Eigenvalues of regular Sturm-Liouville problems

Q. Kong and A. Zettl*

Abstract

The eigenvalues of Sturm-Liouville (SL) problems depend not only continuously but smoothly on the problem. An expression for the derivative of the n-th eigenvalue with respect to a given parameter: an endpoint, a boundary condition constant, a coefficient or weight function, is found.

1 Introduction

For a regular SL problem

$$-(py')' + qy = \lambda wy \tag{1.1}$$

with Dirichlet boundary conditions (BC)

$$y(a) = 0 = y(b) \tag{1.2}$$

Pöschel and Trubowitz in [13], as part of their elegant exposition of inverse spectral theory, consider the n-th eigenvalue $\lambda = \lambda_n(q)$ as a function of q for $q \in L^2(a,b)$, $p = 1 = w$, and show that λ is Frechet differentiable with derivative given by (in our notation)

$$d\lambda_q(h) = \int_a^b u^2 h, \;\; h \in L^2(a,b) \tag{1.3}$$

where u is a normalized eigenfunction of λ. Their proof is long and technical. It is based on functional analysis in the Hilbert space $L^2(a,b)$, complex variable theory, and the asymptotic form of solutions for $|\lambda| \to \infty$.

Dauge and Helffer in [7] show that the Neumann eigenvalues of a regular SL problem on an interval $[a,b]$ are differentiable functions of the right endpoint b satisfying a differential equation of the form

$$\lambda' = u^2(q - \lambda w); \tag{1.4}$$

and they indicate that a similar equation holds for the eigenvalues of other separated BC. These authors also point out that, with the exception of a special result for the Coulomb Hamiltonian, such equations seem not to have been known previously.

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In [9] we give a different proof of the Dauge - Helffer Theorem with substantially weaker hypotheses and we obtained a similar result for coupled BC.

Here we show that the eigenvalues of regular SL problems are differentiable functions of all the data: the endpoints, the boundary conditions, as well as the coefficients and the weight function and we find expressions for their derivatives. Differentiability with respect to a coefficient p, q or weight function w is in the sense of the Frechet derivative in the Banach space $L^1(a,b)$. We maintain that $L^1(a,b)$ - not $L^2(a,b)$ - is the " natural" setting for the regular SL theory. This is because the condition $1/p$, q, $w \in L^1(a,b)$ is necessary and sufficient for initial value problems to have unique solutions - see Everitt and Race [8] and [10].

Our proof is elementary - given the continuous dependence of the eigenvalues. The latter seems to be a part of the folklore of Mathematics and so we provide only an outline of a proof. Besides its theoretical importance, the continuous dependence of the eigenvalues and the eigenfunctions on the data is fundamental from the numerical point of view. The major general purpose codes for the numerical computation of the eigenvalues and eigenfunctions of SL problems - SLEIGN [5], the Fulton and Pruess code SLEDGE, the NAG library code [14] and SLEIGN2 [2], [3] and [4], are based on it.

As a consequence of our main result - Theorem 4.2 - it follows that the convergence of the approximations based on small changes of the data is at least of order $o(h)$ as $h \to 0$.

In section 2 we establish the notation, the continuity of the eigenvalues and eigenfunctions is discussed in section 3, followed by our main result on the differentiability of the eigenvalues in section 4.

2 Notation

Consider the differential equation

$$-(py')' + qy = \lambda w\, y \quad \text{on } (a', b'),\ -\infty \leq a' < b' \leq \infty \text{ with } \lambda \in \mathbb{R} \tag{2.1}$$

where

$$p, q, w : (a', b') \to \mathbb{R}, \quad 1/p,\ q,\ w \in L_{loc}(a', b'), \quad w > 0 \ a.e.\ on\ (a', b')\,. \tag{2.2}$$

Let

$$I = [a, b], \quad a' < a < b < b', \tag{2.3}$$

and consider the BC

$$A \begin{pmatrix} y(a) \\ (py')(a) \end{pmatrix} + B \begin{pmatrix} y(b) \\ (py')(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.4}$$

where the *complex* 2×2 matrices A and B satisfy:

$$\text{The } 2 \times 4 \text{ matrix } (A|B) \text{ has full rank,} \tag{2.5}$$

and

$$AEA^* = BEB^*, \quad E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.6}$$

By a solution of (2.1) on (a', b') is meant a function $y \in AC_{loc}(a', b')$ such that $py' \in AC_{loc}(a', b')$ and the equation (2.1) is satisfied *a.e.* on (a', b'). Here $AC_{loc}(a', b')$ denotes the set of complex valued functions which are absolutely continuous on all compact subintervals of (a', b'). Clearly a solution of (2.1) on (a', b') is also a solution on any subinterval J of (a', b'). Note that the quasi-derivative notation $(py')(t)$ is needed in (2.1) and (2.4) since - under the conditions (2.2), (2.3) - $p(t)$ and $y'(t)$ may not both exist but the product function $(py')(t)$ exists and is continuous for all $t \in (a', b')$.

A SL boundary value problem (BVP) consists of equation (2.1) together with BC (2.4)-(2.6). With conditions (2.2) and (2.3) it is well known that this problem is a regular self-adjoint SL problem which has an infinite but countable number of only real eigenvalues. In this paper we fix all but one of the parameters that determine the SL problem, i.e., all but one of a, b, A, B, p, q, w and study the dependence of the eigenvalues and eigenfunctions on that parameter.

For our purposes here it is convenient to divide these self-adjoint boundary conditions (2.4)-(2.6) into three disjoint and mutually exclusive subclasses and to use the following canonical representations of these subclasses:

1. **Separated self-adjoint BC.** These are

$$A_1 y(a) + A_2 (py')(a) = 0 \quad \text{where } A_1 \text{ and } A_2 \text{ are real and not both zero,} \tag{2.7}$$

$$B_1 y(b) + B_2 (py')(b) = 0 \quad \text{where } B_1 \text{ and } B_2 \text{ are real and not both zero.} \tag{2.8}$$

These separated conditions can be parameterized as follows:

$$\cos\alpha\, y(a) \;-\; \sin\alpha\, (py')(a) \;=\; 0, \quad 0 \le \alpha < \pi; \tag{2.9}$$

$$\cos\beta\, y(b) \;-\; \sin\beta\, (py')(b) \;=\; 0, \quad 0 < \beta \le \pi. \tag{2.10}$$

Note the different normalization in (2.10) for β than that used for α in (2.9). This is for convenience in stating some of the results below.

2. **All *real* coupled self-adjoint BC.** These can be formulated as follows:

$$\begin{pmatrix} y(b) \\ (py')(b) \end{pmatrix} = K \begin{pmatrix} y(a) \\ (py')(a) \end{pmatrix} \tag{2.11}$$

where $K \in SL_2(\mathbb{R})$, i.e. K satisfies

$$K = \begin{pmatrix} k_{11}, & k_{12} \\ k_{21}, & k_{22} \end{pmatrix}, \; k_{ij} \in \mathbb{R}, \; \det K = 1. \tag{2.12}$$

3. **All** *complex* **coupled self-adjoint BC.** These can be formulated as follows:

$$\begin{pmatrix} y(b) \\ (py')(b) \end{pmatrix} = \exp(i\,\theta)\,K \begin{pmatrix} y(a) \\ (py')(a) \end{pmatrix} \tag{2.13}$$

where K satisfies (2.12) and $-\pi < \theta < 0$, or $0 < \theta < \pi$.

Most of the following results are well-known. See [16] for some proofs with only integrable coefficients; see [11] for the case when p changes sign, and see [3] for the case of complex coupled BC.

Basic results and notation. Let (2.2) hold.

- (a) Assume that

$$p \geq 0 \ a.e. \text{ on } [a,b]. \tag{2.14}$$

Then

1. The BVP (2.1), (2.9) and (2.10) has only real and simple eigenvalues; there are an infinite but countable number of them; they are bounded below and can be ordered to satisfy

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \ldots; \text{ with } \lambda_n \to +\infty \text{ as } n \to \infty. \tag{2.15}$$

If u_n is an eigenfunction of λ_n, then u_n is unique up to constant multiples and u_n has exactly n zeros in the open interval (a,b), $n \in \mathbb{N}_0 = \{0,1,2,\ldots\}$.

Notation. Let

$$\lambda_n = \lambda_n(a,b,\alpha,\beta,1/p,q,w); \quad u_n = u_n(\cdot,a,b,\alpha,\beta,1/p,q,w), \quad n \in \mathbb{N}_0, \tag{2.16}$$

to highlight the dependence on these quantities.

2. The BVP (2.1), (2.11) and (2.12) has only real eigenvalues; each of these may be simple or double; there are an infinite but countable number of them and they can be ordered to satisfy

$$-\infty < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots; \text{ with } \lambda_n \to +\infty \text{ as } n \to \infty. \tag{2.17}$$

Notation. Let

$$\lambda_n = \lambda_n(a,b,K,1/p,q,w); \quad u_n = u_n(\cdot,a,b,K,1/p,q,w), \quad n \in \mathbb{N}_0. \tag{2.18}$$

Note that there is some arbitrariness in the indexing of the eigenfunctions corresponding to a double eigenvalue.

3. The BVP (2.1), (2.12), (2.13) has only real and simple eigenvalues; there are an infinite but countable number of them and they can be ordered to satisfy

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \ldots; \text{ and } \lambda_n \to +\infty \text{ as } n \to \infty. \tag{2.19}$$

Notation. Denote these eigenvalues by

$$\lambda_n = \lambda_n(a, b, \theta, K, 1/p, q, w); \quad u_n = u_n(\cdot, a, b, \theta, K, 1/p, q, w), \quad n \in \mathbb{N}_0. \tag{2.20}$$

If we fix all variables except θ and shorten the notation to $\lambda_n = \lambda_n(\theta)$, then we have

$$\lambda_n(-\theta) = \lambda_n(\theta), \tag{2.21}$$

and the complex conjugate of an eigenfunction of $\lambda_n(\theta)$ is an eigenfunction of $\lambda_n(-\theta)$.

- (b) Assume that p changes sign in the interval $[a, b]$, i.e. p is positive on a subset of $[a, b]$ of positive Lebesgue measure and p is negative on a subset of the interval $[a, b]$ of positive Lebesgue measure. Then

1. The BVP (2.1), (2.9) and (2.10) has only real and simple eigenvalues; there are an infinite but countable number of them; they are unbounded below and above and can be ordered to satisfy

$$\ldots < \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < \ldots;$$
$$\lambda_n \to +\infty \text{ as } n \to \infty; \text{ and } \lambda_n \to -\infty \text{ as } n \to -\infty. \tag{2.22}$$

2. The BVP (2.1), (2.11) and (2.12) has only real eigenvalues; each of these may be simple or double; there are an infinite but countable number of them; they are unbounded below and above and can be ordered to satisfy

$$\ldots \le \lambda_{-2} \le \lambda_{-1} \le \lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots;$$
$$\lambda_n \to +\infty \text{ as } n \to \infty, \text{ and } \lambda_n \to -\infty \text{ as } n \to -\infty. \tag{2.23}$$

3. The BVP (2.1), (2.12), (2.13) has only real and simple eigenvalues; there are an infinite but countable number of them; they are unbounded below and above and can be ordered to satisfy:

$$\ldots < \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < \ldots;$$
$$\lambda_n \to +\infty \text{ as } n \to \infty, \text{ and } \lambda_n \to -\infty \text{ as } n \to -\infty. \tag{2.24}$$

The notations for eigenvalues λ_n and eigenfunctions u_n, $n \in \mathbb{Z}$, for part (b) are the same as those introduced in part (a) for $n \in \mathbb{N}_0$.

In the following we denote by λ_n and u_n the n-th eigenvalue and the n-th eigenfunction of a SL problem where $n \in \mathbb{N}_0$ if $p_0 \ge 0\, a.e.$ on $[a, b]$ and $n \in \mathbb{Z}$ if p_0 changes sign on $[a, b]$, respectively.

3 Continuity of Eigenvalues and Eigenfunctions

In this section we show that the eigenvalues are continuous functions of all the parameters of the problem including the coefficients and that normalized eigenfunctions can be found which depend continuously on all parameters in the uniform norm. Let

$$\boldsymbol{\Omega} = \{\boldsymbol{\omega} = (a, b, A, B, 1/p, q, w)\} \tag{3.1}$$

such that (2.2), (2.3), (2.5), (2.6) hold. For the special case of separated BC (2.9), (2.10) we also use the notation

$$\boldsymbol{\Omega}_s = \{\boldsymbol{\omega} = (a, b, \alpha, \beta, 1/p, q, w)\} \tag{3.2}$$

and for the coupled cases (2.11), (2.12) and (2.13) we let

$$\boldsymbol{\Omega}_c = \{\boldsymbol{\omega} = (a, b, \theta, K, 1/p, q, w)\}. \tag{3.3}$$

When $\theta = 0$ we shorten (3.3) to

$$\boldsymbol{\Omega}_{rc} = \{\boldsymbol{\omega} = (a, b, K, 1/p, q, w)\}. \tag{3.4}$$

We want to show that the eigenvalues and eigenfunctions depend continuously on the problem, i.e., a small change of the problem results in a small change of each eigenvalue and each eigenfunction. This means we have to compare the spectrum of different problems which may be defined on different intervals. Each $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ determines a unique SL problem: a, b the interval, A, B the boundary condition, and the restrictions of p, q, w on $[a, b]$ the equation. Observe that the values of p, q, w outside the interval $[a, b]$, i.e. in $(a', b') \setminus [a, b]$, do not affect the spectrum of the problem determined by $\boldsymbol{\omega}$. To account for this and to facilitate comparisons between eigenvalues of problems defined on different intervals we let

$$\tilde{\boldsymbol{\Omega}} = \{\tilde{\boldsymbol{\omega}} = (a, b, A, B, \widetilde{1/p}, \tilde{q}, \tilde{w})\} \tag{3.5}$$

where

$$\tilde{q} = \begin{cases} q & \text{on } [a, b] \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

and $\widetilde{1/p}$, $\tilde{w}$ are defined similarly. Now we introduce the Banach space

$$X = \mathbb{R} \times \mathbb{R} \times M_{2,2}(\mathbb{C}) \times M_{2,2}(\mathbb{C}) \times L^1(a', b') \times L^1(a', b') \times L^1(a', b') \tag{3.7}$$

with its "natural" norm

$$||\boldsymbol{\omega}|| = ||\tilde{\boldsymbol{\omega}}|| = |a| + |b| + ||A|| + ||B|| + \int_{a'}^{b'} \left(|\widetilde{1/p}| + |\tilde{q}| + |\tilde{w}|\right) \tag{3.8}$$

where $||A||$ is any fixed matrix norm. We maintain that this space X is the "natural" setting for the study of regular SL problems. Note that, since $1/p, q, w$ are only assumed to be in $L_{loc}(a', b')$,

$\mathbf{\Omega}$ is not a subset of X but $\tilde{\mathbf{\Omega}}$ is since $\widetilde{1/p}, \tilde{q}, \tilde{w}$ are in $L^1(a', b')$. Now we identify $\mathbf{\Omega}$ with $\tilde{\mathbf{\Omega}}$ as a subset of X. Then $\mathbf{\Omega}$ inherits the norm from X, and the convergence in $\mathbf{\Omega}$ is determined by this norm. It is easy to see that every point in $\mathbf{\Omega}$ is an accumulation point of $\mathbf{\Omega}$ with respect to the norm in X.

The eigenvalues of a regular SL problem depend continuously on the problem. More precisely we have

Theorem 3.1 *Let $\boldsymbol{\omega}_0 = (a_0, b_0, A_0, B_0, 1/p_0, q_0, w_0) \in \mathbf{\Omega}$. Let $\lambda = \lambda_n(\boldsymbol{\omega}_0)$ be the n-th eigenvalue of the SL problem (2.1), (2.4)-(2.6). Then λ is continuous at $\boldsymbol{\omega}_0$. That is, given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\boldsymbol{\omega} \in \mathbf{\Omega}$ satisfies*

$$||\boldsymbol{\omega} - \boldsymbol{\omega}_0|| = |a - a_0| + |b - b_0| + ||A - A_0|| + ||B - B_0|| + \int_{a'}^{b'} \left(|\widetilde{1/p} - \widetilde{1/p_0}| + |\tilde{q} - \tilde{q}_0| + |\tilde{w} - \tilde{w}_0| \right) < \delta, \tag{3.9}$$

then

$$|\lambda(\boldsymbol{\omega}) - \lambda(\boldsymbol{\omega}_0)| \ < \ \epsilon. \tag{3.10}$$

A proof can be based on the Green's function $G(t, s, \lambda)$ using Neuberger's constuction in [12], see also [6], [17], [18]. For fixed a, b, w and assuming that zero in not an eigenvalue, then all the eigenvalues of the SL problem are the reciprocals of the eigenvalues of the integral operator T whose kernel is the Green's function. By [10] the Green's function depends continuously on all the data. Now one can appeal to a result on the spectrum of a convergent sequence of self-adjoint operators in Hilbert space - Theorem 7.35 in [15] - to conclude that the eigenvalues of T, and hence also their reciprocals, which are the eigenvalues of the SL problem, are continuous functions of the data. If zero is an eigenvalue, one translates the problem to an equivalent one which does not have zero as an eigenvalue and proceeds as above.

This argument has to be modified when a, b or w are also allowed to vary since then the Hilbert space changes when a, b or w changes. This can be done as in [1]; see in particular the proof of Theorem 4.1 and Remark 2 on p.16 following this proof.

Next we state two lemmas needed in the later proofs which are also of independent interest. The first states that the unique solution of any initial value problem of equation (2.1) depends continuously on all parameters including the coefficients and the weight function in the "natural" norm.

Lemma 3.1 *Let (2.2) hold, let $c \in (a', b')$ and $h, k \in \mathbb{C}$. Consider the initial value problem consisting of equation (2.1) and the initial conditions*

$$y(c) = h, \ (py')(c) = k.$$

Then the unique solution $y = y(\cdot, c, h, k, 1/p, q, w)$ is a continuous function of all its variables. More *precisely, given $\epsilon > 0$ and any compact subinterval J of (a', b') there exists a $\delta > 0$ such that if*

$$|c - c_0| + |h - h_0| + |k - k_0| + \int_a^b (|1/p - 1/p_0| + |q - q_0| + |w - w_0|) < \delta, \tag{3.11}$$

then

$$|y(t, c, h, k, 1/p, q, w) \; - \; y(t, c_0, h_0, k_0, 1/p_0, q_0, w_0)| \; < \; \epsilon \tag{3.12}$$

and

$$|(py')(t, c, h, k, 1/p, q, w) \; - \; (py')(t, c_0, h_0, k_0, 1/p_0, q_0, w_0)| \; < \; \epsilon \tag{3.13}$$

for all $t \in J$.

Proof: This follows from Theorem 2.7 in [10]. ∎

As a consequence of Theorem 3.1 and Lemma 3.1 we obtain

Lemma 3.2 *Let $\boldsymbol{\omega}_0 = (a_0, b_0, A_0, B_0, 1/p_0, q_0, w_0) \in \mathbf{\Omega}$. Let $\lambda = \lambda_n(\boldsymbol{\omega})$ be the n-th eigenvalue of the SL problem (2.1), (2.4)-(2.6). If $\lambda(\boldsymbol{\omega}_0)$ is simple, then there exists a neighborhood M of $\boldsymbol{\omega}_0$ in $\mathbf{\Omega}$ such that $\lambda(\boldsymbol{\omega})$ is simple for every $\boldsymbol{\omega}$ in M. In particular we have the following:*

1. *Fix $a, b, 1/p, q, w$ and consider $\lambda = \lambda(K)$ as a function of K for $K \in SL_2(\mathbb{R})$. Assume that for some $K_0 \in SL_2(\mathbb{R})$, $\lambda(K_0)$ is a simple eigenvalue. Then there exists a neighborhood M of K_0 in $SL_0(\mathbb{R})$ such that $\lambda(K)$ is simple for every $K \in M$.*

2. *Fix a, b, K, q, w and consider $\lambda = \lambda(1/p)$ as a function of $1/p$ for $1/p \in L^1(a, b)$. Assume that for some $1/p_0 \in L^1(a, b)$, $\lambda(1/p_0)$ is a simple eigenvalue. Then there exists a neighborhood M of $1/p_0$ in $L^1(a, b)$ such that $\lambda(1/p)$ is simple for every $1/p \in M$.*

3. *Fix $a, b, 1/p, K, w$ and consider $\lambda = \lambda(q)$ as a function of q for $q \in L^1(a, b)$. Assume that for some $q_0 \in L^1(a, b)$, $\lambda(q_0)$ is a simple eigenvalue. Then there exists a neighborhood M of q_0 in $L^1(a, b)$ such that $\lambda(q)$ is simple for every $q \in M$.*

4. *Fix $a, b, 1/p, q, K$ and consider $\lambda = \lambda(w)$ as a function of w for $w \in L^1(a, b)$. Assume that for some $w_0 \in L^1(a, b)$, $\lambda(w_0)$ is a simple eigenvalue. Then there exists a neighborhood M of w_0 in $L^1(a, b)$ such that $\lambda(w)$ is simple for every $w \in M$.*

Remark 3.1 *It has been shown in [9] that for fixed $b, A, B, 1/p, q, w$, $\lambda(a)$ is simple for some a if and only if it is simple for all a and similarly for b. Also, only for real coupled BC is Lemma 3.2 nontrivial since only for these is it possible to have double eigenvalues. By part (1) the set S of points $K \in SL_2(\mathbb{R})$ such that $\lambda(K)$ is simple is an open set in $SL_2(\mathbb{R})$. Hence its complement - the set D of points K for which $\lambda(K)$ is a double eigenvalue - is a closed set in $SL_2(\mathbb{R})$.*

By part (2) the set S of points $q \in L^1(a, b)$ such that $\lambda(q)$ is simple is an open set in $L^1(a, b)$. Hence its complement - the set D of points $q \in L^1(a, b)$ for which $\lambda(q)$ is a double eigenvalue - is *a closed set in $L^1(a,b)$. We will see by Theorem 4.3 that this set D is nowhere dense in the space $L^1(a,b)$. This remark also applies to cases (3) and (4).*

Proof of Lemma 3.2: For a solution y of (2.1) and an eigenfunction $u(\cdot,\boldsymbol{\omega})$ of a SL problem define

$$Y = \begin{pmatrix} y \\ py' \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} u \\ pu' \end{pmatrix} \tag{3.14}$$

to be the corresponding vector solution and vector eigenfunction, respectively. Since the eigenvalues for separated BC and for complex coupled BC are always simple, Lemma 3.2 is clear for these cases. Assume that $\lambda(\boldsymbol{\omega}_0)$ is simple for $\boldsymbol{\omega}_0 = (a_0, b_0, K_0, 1/p_0, q_0, w_0) \in \boldsymbol{\Omega}_{rc}$. Suppose the conclusion is false, then we have $\boldsymbol{\omega}_k \to \boldsymbol{\omega}_0$ for some sequence $\{\boldsymbol{\omega}_k\} \subset \boldsymbol{\Omega}_{rc}$ with $\lambda(\boldsymbol{\omega}_k)$ a double eigenvalue for each $k \in \mathbb{N}$. Choose linearly independent vectors v_1, v_2 in $\mathbb{R}^2$ and determine the vector solutions $U^1(\cdot,\boldsymbol{\omega}_k)$ and $U^2(\cdot,\boldsymbol{\omega}_k)$ of (2.1) with $\lambda = \lambda(\boldsymbol{\omega}_k)$ and the initial conditions

$$U^1(a,\boldsymbol{\omega}_k) = v_1, \quad U^2(a,\boldsymbol{\omega}_k) = v_2.$$

Then $U^1(\cdot,\boldsymbol{\omega}_k)$ and $U^2(\cdot,\boldsymbol{\omega}_k)$ are vector eigenfunctions satisfying the boundary condition

$$U^j(b,\boldsymbol{\omega}_k) = K_k U^j(a,\boldsymbol{\omega}_k), \quad k \in \mathbb{N}, \quad j = 1,2. \tag{3.15}$$

Letting $k \to \infty$ in (3.15) and using Theorem 3.1 and Lemma 3.1 we may conclude that

$$Y^j(b,\boldsymbol{\omega}_k) = K_0 Y^j(a,\boldsymbol{\omega}_0), \quad j = 1,2 \tag{3.16}$$

where Y^j is the uniform limit of U^j as $k \to \infty$ for $j = 1,2$. Thus the top components of Y_j, $j = 1,2$, are two linearly independent eigenfunctions of $\lambda(\boldsymbol{\omega}_0)$. This contradiction completes the proof. ▮

By a normalized eigenfunction u of an SL problem we mean an eigenfunction u that satisfies

$$\int_a^b |u|^2 w = 1. \tag{3.17}$$

Next we prove a result for normalized eigenfunctions. Note that these are not uniquely determined by (3.17). In the case of a simple eigenvalue they are unique up to sign, but for a double eigenvalue there are pairs of linearly independent normalized eigenfunctions.

Theorem 3.2 *Let the notation and hypotheses of Theorem 3.1 hold.*

(i) Assume the eigenvalue $\lambda(\boldsymbol{\omega}_0)$ is simple for some $\boldsymbol{\omega}_0 \in \boldsymbol{\Omega}$ and let $u = u_n(\cdot,\boldsymbol{\omega}_0)$ denote a normalized eigenfunction of $\lambda(\boldsymbol{\omega}_0)$. Then there exist normalized eigenfunctions $u = u_n(\cdot,\boldsymbol{\omega})$ of $\lambda(\boldsymbol{\omega})$ for $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ such that

$$u(\cdot,\boldsymbol{\omega}) \to u(\cdot,\boldsymbol{\omega}_0), \quad (pu')(\cdot,\boldsymbol{\omega}) \to (pu')(\cdot,\boldsymbol{\omega}_0), \quad \textit{as } \boldsymbol{\omega} \to \boldsymbol{\omega}_0 \textit{ in } \boldsymbol{\Omega}, \tag{3.18}$$

both uniformly on any compact subinterval J of (a',b').

(ii) Assume that $\lambda(\boldsymbol{\omega})$ is a double eigenvalue for all $\boldsymbol{\omega}$ in some neighborhood M of $\boldsymbol{\omega}_0$ in $\boldsymbol{\Omega}$. Let $u = u(\cdot, \boldsymbol{\omega}_0)$ be any normalized eigenfunction of $\lambda(\boldsymbol{\omega}_0)$. Then there exist normalized eigenfunctions $u = (\cdot, \boldsymbol{\omega})$ of $\lambda(\boldsymbol{\omega})$ such that

$$u(\cdot,\, \boldsymbol{\omega}) \to u(\cdot,\, \boldsymbol{\omega}_0), \quad (pu')(\cdot,\, \boldsymbol{\omega}) \to (pu')(\cdot,\, \boldsymbol{\omega}_0), \quad \text{as } \boldsymbol{\omega} \to \boldsymbol{\omega}_0 \text{ in } \boldsymbol{\Omega}, \tag{3.19}$$

both uniformly on any compact subinterval J of (a', b'). Note that in this case, given two linearly independent normalized eigenfunctions u_j of $\lambda(\boldsymbol{\omega}_0)$ there exist a pair of linearly independent normalized eigenfunctions of $\lambda(\boldsymbol{\omega})$ one of which converges to u_1 and the other to u_2 as $\boldsymbol{\omega} \to \boldsymbol{\omega}_0$ in $\boldsymbol{\Omega}$.

Proof: (i) First we show that there exist (not necessarily normalized) eigenfunctions $u_n(\cdot, \boldsymbol{\omega})$ such that (3.18) holds uniformly on J. As before, for a solution y of (2.1) and an eigenfunction $u(\cdot, \boldsymbol{\omega})$ let (3.14) hold. Assume the boundary conditions are separated. Choose eigenfunctions $u = u_n(\cdot, \boldsymbol{\omega})$ for $\boldsymbol{\omega} \in \boldsymbol{\Omega}$, $\boldsymbol{\omega}$ near $\boldsymbol{\omega}_0$, all satisfying the same initial condition at $c \in (a, b)$. Then the uniform convergence $U(\cdot, \boldsymbol{\omega}) \to U(\cdot, \boldsymbol{\omega}_0)$ on J follows from Lemma 3.1 and Theorem 3.1.

Assume the boundary conditions are coupled with $-\pi < \theta \leq \pi$. Suppose $\lambda_n(\boldsymbol{\omega}_0)$ is simple. Then by Lemma 3.2 there exists a neighborhood M of $\boldsymbol{\omega}_0$ such that $\lambda(\boldsymbol{\omega})$ is simple for all $\boldsymbol{\omega} \in M$. For all $\boldsymbol{\omega} \in M$ choose an eigenfunction $u = u_n(\cdot, \boldsymbol{\omega})$ of $\lambda(\boldsymbol{\omega})$ satisfying

$$\|U(a_0, \boldsymbol{\omega})\| = |u(a_0,\, \boldsymbol{\omega})| + |(pu')(a_0,\, \boldsymbol{\omega})| = 1, \quad \text{and } u(t, \boldsymbol{\omega}) > 0 \text{ for } t \text{ near } a_0.$$

It suffices to show that

$$U(a_0, \boldsymbol{\omega}) \to U(a_0, \boldsymbol{\omega}_0) \text{ as } \boldsymbol{\omega} \to \boldsymbol{\omega}_0 \text{ in } \boldsymbol{\Omega} \tag{3.20}$$

since the uniform convergence on $[a_0, b_0]$ then follows from Theorem 3.1 and Lemma 3.1. If (3.20) does not hold, then there exists a sequence $\boldsymbol{\omega}_k \to \boldsymbol{\omega}_0$ such that

$$U(a_0, \boldsymbol{\omega}_0) - U(a_0, \boldsymbol{\omega}_k) := v_k \to v_0 \neq 0, \text{ as } \boldsymbol{\omega} \to \boldsymbol{\omega}_0. \tag{3.21}$$

Let Y_k, Z_k, Y be the vector solutions of (2.1) with the same $\boldsymbol{\omega} = \boldsymbol{\omega}_0$, $\lambda = \lambda(\boldsymbol{\omega}_0)$ determined by the initial conditions

$$Y_k(a_0) = v_k, \ Z_k(a_0) = U(a_0, \boldsymbol{\omega}_k), \ Y(a_0) = v_0, \quad k \in \mathbb{N},$$

respectively. Then by the uniqueness of solutions to initial value problems we have

$$Y_k = U(\cdot, \boldsymbol{\omega}_0) - Z_k, \quad \text{on the interval } [a_0, b_0]. \tag{3.22}$$

Using the BC (2.13) we get

$$\begin{aligned} Y_k(b_0) &= U(b_0, \boldsymbol{\omega}_0) - Z_k(b_0) = U(b_0, \boldsymbol{\omega}_0) - U(b_k, \boldsymbol{\omega}_k) + U(b_k, \boldsymbol{\omega}_k) - Z_k(b_0) \\ &= \exp(i\theta_0) K_0 U(a_0, \boldsymbol{\omega}_0) - \exp(i\theta_k) K_k U(a_k, \boldsymbol{\omega}_k) + U(b_k, \boldsymbol{\omega}_k) - Z_k(b_0) \\ &= \exp(i\theta_0) K_0 [U(a_0, \boldsymbol{\omega}_0) - U(a_0, \boldsymbol{\omega}_k)] + \exp(i\theta_0) K_0 U(a_0, \boldsymbol{\omega}_k) - \exp(i\theta_k) K_k U(a_k, \boldsymbol{\omega}_k) \end{aligned}$$

$$
\begin{aligned}
&+U(b_k,\boldsymbol{\omega}_k) - Z_k(b_0)\\
= \quad & \exp(i\theta_0)K_0Y_k(a_0) + \exp(i\theta_0)K_0U(a_0,\boldsymbol{\omega}_k) - \exp(i\theta_k)K_kU(a_k,\boldsymbol{\omega}_k)\\
&+U(b_k,\boldsymbol{\omega}_k) - Z_k(b_0). \qquad (3.23)
\end{aligned}
$$

Letting $k \to \infty$ in (3.23) and using Lemma 3.1 we get

$$Y(b_0) = \exp(i\theta_0)K_0Y(a_0).$$

Since $Y(a_0) = v_0 \neq 0$, Y is a nontrivial vector eigenfunction corresponding to the eigenvalue $\lambda(\boldsymbol{\omega}_0)$. Since $\lambda(\boldsymbol{\omega}_0)$ is simple, there is a constant $h \neq 0$ such that $Y = h\,U(\cdot,\boldsymbol{\omega}_0)$. In particular, $v_0 = Y(a_0) = h\,U(a_0,\boldsymbol{\omega}_0)$. Letting $k \to \infty$ in (3.21) we obtain that

$$U(a_0,\boldsymbol{\omega}_0) - \lim_{k\to\infty} U(a_0,\boldsymbol{\omega}_k) = v_0 = h\,U(a_0,\boldsymbol{\omega}_0),$$

i.e.

$$\lim_{k\to\infty} U(a_0,\boldsymbol{\omega}_k) = (1-h)U(a_0,\boldsymbol{\omega}_0).$$

Noting that $u(t,\boldsymbol{\omega}_k)$ and $u(t,\boldsymbol{\omega}_0)$ have the same sign for t near a_0, we have that $1-h > 0$ and hence

$$\lim_{k\to\infty} \|U(a_0,\boldsymbol{\omega}_k)\| = (1-h)\,\|U(a_0,\boldsymbol{\omega}_0)\|$$

which contradicts

$$\|U(a_0,\boldsymbol{\omega}_k)\| = \|U(a_0,\boldsymbol{\omega}_0)\| = 1.$$

(ii) Suppose $\lambda(\boldsymbol{\omega})$ is double for all $\boldsymbol{\omega}$ in some neighborhood M of $\boldsymbol{\omega}_0$. Then we can argue as before by choosing eigenfunctions $u(\cdot,\boldsymbol{\omega})$ of $\lambda(\boldsymbol{\omega})$ all of which satisfy the same initial condition at c for some $c \in (a,b)$ since a linear combination of two linearly independent eigenfunctions can be chosen to satisfy arbitrary initial conditions.

The above discussion shows that for every self-adjoint boundary condition and every fixed index n the eigenfunction $u_n(\cdot,\boldsymbol{\omega})$ and its quasi-derivative $(pu_n')(\cdot,\boldsymbol{\omega})$ are uniformly convergent in $\boldsymbol{\omega}$ on every compact subinterval of (a',b'). By normalizing the eigenfunctions we complete the proof. ∎

4 Differentiability Properties of Eigenvalues

In this section we show that the eigenvalues are differentiable functions of all the parameters of the problem including the coefficients. Recall the definition of the Frechet derivative:

Definition 1 A map T from a Banach space X into a Banach space Y is differentiable at a point $x \in X$ if there exists a bounded linear operator $dT_x : X \to Y$ such that for $h \in X$

$$|T(x+h) - T(x) - dT_x(h)| \;=\; o(h) \text{ as } h \to 0. \qquad (4.1)$$

Definition 2 We say that a function $f : (a, b) \to \mathbb{R}$ is nonoscillatory, or NO for short, at a point $c \in (a, b)$ if there is some positive number δ such that f is nonnegative or nonpositive on $(c - \delta, c)$ and on $(c, c + \delta)$; the sign of f need not be the same on these two intervals.

Theorem 4.1 *Let $\boldsymbol{\omega} = (a, b, A, B, 1/p, q, w) \in \boldsymbol{\Omega}$. Let $\lambda = \lambda_n(\boldsymbol{\omega})$ and let $u = u_n(\cdot, \boldsymbol{\omega})$ be a normalized eigenfuntion of λ for the BVP (2.1), (2.4)-(2.6).*

1. *Fix all the components of $\boldsymbol{\omega}$ except the left endpoint a and let $\lambda = \lambda(a)$ and $u = u(\cdot, a)$. Assume both p and q are nonoscillatory a.e. in $(a', b]$. Then λ is differentiable a.e. and*
$$\lambda'(a) = \frac{1}{p(a)}(pu')^2(a) - u^2(a)[q(a) - \lambda(a)w(a)] \quad a.e.\ in \ \ (a', b]. \tag{4.2}$$
In particular, if p, q, w are continuous and nonoscillatory at $a \in (a', b]$, then
$$\lambda'(a) = \frac{1}{p(a)}(pu')^2(a) - u^2(a)\,[q(a) - \lambda(a)w(a)]. \tag{4.3}$$

2. *Fix all the components of $\boldsymbol{\omega}$ except b and let $\lambda = \lambda(b)$ and $u = u(\cdot, b)$. Assume both p and q are nonoscillatory a.e. in $[a, b')$. Then λ is differentiable a.e. and*
$$\lambda'(b) = -\frac{1}{p(b)}(pu')^2(b) + u^2(b)\,[q(b) - \lambda(b)w(b)] \quad a.e.\ in \ \ [a, b'). \tag{4.4}$$
In particular, if p, q, w are continuous and nonoscillatory at $b \in [a, b')$, then
$$\lambda'(b) = -\frac{1}{p(b)}(pu')^2(b) + u(b)^2\,[q(b) - \lambda(b)w(b)]. \tag{4.5}$$

Proof: A proof is given in [9] for the case of separated BC - see Theorem 3.4 - and for the case of real coupled BC - see Theorem 3.5. The proof for the complex coupled case (2.13) is similar to that of the real coupled case after taking into account the fact that the eigenfunctions of $\lambda_n(\theta, K)$ and $\lambda_n(-\theta, K)$ are complex conjugates of each other. ▮

Remark 4.1 *It is interesting to note that (4.2) - (4.5) are independent of the particular normalized eigenfunction u. This is not surprising in the case of a simple eigenvalue since then the normalization condition (3.15) determines u uniquely up to sign. But in the case of a double eigenvalue this is rather surprising since then the normalization condition is satisfied by two linearly independent eigenfunctions.*

We now come to our main result.

Theorem 4.2 *Let $\boldsymbol{\omega} = (a, b, A, B, 1/p, q, w) \in \boldsymbol{\Omega}$. Let $\lambda = \lambda_n(\boldsymbol{\omega})$ and let $u = u_n(\cdot, \boldsymbol{\omega})$ be a normalized eigenfuntion of λ for the BVP (2.1),(2.4)-(2.6). Assume that either (i) $\lambda(\boldsymbol{\omega})$ is a simple eigenvalue or (ii) that $\lambda(\rho)$ is a double eigenvalue for each ρ in some neighborhood $M \subset \boldsymbol{\Omega}$ of $\boldsymbol{\omega}$. Then*

λ is continuously differentiable with respect to each variable α, β for the separated BC (2.9)(2.10), continuously differentiable with respect to each variable θ, K for the coupled BC (2.12)(2.13), and continuously differentiable with respect to each variable $1/p, q, w$ for the general BC (2.4)-(2.6) in the appropriate sense. The derivatives are given by:

1. *Fix all components of $\boldsymbol{\omega}$ except α and let $\lambda = \lambda(\alpha)$ and $u = u(\cdot, \alpha)$. Then λ is differentiable and*
$$\lambda'(\alpha) = -u^2(a) - (pu')^2(a), \quad 0 \leq \alpha < \pi. \tag{4.6}$$
2. *Fix all components of $\boldsymbol{\omega}$ except β and let $\lambda = \lambda(\beta)$ and $u = u(\cdot, \beta)$. Then λ is differentiable and*
$$\lambda'(\beta) = u^2(b) + (pu')^2(b), \quad 0 < \beta \leq \pi. \tag{4.7}$$
3. *Fix all components of $\boldsymbol{\omega}$ except θ and let $\lambda = \lambda(\theta)$ and $u = u(\cdot, \theta)$. Then λ is differentiable at θ for any θ satisfying $-\pi < \theta < 0$ or $0 < \theta < \pi$ and*
$$\lambda'(\theta) = -2\ \mathrm{Im}[u(b)(p\overline{u}')(b)], \tag{4.8}$$
where $\mathrm{Im}(z)$ denotes the imaginary part of z.
4. *Fix all components of $\boldsymbol{\omega}$ except K and let $\lambda = \lambda(K)$ and $u = u(\cdot, K)$. Assume K satisfies (2.12). Then λ is differentiable within $SL_2(\mathbb{R})$ and its Frechet derivative is given by:*
$$d\lambda_K(H) \;=\; [p\overline{u}'(b), -\overline{u}(b)]HK^{-1}\begin{pmatrix} u(b) \\ (pu')(b) \end{pmatrix},\ H \in M_{2,2}(\mathbb{R}) \text{ such that } K + H \in SL_2(\mathbb{R}). \tag{4.9}$$
Note that the phrase "differentiable within $SL_2(\mathbb{R})$" above is given as an indication that Definition 1 needs to be modified since (4.9) does not hold for all H in the Banach space $M_{2,2}(\mathbb{R})$ but only for those as indicated.
5. *Fix all components of $\boldsymbol{\omega}$ except $1/p$ and consider λ as a function of $1/p \in L^1(a,b)$. Then λ is Frechet differentiable and its Frechet derivative is given by:*
$$d\lambda_{(1/p)}(h) = \int_a^b |pu'|^2 h,\ h \in L^1(a,b). \tag{4.10}$$
6. *Fix all components of $\boldsymbol{\omega}$ except q and consider λ as a function of $q \in L^1(a,b)$. Then λ is Frechet differentiable and its Frechet derivative is given by:*
$$d\lambda_q(h) = \int_a^b |u|^2 h,\ h \in L^1(a,b). \tag{4.11}$$
7. *Fix all components of $\boldsymbol{\omega}$ except w and consider λ as a function of $w \in L^1(a,b)$. Then λ is differentiable and its Frechet derivative is given by :*
$$d\lambda_w(h) = \lambda \int_a^b |u|^2 h,\ h \in L^1(a,b). \tag{4.12}$$

Remark 4.2 *1. The assumption that either $\lambda(\boldsymbol{\omega})$ is simple or $\lambda(\rho)$ is double for each ρ in a neighborhood of $\boldsymbol{\omega}$ is automatically satisfied for any separated BC and any complex coupled BC. However, it is not always satisfied for the real coupled BC (2.11), (2.12). In that case, the question arises: Do (4.9)-(4.12) hold at $\boldsymbol{\omega}_0$ when $\lambda(\boldsymbol{\omega}_0)$ is a double eigenvalue but there does not exist a neighborhood M of $\boldsymbol{\omega}_0$ such that $\lambda(\boldsymbol{\omega})$ is double for every $\boldsymbol{\omega} \in M$?*

2. If $q_k \to q$ in $L^1(a,b)$ then, by Theorem 3.1 we have that $\lambda(q_k) \to \lambda(q)$. Similarly for $1/p$ and w. Theorem 4.2 implies that this convergence is at least $o(h)$. As mentioned in the introduction this remark is of some significance in numerical analysis. For example, see the Fulton and Pruess software package SLEDGE (available from netlib) which computes numerical approximations of the eigenvalues of SL problems by using piecewise constant approximations of the coefficients p, q and the weight function w. Theorem 4.2 is of similar significance for other software packages for the numerical computation of eigenvalues (and eigenfunctions) of SL problems; in particular, the Bailey, Gordon and Shampine designed code SLEIGN [5], the NAG code designed by Pryce and Marletta - see [14], and the Bailey, Garbow, Everitt, Zettl code SLEIGN2 [2], see also [3], [4]. These are the major general purpose and state of the art codes for the numerical computation of the eigenvalues and eigenfunctions of SL problems.

Proof of Theorem 4.2: First we establish (4.6) to (4.12). Since the proofs of (4.6) and (4.7) are similar we just prove (4.7). Also we assume $\beta \neq \pi/2$, the proof for the case $\beta = \pi/2$ is similar. Let $u = u_n(\cdot, \beta)$ and $v = u_n(\cdot, \beta + h)$ denote normalized real valued eigenfunctions of $\lambda = \lambda_n(\beta)$ and $\lambda = \lambda_n(\beta + h)$, respectively, for $h \in \mathbb{R}$ sufficiently small. From (2.1) we get

$$[\lambda(\beta + h) - \lambda(\beta)]uvw = -u(pv')' + v(pu')' = -[u, v]'$$

where $[u, v] := u(pv') - v(pu')$ is the usual Lagrange bracket. From the BC (2.9), (2.10) and an integration we get

$$[\lambda(\beta + h) - \lambda(\beta)] \int_a^b uvw = -[u, v](b) = [v(pu') - u(pv')](b) = [\tan(\beta + h) - \tan\beta](pu')(b)(pv')(b). \tag{4.13}$$

Dividing both sides of (4.13) by h and taking the limit as $h \to 0$ we obtain

$$\lambda'(\beta) = \sec^2\beta\,(pu')^2(b) = \tan^2\beta\,(pu')^2(b) + (pu')^2(b) = u^2(b) + (pu')^2(b). \tag{4.14}$$

This completes the proof of (4.7).

To establish (4.8) and (4.9) note that (2.13) implies that for any eigenfunction u of the BVP (2.1), (2.12) and (2.13)

$$(pu', -u)(b) = (u, pu')(b) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp(i\theta)(u, pu')(a)\, K^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \exp(i\theta)(pu', -u)(a) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} K^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp(i\theta)(pu', -u)(a)\, K^{-1},$$

or

$$(p\overline{u}', -\overline{u})(a) = \exp(i\theta)(p\overline{u}', -\overline{u})(b)\,K.$$

To prove (4.8), let $\lambda(\theta) = \lambda_n(\theta, K)$, and let $u = u_n(\cdot,\,\theta),\ v = u_n(\cdot,\,\theta + h)$ for small $h \in \mathbb{R}$. Similar to (4.13) we have

$$\begin{aligned}
&[\lambda(\theta+h) - \lambda(\theta)] \int_a^b uvw = -\,[u,\overline{v}]_a^b = -[u(p\overline{v}') - \overline{v}(pu')]_a^b \\
&= -\left[(p\overline{v}', -\overline{v}) \begin{pmatrix} u \\ pu' \end{pmatrix}\right]_a^b = -(p\overline{v}', -\overline{v})(b) \begin{pmatrix} u \\ pu' \end{pmatrix}(b) + (p\overline{v}', -\overline{v})(a) \begin{pmatrix} u \\ pu' \end{pmatrix}(a) \\
&= -\exp(i\theta)(p\overline{v}', -\overline{v})(b)\,K \begin{pmatrix} u \\ pu' \end{pmatrix}(a) + \exp(i(\theta+h))(p\overline{v}', -\overline{v})(b)\,K \begin{pmatrix} u \\ pu' \end{pmatrix}(a) \\
&= \exp(i\theta)(p\overline{v}', -\overline{v})(b)\,K \begin{pmatrix} u \\ pu' \end{pmatrix}(a)[\exp(ih) - 1].
\end{aligned}$$

Dividing both sides by h and letting $h \to 0$ we get

$$\begin{aligned}
\lambda'(\theta) &= i\exp(i\theta)(p\overline{u}', -\overline{u})(b)\,K \begin{pmatrix} u \\ pu' \end{pmatrix}(a) = i(p\overline{u}', -\overline{u})(b) \begin{pmatrix} u \\ pu' \end{pmatrix}(b) \\
&= i[u(p\overline{u}' - \overline{u}(pu')](b) = -2\,\mathrm{Im}\left(u(p\overline{u}')(b)\right).
\end{aligned}$$

To establish (4.9), let $u = u_n(\cdot, K), v = u_n(\cdot,\, K + H)$ for $K, K + H \in SL_2(\mathbb{R})$. Proceeding similarly to the argument above we obtain

$$\begin{aligned}
&[\lambda(K+H) - \lambda(K)] \int_a^b u\overline{v}w = -((p\overline{v}') - \overline{v})(b) \begin{pmatrix} u, \\ pu' \end{pmatrix}(b) + (p\overline{v}', -\overline{v})(a) \begin{pmatrix} u \\ pu' \end{pmatrix}(a) \\
&= -\exp(i\theta)(p\overline{v}', -\overline{v})(b)\,K \begin{pmatrix} u \\ pu' \end{pmatrix}(a) + \exp(i\theta)(p\overline{v}', -\overline{v})(b)\,(K+H) \begin{pmatrix} u \\ pu' \end{pmatrix}(a) \\
&= \exp(i\theta)(p\overline{v}', -\overline{v})(b)\,H \begin{pmatrix} u \\ pu' \end{pmatrix}(a) = (p\overline{v}', -\overline{v})(b)\,HK^{-1} \begin{pmatrix} u \\ pu' \end{pmatrix}(b) \\
&= (p\overline{u}', -\overline{u})(b)\,HK^{-1} \begin{pmatrix} u \\ pu' \end{pmatrix}(b) + (p\overline{v}' - p\overline{u}', \overline{v} - \overline{u})(b)\,HK^{-1} \begin{pmatrix} u \\ pu' \end{pmatrix}(b) \\
&= (p\overline{u}', -\overline{u})(b)\,HK^{-1} \begin{pmatrix} u \\ pu' \end{pmatrix}(b) + o(H).
\end{aligned}$$

Hence

$$\lambda(K+H) - \lambda(K) = (p\overline{u}', -\overline{u})(b)\,HK^{-1} \begin{pmatrix} u \\ pu' \end{pmatrix}(b) + o(H)$$

and (4.9) follows.

To prove (4.10), let $u = u_n(\cdot,\, 1/p),\ v = u_n(\cdot,\, 1/p_h)$ where $1/p_h = 1/p + h,\ h \in L^1(a, b)$. Then $1/p \in L^1(a, b)$ implies that $1/p_h \in L^1(a, b)$ and

$$p - p_h = pp_hh.$$

Using (2.1) and integration by parts we obtain

$$[\lambda(1/p_h) - \lambda(1/p)] \int_a^b u\overline{v}w = [-u(p_h\overline{v}') + \overline{v}(pu')]_a^b + \int_a^b (pu')(p_h\overline{v}')h.$$

For all boundary conditions we have that

$$[-u(p_h\overline{v}') + \overline{v}(pu')]_a^b = 0.$$

Noting that $1/p_h \to 1/p$ as $h \to 0$ in $L^1(a,b)$ and using Theorem 3.1 and Lemma 3.1 we have

$$[\lambda(1/p+h) - \lambda(1/p)](1+o(1)) = \int_a^b |pu'|^2 h + o(h)$$

and consequently

$$\lambda(1/p+h) - \lambda(1/p) = (\int_a^b |pu'|^2 h + o(h))(1+o(1))^{-1} = \int_a^b |pu'|^2 h + o(h)$$

as $h \to 0$ in $L^1(a,b)$. This completes the proof of (4.10).

To show (4.11), we let $u = u_n(\cdot, q)$, $v = u_n(\cdot, q+h)$ where $h \in L^1(a,b)$. Using (2.1) and integration by parts we obtain

$$[\lambda(q+h) - \lambda(q)] \int_a^b u\overline{v}w = [-u(p\overline{v}') + \overline{v}(pu')]_a^b + \int_a^b u\overline{v}h = \int_a^b u\overline{v}h.$$

Using Theorem 3.1 and Lemma 3.1 we have

$$[\lambda(q+h) - \lambda(q)](1+o(1)) = \int_a^b |u|^2 h + o(h)$$

and consequently

$$\lambda(q+h) - \lambda(q) = (\int_a^b |u|^2 h + o(h))(1+o(1))^{-1} = \int_a^b |u|^2 h + o(h)$$

as $h \to 0$ in $L^1(a,b)$. This completes the proof of (4.11).

The proof of (4.12) is similar to that of (4.11) and hence omitted. Now that (4.6) - (4.12) have been established, the continuous differentiability follows from these and Theorem 3.2. ∎

Theorem 4.3 *Fix all compoments of* $\boldsymbol{\omega}$ *except* q *and let* $\lambda = \lambda_n(q)$ *for a fixed* n. *Let* S_1 *and* S_2 *be the subsets of* $L^1(a,b)$ *such that* $\lambda(q)$ *is simple for all* $q \in S_1$ *and* $\lambda(q)$ *is double for all* $q \in S_2$. *Then* S_1 *is open in* $L^1(a,b)$ *and* S_2 *is closed and nowhere dense in* $L^1(a,b)$.

The above conclusion also holds if q *is replaced by* $1/p$ *or* w.

Proof: By Lemma 3.2 we see that S_1 is open in $L^1(a,b)$ and hence S_2 is closed in $L^1(a,b)$. Assume that S_2 is not nowhere dense. Then there exist a $q \in S_2$ and a neighborhood M of q in $L^1(a,b)$ which is totally contained in S_2. Since $\lambda(q)$ is a double eigenvalue, there are two linearly independent normalized eigenfunctions $u_1(\cdot, q)$ and $u_2(\cdot, q)$ of $\lambda(q)$. By Theorem 4.2, part 6, (4.11) holds with $u = u_1(\cdot, q)$ and with $u = u_2(\cdot, q)$. This contradicts the uniqueness of the Frechet derivative of a Frechet differentiable function.

The proofs for the other cases are similar and hence are omitted. ∎

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