatrix} u \\ pu' \end{pmatrix}\right]_a^b = -(p\bar{v}', -\bar{v})(b)\begin{pmatrix} u \\ pu' \end{pmatrix}(b) + (p\bar{v}', -\bar{v})(a)\begin{pmatrix} u \\ pu' \end{pmatrix}(a) \\
=& -(p\bar{v}', -\bar{v})(b)\begin{pmatrix} u \\ pu' \end{pmatrix}(b) + (p\bar{v}', -\bar{v})(b+h)\begin{pmatrix} u \\ pu' \end{pmatrix}(b) \\
=& [(p\bar{v}', -\bar{v})(b+h) - (p\bar{v}', -\bar{v})(b)]\begin{pmatrix} u \\ pu' \end{pmatrix}(b)
\end{aligned} \tag{3.19}$$

Now proceeding as in Theorems 3.2 and 3.3 we have as $h \to 0$

$$\frac{1}{h}[(p\bar{v})(b+h) - (p\bar{v}')(b)] \to \bar{u}(b)[q(b) - \lambda(b)w(b)] \quad a.e. \tag{3.20}$$

and

$$\frac{1}{h}[\bar{v}(b+h) - \bar{v}(b)] \to \frac{1}{p(b)}(p\bar{u}')(b) \quad a.e. \tag{3.21}$$

Now dividing (3.19) by h, taking the limit as $h \to 0$ and using (3.20), (3.21) we get

$$\lambda'(b) = (\bar{u}(q - \lambda w), -\frac{1}{p}(p\bar{u}'))(b)\begin{pmatrix} u \\ pu' \end{pmatrix}(b) \quad a.e. \tag{3.22}$$

and this concludes the proof. ∎

Combining Theorems 3.4 and 3.5 we see that the eigenvalues of the general separated BVPs as well as the eigenvalues of the general coupled BVPs all satisfy differential equations of the same form - namely (3.18) - where u is the corresponding eigenfunction.

4 Behavior of Eigenvalues as Functions of the Boundary Points

Based on the differential equations we obtained in the previous section we discuss the behavior of the eigenvalues as functions of the endpoint b. Theorem 4.1 is a result for the Dirichlet eigenvalues. Theorems 4.2-4.4 reveal various properties of the Neumann eigenvalues.

Theorem 4.1 *Let (2.2) hold. Fix a and consider the Dirichlet eigenvalues $\lambda_n^D(b) = \lambda_n^D(0, \pi, a, b)$ for b in (a, B) defined as in (2.17). If*

$$p \geq 0 \ a.e. \quad and \quad q^2/w \in L_{loc}(A, B), \tag{4.1}$$

then, for $n \in N_0$, $\lambda_n(b)$ is strictly decreasing on (a, B) and

$$\lambda_n^D(b) \to +\infty \quad as \quad b \to a^+. \tag{4.2}$$

Proof: The decreasing property of λ_n^D as a function of b follows directly from Theorem 3.2. Assume (4.2) is false. Then by Theorem 3.2 $\lambda(b) = \lambda_0^D(b)$ has a finite limit, say $\lambda^+(a)$, as $b \to a^+$ and hence is bounded on $(a, B_1]$ for any $B_1 < B$. Let $u = u_0(\cdot, b)$ be an eigenfunction of $\lambda(b)$ normalized to satisfy

$$\int_a^b u^2 w = 1 \ \text{ and } \ (pu')(b, b) > 0. \tag{4.3}$$

First we show that

$$(pu')(a, b) \to 0 \ \text{ as } b \to a^+. \tag{4.4}$$

To see this choose c in (a, b) such that $(pu')(c, b) = 0$; such a choice is possible for otherwise we would have that $pu' > 0$ on (a, b). This would imply $u' > 0 \, a.e.$ which is impossible since u is zero at a and at b. Using $(pu')(c, b) = 0$, the boundedness of λ_0 and the Schwarz inequality we get

$$\begin{aligned}
[(pu')(a, b)]^2 &= [-(pu')(a, b) + (pu')(c, b)]^2 = [\int_a^c (pu')']^2 = [\int_a^c (q - \lambda w) u]^2 \\
&= [\int_a^c (q w^{-1/2} - \lambda w^{1/2}) w^{1/2} u]^2 \leq \int_a^c (q w^{-1/2} - \lambda w^{1/2})^2 \int_a^c u^2 w \\
&\leq \int_a^b (q^2/w - 2\lambda q + \lambda^2 w) \int_a^b u^2 w \to 0 \ as \ b \to a^+.
\end{aligned}$$

Noting that $\lambda(b) \to \lambda^+(a)$ as $b \to a^+$, by (4.4) and the continuous dependence of solutions of (1.1) on initial conditions and on the parameter λ we conclude that $u(\cdot, b) \to 0$ uniformly on any compact subinterval of $[a, B)$. Therefore, for $\epsilon > 0$ there exists a $b_0 \in (a, B)$, such that

$$|u(t, b)| < \epsilon, \ t \in [a, b], \ a < b < b_0. \tag{4.5}$$

This implies that

$$\int_a^b u^2 w < \epsilon^2 \int_a^b w \tag{4.6}$$

for ϵ sufficiently small this contradicts the normalization (4.3) and completes the proof. ∎

Lemma 4.1 *In addition to the conditions and notation in Theorem 3.3 assume that* $Q := q/w \in AC_{loc}(a,B)$. *Let* $\lambda = \lambda_n^N$ *be the nth Neumann eigenvalue. Then for* $c, b \in (a,B)$ *we have*

$$\lambda'(b) = u^2 w \exp(-\int_c^b u^2 w) \left(\int_c^b \exp(\int_c^s u^2 w)\, Q'(s) ds + Q(c) - \lambda(c) \right) \tag{4.7}$$

where $u = u(b,b)$.

Proof: By (3.12)

$$\begin{aligned}
\lambda(b) &= \exp(-\int_c^b u^2 w) \left(\int_c^b u^2 q \exp(\int_c^s u^2 w)\, ds + \lambda(c) \right) \\
&= \exp(-\int_c^b u^2 w) \left(\int_c^b Q(s)\, d \exp(\int_c^s u^2 w) + \lambda(c) \right) \qquad (Q = q/w) \\
&= \exp(-\int_c^b u^2 w) \left(Q(b) \exp(\int_c^b u^2 w) - Q(c) - \int_c^b \exp(\int_c^s u^2 w) Q'(s)\, ds + \lambda(c) \right) \\
&= Q(b) - \exp(-\int_c^b u^2 w) \left(\int_c^b \exp(\int_c^s u^2 w) Q'(s)\, ds + Q(c) - \lambda(c) \right).
\end{aligned} \tag{4.8}$$

Now substitute this expression for $\lambda(b)$ into (3.12) to complete the proof. ▮

The next theorems extend the results of Dauge and Helffer for the properties of the Neumann eigenvalues. They are obtained as consequences of the equation (4.7) satisfied by the Neumann eigenvalues. First we introduce a definition which extends the concept of a relative extremum of a function.

Definition Let $f : (a,B) \to \mathbb{R}$, let $c, d \in (a,B)$ with $c \le d$ and let $I = [c,d]$. Then the point or interval $[c,d]$ is said to be a relative maximum of f if f is constant on I and f is increasing on $(c-\delta, c)$ and decreasing on $(d, d+\delta)$ for some $\delta > 0$. A relative minimum is defined similarly, and a relative extremum is either a relative minimum or a relative maximum. We speak of a strict extremum when the extremum "interval" is just a point.

Notation. Denote by $E(f,B)$ the set of all extrema of f on (a,B) and note that for any I_1 and I_2 in $E(f,B)$ we have that either $I_1 = I_2$ or $I_1 \cap I_2 = \emptyset$. Let $I_i = [c_i, d_i], i = 1,2$. Then we denote $(I_1, I_2) = (d_1, c_2)$. Denote by $\mathcal{N}(\{,\mathcal{B})$ the cardinality of the set $E(f,B)$ so that $\mathcal{N}(\{,\mathcal{B})$ is either 0, a positive integer, or $+\infty$. Let I_f and I_g be relative extrema of f and g respectively, then by $I_f < I_g$ we mean that the right endpoint of I_f is to the left of the left endpoint of I_g.

Theorem 4.2 *Assume the conditions and notation of Lemma 4.1 hold. Let* $\lambda = \lambda_n^N(b)$ *be the nth Neumann eigenvalue. Then*

1. *For each* $I_\lambda \in E(\lambda, B)$ *and each* $I_Q \in E(Q,B)$ *we have* $I_\lambda \cap I_Q = \emptyset$.

2. *If* $I_1, I_2 \in E(\lambda, B)$ *and* $I_1 < I_2$ *then there exists an* $I_Q \in E(Q,B)$ *such that* $I_1 < I_Q < I_2$.

3. *If I_1 and I_2 are two consecutive extrema of Q in (a,B) - here the singleton set $\{a\}$ is considered an extremum of Q in (a,B) - and $I_1 < I_2$, then there exists at most one $I_\lambda \in E(\lambda, B)$ such that*

$$I_1 < I_\lambda < I_2;$$

furthermore, if $Q' \geq 0$ $[Q' \leq 0]$ a.e. in (I_1, I_2), then I_λ is a relative minimum [maximum] of λ in (a,B).

4. *Regardless of whether these quantities are finite or infinite we have*

$$\mathcal{N}(\lambda,\mathcal{B}) \leq \mathcal{N}(\mathcal{Q},\mathcal{B}) + \infty.$$

Proof: 1. Without loss of generality assume $I_Q = [c,d] \subset (a,B)$ is a relative maximum of Q. Then there exist $k \in \mathbb{R}$ and $\delta > 0$ such that $Q \equiv k$ in I_Q, Q is increasing in $(c-\delta, c)$ and decreasing in $(d, d+\delta)$. Hence $Q' = 0$ in I_Q, $Q' \geq 0\, a.e.$ in $(c-\delta, c)$ and $Q' \leq 0\, a.e.$ in $(d, d+\delta)$. Choose $h \in I_Q$. Then

$$\int_h^b \exp(\int_h^s u^2 w) Q'(s)\, ds \geq 0 \text{ for } b \in (c-\delta, d+\delta). \tag{4.9}$$

By (4.7) and the continuity of Q and λ we see that λ' cannot change sign in $(c-\delta, d+\delta)$; also λ' cannot be zero *a.e.* in this interval, otherwise $Q'(t) = 0\, a.e.$ in this interval, contradicting the fact that I_Q is a relative maximum. Therefore $\lambda(b)$ is monotone and not constant in $(c-\delta, d+\delta)$. Hence $I_Q \cap I_\lambda = \emptyset$.

2. We show first that if $I \in E(\lambda, B)$, then for each $h \in I$ we have $\lambda(h) = Q(h)$. If $\lambda(h) < Q(h)$, then from (4.7), $\lambda'(b) > 0$ *a.e.* for b in a neighbourhood of h , contradicting that $h \in I$. Similarly for $\lambda(h) > Q(h)$. Define λ' to be zero at all points, if any, in I where it was not defined, we have from (4.7)

$$\lambda'(b) = u^2(b) w(b) \exp(-\int_h^b u^2 w) \int_h^b \exp(\int_h^s u^2 w) Q'(s)\, ds, \quad h \in I,\ b \in (a,B). \tag{4.10}$$

Let $I_i = [a_i, b_i]$, $i = 1,2$. Since $\lambda'(b_1) = \lambda'(a_2) = 0$, letting $h = a_2$ and $b = b_1$ in (4.10) and noting that $u(b) \neq 0$ we have

$$\int_{a_2}^{b_1} \exp(\int_{a_2}^s u^2 w) Q'(s)\, ds = 0. \tag{4.11}$$

Assume Q has no extrema in (a_2, b_1), then Q is monotone in (a_2, b_1). If $Q' \geq 0\, a.e.$ and Q' is not equal to $0\, a.e.$ in (a_2, b_1), then

$$\int_{a_2}^{b_1} \exp(\int_{a_2}^s u^2 w) Q'(s)\, ds > 0. \tag{4.12}$$

This is a contradiction. We reach a similar contradiction if we assume that $Q' \leq 0$. If $Q' = 0\, a.e.$ in (a_2, b_1), then by (4.10) $\lambda'(b) = 0$ in (a_2, b_1) and λ is constant in $I_1 \cup I_2$, contradicting that I_1 and I_2 are different relative extrema of λ.

3. By part 1, we have if there is an I_λ between I_1 and I_2, then $I_1 < I_\lambda < I_2$. Assume there exist $J_1, J_2 \in E(\lambda, B)$ and $I_1 < J_1 < J_2 < I_2$. Then by part 2, there exists an $I_3 \in E(Q, B)$ such that $J_1 < I_3 < J_2$ contradicting the fact that I_1 and I_2 are consecutive extrema. The second statement follows immediately from (4.10).

4. This follows from parts 2 and 3. ∎

The next two theorems extend the results of sections 3 and 4 of [3]. More specifically Propositions 3.4, 3.7, 3.8, 4.9, Theorems 4.1, 4.2, 4.3 and 4.8. We notice that since our theorems are based on Lemma 4.1, the conditions involved are consequently much weaker than those in [3].

Theorem 4.3 *Assume the conditions and notation of Lemma 4.1 hold and $p(b) > 0$ a.e. in (a, B). Let $\lambda_n = \lambda_n^N(b)$ be the nth Neumann eigenvalue. Then for $n \in \mathbb{N}_0$*

1. *There exists a positive number K_n such that $\lambda_n(b) < K_n$ for all $b \in (a_1, B)$ where $a_1 > a$.*

2. *If $\lim_{b \to B^-} Q(b) = Q(B^-)$ exists, then $\lim_{b \to B^-} \lambda_n(b) = \lambda_n(B^-)$ exists, and $\lambda_n(B^-) \in [Q(B^-), K_n]$.*

Proof: This follows from Lemma 4.1 and Theorem 4.2 by an argument similar to that in [3], see p. 257. ∎

Theorem 4.4 *Assume the conditions and notation of Lemma 4.1 hold and $p(b) \geq k > 0$ for $b \in (a, B)$. Let $\lambda_n = \lambda_n^N(b)$ be the nth Neumann eigenvalue. Then*

1. $\lim_{b \to a+} \lambda_0(b) = Q(a) = q(a)/w(a)$.

2. $\lim_{b \to a+} \lambda_n(b) = +\infty$, *for* $n = 1, 2, 3, \ldots$.

3. *If Q is decreasing in (a, B), then for $n \in \mathbb{N}_0$, $\lambda_n(b)$ is decreasing in (a, B) and $\lambda_n(b) \geq Q(b)$.*

4. *If Q is increasing in (a, B) and $\lim_{b \to B^-} Q(b) = +\infty$, then $\lambda_0(b)$ is increasing in (a, B) and $\lambda_0(b) \leq Q(b)$, and for $n \in \mathbb{N}_0$, λ_n have a unique extremum in (a, B) and this extremum is a strict minimum.*

5. *If Q has a unique extremum in (a, B) and this extremum is a strict minimum, and $\lim_{b \to B^-} Q(b) = +\infty$, then for $n \in \mathbb{N}_0$, $\lambda_n(b)$ has a unique extremum in (a, B) and this extremum is a strict minimum.*

Proof: This follows from Lemma 4.1 and Theorems 4.2 and 4.3 by an argument similar to that in [3], see p. 256-259. ∎

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