MULTIPLICITY OF STURM-LIOUVILLE EIGENVALUES

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This paper is dedicated to Norrie Everitt on the occasion of his 80th birthday.

ABSTRACT. The geometric multiplicity of each eigenvalue of a self-adjoint Sturm-Liouville problem is equal to its algebraic multiplicity. This is true for regular problems and for singular problems with limit-circle endpoints, including the case when the leading coefficient changes sign.

1. INTRODUCTION

The equivalence between the algebraic and geometric multiplicities of any eigenvalue of regular self-adjoint Sturm-Liouville problems (SLP's) was recently established by Eastham, Kong, Wu and Zettl in [8] Theorem 4.2 for coupled boundary conditions and by Kong, Wu and Zettl in [18] Theorem 5.5 for separated boundary conditions.

In this paper we prove this equivalence for self-adjoint singular SLP with limit-circle (LC) endpoints. This for endpoints which are nonoscillatory or oscillatory and for a leading coefficient which may change sign.

The geometric multiplicity of an eigenvalue is the dimension of its eigenspace i.e. the number of its linearly independent eigenfunctions. For SLP this number is either one or two. The algebraic multiplicity is defined in terms of a characteristic function. This is a function whose zeros are precisely the eigenvalues of the problem. The order of a zero is the algebraic multiplicity of the corresponding eigenvalue. For regular problems there is a standard, natural, and well known construction of such a characteristic function [24]. This construction depends on the fact that all solutions of the equation and their quasi-derivatives exist, at least as finite limits, at regular endpoints. Since this is not true for singular problems, the extension of this construction to the singular case is not routine. Bailey, Everitt and Zettl [3] gave a construction for the case of coupled boundary conditions (BC) and positive leading coefficient, but did not consider the question of the equivalence between the algebraic and geometric multiplicities.

The (regular or singular) characteristic function depends on the equation and the boundary conditions. Hence it is necessary to discuss the self-adjoint BC. We do this in some detail, particularly for the less well known singular case. For this case, since the solutions and their quasi-derivatives are, in general, not defined at the endpoints, the BC are defined with the aid of a 'Lagrange form'. This form utilizes a pair of maximal domain functions, which we designate as a 'BC basis', to 'steer' all solutions for all values of the spectral parameter to finite limits at the endpoints. The BC and the characteristic function are defined in terms of these limits. This raises the question of the dependence of the BC and the characteristic function on the BC bases. We also study this question in some detail.

Further, we give a detailed proof of the canonical representation of the coupled regular and singular boundary conditions. Although these representations have been used by other authors [2],

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[24] we don't know of a detailed proof in the literature. A characterization of all *real* self-adjoint Sturm-Liouville operators is derived from this canonical representation of the BC.

For an application of the equivalence between the algebraic and geometric multiplicities of eigenvalues of regular and singular Sturm-Liouville problems to Lagrange interpolation series, see the paper by Everitt and Nasri-Roudsari [12]; for another application to the approximation of singular problems by regular ones see [19].

The organization of this paper is as follows: This introductory section is followed by a discussion of regular problems in Section 2. Section 3 contains statements of the main results for singular problems with proofs postponed to Section 4. A canonical form of singular coupled boundary conditions and a corresponding (alternative) characteristic function which is analogous to the characteristic function used in Floquet theory for regular problems, are discussed in Section 5.

2. Regular Endpoints

Although our primary focus is on singular limit-circle endpoints we review the regular case in this section for the convenience of the reader and because the singular case will be based on it. We consider the equation

$$My = -(py')' + qy = \lambda wy \text{ on } J = (a, b),\ -\infty \le a < b \le \infty,\ \lambda \in \mathbb{C}, \tag{2.1}$$

and assume, throughout this section, that the coefficients satisfy

$$\frac{1}{p}, q, w \in L^1(J, \mathbb{R}),\ w > 0 \text{ a.e. on } J. \tag{2.2}$$

Remark 2.1. *Under condition (2.2) both endpoints and the equation are said to be regular. Note that $a = -\infty$ or $b = \infty$ have not been ruled out; this contrasts with much of the literature, including Naimark* [22]*, where an infinite endpoint is automatically classified as singular. As the next lemma will show, the significance of an endpoint being regular is that all solutions, together with their quasi-derivatives, have finite limits at such an endpoint and can therefore be continuously extended to this endpoint. This is not true at a singular endpoint* [13]*. Thus this seems to us to be a natural meaning of 'regular'.*

Remark 2.2. *Note that no sign restriction is placed on p. The reason for the sign restriction on w is so that the well developed and beautiful operator theory in the weighted Hilbert space $H = L^2(J, w)$ can be applied.*

Lemma 2.1. *Let (2.2) hold and let $d = a$ or $d = b$. Then the limits*

$$y(d) = \lim_{t \to d} y(t),\ (py')(d) = \lim_{t \to d} (py')(t)$$

both exist and are finite for any solution y of the nonhomogeneous equation

$$-(py')' + qy = f,\ f \in L^1(J).$$

Proof. See [24], for the last statement see [13]. □

Let $A, B \in M_2(\mathbb{C})$, the set of 2×2 matrices over $\mathbb{C}$, satisfy the following conditions:

$$rank(A|B) = 2; \tag{2.3}$$

$$AEA^* = BEB^*,\ E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.4}$$

When written in terms of components condition (2.4) becomes

$$\begin{aligned} a_{11}\overline{a_{22}} - a_{12}\overline{a_{21}} &= b_{11}\overline{b_{22}} - b_{12}\overline{b_{21}} \\ a_{11}\overline{a_{12}} - a_{12}\overline{a_{11}} &= b_{11}\overline{b_{12}} - b_{12}\overline{b_{11}} \\ a_{21}\overline{a_{22}} - a_{22}\overline{a_{21}} &= b_{21}\overline{b_{22}} - b_{22}\overline{b_{21}} \\ a_{22}\overline{a_{11}} - a_{21}\overline{a_{12}} &= b_{22}\overline{b_{11}} - b_{21}\overline{b_{12}}. \end{aligned}$$

We consider the two point BC:

$$AY(a) + BY(b) = 0,\ Y = \begin{pmatrix} y \\ py' \end{pmatrix}. \tag{2.5}$$

Definition 2.1. *A complex number λ is an eigenvalue of the (SLP) (2.1), (2.5) if the equation (2.1), for this value of λ, has a nontrivial solution y which satisfies the BC (2.5).*

It turns out that the eigenvalues can be characterized as the zeros of an entire function called a characteristic function of the problem.To construct such a function it is convenient to consider the system form of equation (2.1):

$$Y' = (P - \lambda W)Y \text{ on } J, \tag{2.6}$$

where

$$Y = \begin{pmatrix} y \\ py' \end{pmatrix},\ P = \begin{bmatrix} 0 & 1/p \\ q & 0 \end{bmatrix},\ W = \begin{bmatrix} 0 & 0 \\ w & 0 \end{bmatrix}. \tag{2.7}$$

For each $\lambda \in \mathbb{C}$ and each s, $a \le s \le b$, let $\Phi(\cdot, s, \lambda)$ be the fundamental matrix of (2.7) determined by the initial condition

$$\Phi(s, s, \lambda) = I \tag{2.8}$$

where I denotes the identity matrix. Define

$$\delta(\lambda) = \det(A + B\,\Phi(b, a, \lambda)),\ \lambda \in \mathbb{C}. \tag{2.9}$$

We can now state

Lemma 2.2.
(1) *For each t, s with $a \le t, s \le b$, and each $\lambda \in \mathbb{C}$, $\Phi(t, s, \lambda)$ is well defined and for fixed t, s, $\Phi(t, s, \lambda)$ is an entire function of λ.*
(2) *A number λ is an eigenvalue of the SLP (2.1), (2.5) if and only if $\delta(\lambda) = 0$.*

Proof. This is well known, see [24]. □

Proposition 2.1. *Let (2.2) to (2.4) hold. Then all eigenvalues of the SLP (2.1), (2.5) are real and there are an infinite but countable number of them. Moreover,*

(1) *If $p > 0$ a.e. on J, then the eigenvalues are unbounded above but bounded below and can be ordered and indexed to satisfy*

$$-\infty < \lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots$$

where equality cannot occur in two consecutive terms.

(2) *If p changes sign, then the eigenvalues are unbounded above and below and can be ordered and indexed to satisfy*

$$\ldots \le \lambda_{-2} \le \lambda_{-1} \le \lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots$$

In this case the index is not unique; one way to define it uniquely is to define λ_0 to be the smallest nonnegative eigenvalue.

Proof. This is well known, see [24]. □

Based on Lemma 2.2 we can now define the algebraic multiplicity of an eigenvalue.

Definition 2.2. *The algebraic multiplicity of an eigenvalue λ of the SLP (2.1), (2.5) is the order of it as a root of the characteristic equation $\delta(\lambda) = 0$.*

Theorem 2.1. *Let (2.2) to (2.4) hold and suppose that λ is an eigenvalue of (2.1), (2.5). Then the algebraic and geometric multiplicities of λ are the same. In particular, λ is a simple eigenvalue (i.e. its geometric multiplicity is one) if and only if $\delta(\lambda) = 0$ and $\delta^{'}(\lambda) \neq 0$; the geometric multiplicity of λ is two if and only if $\delta(\lambda) = 0$, $\delta^{'}(\lambda) = 0$ and $\delta^{''}(\lambda) \neq 0$. No eigenvalue has an algebraic or geometric multiplicity greater than 2. Furthermore, the geometric (and therefore also the algebraic) multiplicity of an eigenvalue λ is two if and only if the BC (2.5) is equivalent to the condition with*

$$B = -I \text{ and } A = \Phi(b, a, \lambda). \tag{2.10}$$

Proof. This is proved in [18] Theorem 5.5 for separated BC and in [8] Theorem 4.2 for coupled BC under the additional hypothesis that p is positive. However the proof given in [8] is valid without this additional hypothesis and therefore we will not repeat the details here. The furthermore statement is proved in Theorem 4.1 of [16] . □

Remark 2.3. *We comment on the furthermore statement and (2.10). Clearly the homogeneous BC (2.5) is invariant under multiplication on the left by a nonsingular matrix. Since all eigenvalues are real, A is real and, by Abel's Theorem $\det \Phi(b, a, \lambda) = 1$. So any given real number λ is a double eigenvalue for exactly one boundary condition satisfying (2.3), (2.4), namely the one given by (2.10).*

Lemma 2.3. *The matrices satisfying the self-adjointness conditions can be classified into two mutually exclusive classes. Let $A, B \in M_2(\mathbb{C})$ satisfy (2.3), (2.4).*

(I) Suppose A is singular. Then (2.4) implies that B is singular and (2.5) can be represented as follows:

$$\begin{aligned} A_1 y(a) + A_2 (py')(a) &= 0,\ A_1, A_2 \in \mathbb{R},\ (A_1, A_2) \neq (0,0); \\ B_1 y(b) + B_2 (py')(b) &= 0,\ B_1, B_2 \in \mathbb{R},\ (B_1, B_2) \neq (0,0). \end{aligned} \tag{2.11}$$

These conditions are called separated and have the canonical representation

$$\begin{aligned} \cos(\alpha) y(a) - \sin(\alpha)(py')(a) &= 0,\ 0 \leq \alpha < \pi; \\ \cos(\beta) y(b) - \sin(\beta)(py')(b) &= 0,\ 0 < \beta \leq \pi. \end{aligned} \tag{2.12}$$

(The slightly different normalization for α and β is for convenience in studying the continuous dependence of the eigenvalues on α and β.)

(II) Assume A is nonsingular. Then (2.4) implies that B is nonsingular and (2.5) has the canonical representation:

$$Y(b) = e^{i\gamma} K\, Y(a),\ Y = \begin{pmatrix} y \\ py' \end{pmatrix},\ -\pi < \gamma \leq \pi,\ K = (k_{ij}),\ k_{ij} \in \mathbb{R},\ \det K = 1. \tag{2.13}$$

Proof. This is elementary, but since we don't know of a detailed proof of (2.13) in the literature, we give one here. We have that (2.5) is equivalent with $CY(a) - Y(b) = 0$ where $C = -B^{-1}A$. Note that (2.3) and (2.4) hold with A replaced by C and B by $-I$. In other words, in this case we can assume that $B = -I$. For simplicity we continue to use the notation $C = A$ and $B = -I$. Then A is nonsingular, hence $a_{11}a_{21} \neq 0$ and $a_{21}a_{22} \neq 0$.

Condition (2.4) becomes

$$\begin{aligned} a_{11}\overline{a_{22}} - a_{12}\overline{a_{21}} &= 1 \\ a_{11}\overline{a_{12}} - a_{12}\overline{a_{11}} &= 0 \\ a_{21}\overline{a_{22}} - a_{22}\overline{a_{21}} &= 0 \\ a_{22}\overline{a_{11}} - a_{21}\overline{a_{12}} &= 1. \end{aligned} \tag{2.14}$$

Set

$$a_{jr} = e^{i\gamma_{jr}} k_{jr},\ k_{jr} \in \mathbb{R},\ -\pi < \gamma_{jr} \leq \pi,\ j, r = 1, 2. \tag{2.15}$$

Then $k_{11}k_{12} \neq 0$ and $k_{21}k_{22} \neq 0$.

From $a_{11}\overline{a_{12}} = a_{12}\overline{a_{11}}$ it follows that

$$e^{i(\gamma_{11}-\gamma_{22})} k_{11}k_{12} = e^{-i(\gamma_{11}-\gamma_{22})} k_{11}k_{12}$$

and hence $\gamma_{11} = \gamma_{12}$. Similarly we get $\gamma_{21} = \gamma_{22}$. From this and from the first and last equations of (2.15) we get

$$e^{i(\gamma_{11}-\gamma_{22})}[k_{11}k_{22} - k_{21}k_{12}] = 1 \textit{ and } e^{-i(\gamma_{11}-\gamma_{22})}[k_{11}k_{22} - k_{21}k_{12}] = 1. \tag{2.16}$$

Thus we may conclude that

$$\gamma_{11} = \gamma_{22} = \gamma_{12} = \gamma_{21} \textit{ and } \det K = k_{11}k_{22} - k_{21}k_{12} = 1.$$

□

We illustrate one theoretical application of the canonical representation of the coupled BC (2.13) by characterizing all *real* self-adjoint extensions of the minimal operator.

Definition 2.3. *Suppose S is a symmetric densely defined linear operator in a Hilbert space H. A linear operator T, with domain $D(T)$, is called a real self-adjoint extension of S if T is a self-adjoint extension of S with the following properties:*

(1) *$g \in D(T)$ implies $\overline{g} \in D(T)$,*
(2) *$T(\overline{g}) = \overline{Tg}$.*

Corollary 2.1. *Let $S_{\min}$ denote the minimal operator associated with (2.1) and let S be a self-adjoint extension of $S_{\min}$ in the weighted complex Hilbert space $H = L^2(J, w)$ determined by a BC (2.5) with A, B satisfying (2.3), (2.4). Then S is a real self-adjoint extension of $S_{\min}$ in H if its domain $D(S)$ is given by either (i) a separated BC (2.12) or (ii) a coupled BC (2.13) with $\gamma = 0$. (Note that $\gamma = \pi$ reduces to $\gamma = 0$ by replacing K by $-K$.)*

Proof. This follows directly from the representations (2.12) and (2.13) and the reality of the coefficients of equation (2.1). □

The next theorem gives an alternative characterization of the eigenvalues for coupled BC in terms of a different characteristic function used in Floquet theory [7], [23].

Definition 2.4. *Let $B = -I$ and $A = e^{i\gamma}K$, $-\pi < \gamma \leq \pi$, $K \in SL(2, \mathbb{R})$, i.e. $K = (k_{ij})$, $k_{ij} \in \mathbb{R}$, $\det K = 1$. Define for $\lambda \in \mathbb{C}$,*

$$D(\lambda, K) = k_{11}\phi_{22}(b, a, \lambda) + k_{22}\phi_{11}(b, a, \lambda) - k_{12}\phi_{21}(b, a, \lambda) - k_{21}\phi_{12}(b, a, \lambda). \tag{2.17}$$

Theorem 2.2. *Let (2.2), (2.13), (2.17) hold. Then λ is an eigenvalue of the SLP (2.1), (2.13) if and only if*

$$D(\lambda, K) = 2\cos(\gamma). \tag{2.18}$$

Proof. Note that det $\Phi(b,a,\lambda)=1$ by Abel's Theorem since $trace(P-\lambda W)=0$. Expanding (2.9) and letting $\Phi=(\phi_{ij})=(\phi_{ij}(b,a,\lambda))$ we get

$$\begin{aligned}
\delta(\lambda) &= \det[e^{i\gamma}K-\Phi(b,a,\lambda)] \\
&= \begin{vmatrix} e^{i\gamma}k_{11}-\phi_{11} & e^{i\gamma}k_{12}-\phi_{12} \\ e^{i\gamma}k_{21}-\phi_{21} & e^{i\gamma}k_{22}-\phi_{22} \end{vmatrix}(b,a,\lambda) \\
&= [(e^{i\gamma}k_{11}-\phi_{11})(e^{i\gamma}k_{22}-\phi_{22})-(e^{i\gamma}k_{12}-\phi_{12})(e^{i\gamma}k_{21}-\phi_{21})](b,a,\lambda) \\
&= \{e^{2i\gamma}[k_{11}k_{22}-k_{12}k_{21}]-e^{i\gamma}[k_{11}\phi_{22}+k_{22}\phi_{11}-k_{12}\phi_{21}-k_{21}\phi_{12}]+\phi_{11}\phi_{22}-\phi_{12}\phi_{21}\}(b,a,\lambda) \\
&= e^{2i\gamma}-e^{i\gamma}D(\lambda,K)+1.
\end{aligned}$$

Dividing by $e^{i\gamma}$ we obtain

$$e^{-i\gamma}\delta(\lambda)=e^{i\gamma}+e^{-i\gamma}-D(\lambda,K)=2\cos(\gamma)-D(\lambda,K).$$

The conclusion follows from part (4) of Lemma 2.2. □

3. LC Endpoints

Proofs of results stated in this section will be given in Section 4. We study boundary value problems for the equation

$$My=-(py')'+qy=\lambda wy \text{ on } J=(a,b),\ -\infty\le a<b\le\infty,\ \lambda\in\mathbb{C}, \tag{3.1}$$

with coefficients which are only locally Lebesgue integrable:

$$\frac{1}{p},q,w\in L_{loc}(J,\mathbb{R}),\ w>0 \text{ a.e. on } J. \tag{3.2}$$

In this case the equation (3.1) and its equivalent system (2.6) may be singular at the endpoints a or b.

Definition 3.1. *Let (3.2) hold and let $c\in J$. The endpoint a of the underlying interval J is said to be in the limit-circle case, or a is LC for short, if for some $\lambda\in\mathbb{C}$, all solutions of equation (3.1) are in $L^2((a,c),w)$. Similarly, the endpoint b of J is LC, if for some $\lambda\in\mathbb{C}$, all solutions of equation (3.1) are in $L^2((c,b),w)$.*

Remark 3.1. *It is clear from (3.2) that the LC classification is independent of the point $c\in J$. Also it is well known* [24] *that if all solutions of (3.1) are in $L^2((a,c),w)$ for some $\lambda\in\mathbb{C}$, then this is true for all $\lambda\in\mathbb{C}$. Similarly for the endpoint b. Therefore the LC classification at each endpoint is independent of c and of λ and depends only on the behavior of the coefficients p,q,w near that endpoint.*

Remark 3.2. *Note that we include regular endpoints in the LC classification; this is done for simplicity of exposition only. The case when both endpoints are regular is discussed in Section 2 above, so in this section we focus on the cases when one or both endpoints are LC singular.*

(3.3) **Throughout this paper we assume that each endpoint is LC.**

Our main goal in this paper is to prove that the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity. Since the algebraic multiplicity is defined in terms of a characteristic function we must first construct such a function. Note that the construction of the regular characteristic function $\delta(\lambda)$ given by (2.9) does not make sense here since the fundamental matrix $\Phi(b,a,\lambda)$ is not defined, in general, at a limit circle endpoint a or b. Similarly the BC (2.5) does not make sense, in general, at a limit circle endpoint. We overcome these obstacles by replacing the entries of Φ with Lagrange sesquilinear forms which exist as finite limits at all limit-circle endpoints.

We start with a representation of self-adjoint BC. This depends on a 'BC basis' at each endpoint. In general these bases are different at the two endpoints.

Let

$$D_{\max} = \{f \in H = L^2(J, w) : f,\ pf' \in AC_{loc}(J),\ w^{-1}Mf \in H\} \tag{3.4}$$

Of critical importance to the description self-adjoint boundary conditions is the Lagrange sesquilinear form given by

$$[f, g] = fp\overline{g}' - \overline{g}pf',\ (f, g \in D_{\max}). \tag{3.5}$$

Observe that the Green's formula:

$$\int_\alpha^\beta \{\overline{g}Mf - f\overline{Mg}\} = [f, g](\beta) - [f, g](\alpha),\ (f, g \in D_{\max};\ \alpha, \beta \in J), \tag{3.6}$$

holds and that it follows from (3.6) that the limits

$$\lim_{\beta \to b^-} [f, g](\beta)\ ;\ \lim_{\alpha \to a^+} [f, g](\alpha) \tag{3.7}$$

exist and are finite for all $f, g \in D_{\max}$, and in particular, for all solutions y of (3.1) for any $\lambda \in \mathbb{C}$.

Definition 3.2. *A pair of real-valued functions $\{f, g\}$ is called a (BC) basis at a if $f, g \in D_{\max}$ and satisfy $[g, f](a) = 1$. Similarly a pair of real-valued functions $\{h, k\}$ is called a BC basis at b if $h, k \in D_{\max}$ and satisfy $[k, h](b) = 1$.*

Such BC bases exist: just take real-valued linearly independent solutions of (3.1) for any particular real value of λ and normalize their Wronskian to be 1 to get a basis for both endpoints. Or, more generally, take real-valued linearly independent solutions of (3.1) for some real $\lambda = \lambda_a$ on some interval (a, c), $c \in J$ and normalize their Wronskian to be 1; then take real-valued linearly independent solutions of (3.1) for some real $\lambda = \lambda_b$ on some interval (d, b), $d \in J$ and normalize their Wronskian to be 1. But note that, while this construction provides a plethora of BC bases in terms of solutions, such bases, in general, need not be solutions near the endpoints. For example f, g might be constructed from the first term of the asymptotic expansion of solutions when solutions are not known in closed form for any λ, see Examples 2 and 4 of the SLEIGN2 code [1] for an illustration.

Let $\{f, g\}$ be a BC basis at a and $\{h, k\}$ a BC basis at b. For matrices $A, B \in M_2(\mathbb{C})$ we consider the boundary condition

$$A \begin{pmatrix} [y, f](a) \\ [y, g](a) \end{pmatrix} + B \begin{pmatrix} [y, h](b) \\ [y, k](b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.8}$$

Definition 3.3. *A complex number λ is an eigenvalue of the SLP (3.1), (3.8) if the equation (3.1) has a nontrivial solution y, for this value of λ, satisfying the BC (3.8).*

Note that for any λ and any solution y of (3.1) it is meaningful to ask the question of whether the BC (3.8) is satisfied since the Lagrange brackets $[y, f]$, $[y, h]$ etc exist as finite limits at LC endpoints.

Proposition 3.1. *Let (3.2), (3.3) hold. Let $\{f, g\}$ be a BC basis at a and $\{h, k\}$ a boundary condition basis at b and let matrices $A, B \in M_2(\mathbb{C})$ satisfy (2.3), (2.4). Then all eigenvalues of the SLP (3.1), (3.8) are real and there are an infinite but countable number of them.*

For a fixed BC basis $\{f, g\}$ at a the Lagrange brackets $[y, f](a, \lambda)$, $[y, g](a, \lambda)$ exist as finite limits for any solution y of (3.1) for any λ. Can these brackets assume arbitrary values and is their dependence on λ analytic?

Lemma 3.1. *Let (3.2), (3.3) hold. Let $\{f, g\}$ be BC basis at a. Let $c, d \in \mathbb{C}$. For any $\lambda \in \mathbb{C}$ there exists a unique solution $y = y(\cdot, \lambda)$ of (3.1) such that*

$$[y, f](a, \lambda) = c \text{ and } [y, g](a, \lambda) = d. \tag{3.9}$$

Furthermore the brackets $[y, f](t, \lambda)$ and $[y, g](t, \lambda)$ exist and are entire functions of λ for any fixed t, $a \le t \le b$. There is a similar result for the endpoint b.

Remark 3.3. *The unique solution $y(\cdot, \lambda)$ of (3.1) satisfying (3.9) is defined on the open interval (a, b) but not, in general, at the endpoints a, b. Thus $[y, f](a, \lambda)$ may be viewed as a substitute for $y(a, \lambda)$; similarly $[y, g](a, \lambda)$ may be viewed as a replacement for $(py')(a, \lambda)$. Note that we are using the notation $[y, f](a, \lambda)$ for $[y(\cdot, \lambda), f](a)$, etc. If a is regular, then f, g can be chosen so that $[y, f](a, \lambda) = y(a, \lambda)$ and $[y, g](a, \lambda) = (py')(a, \lambda)$. Similar remarks apply at the endpoint b.*

Theorem 3.1. *Let the hypotheses and notation of Proposition 3.1 hold. Let $\psi_1 = \psi_1(\cdot, \lambda)$, $\psi_2 = \psi_2(\cdot, \lambda)$ be the unique solutions of (3.1) satisfying, for each $\lambda \in \mathbb{C}$,*

$$[\psi_1, f](a, \lambda) = 1 \text{ and } [\psi_1, g](a, \lambda) = 0; \; [\psi_2, f](a, \lambda) = 0 \text{ and } [\psi_2, g](a, \lambda) = 1. \tag{3.10}$$

Define for all $\lambda \in \mathbb{C}$,

$$\Delta(\lambda) = \det \left(A + B \begin{pmatrix} [\psi_1, h] & [\psi_2, h] \\ [\psi_1, k] & [\psi_2, k] \end{pmatrix} \right) (b, \lambda). \tag{3.11}$$

Then Δ is an entire function, and λ is an eigenvalue of the SLP (3.1), (3.8) if and only if $\Delta(\lambda) = 0$.

Definition 3.4. *The function $\Delta(\lambda)$ given by (3.11) is a characteristic function of the SLP (3.1), (3.8). The algebraic multiplicity of an eigenvalue λ is the order of it as a root of the characteristic equation $\Delta(\lambda) = 0$. We write $\Delta(\lambda) = \Delta(\lambda, A, B)$ to indicate the dependence of Δ on A, B.*

Theorem 3.2. *Let the hypotheses and notation of Theorem 3.1 hold. Then the geometric multiplicity of any eigenvalue is equal to its algebraic multiplicity.*

4. Proofs

The proofs of Lemma 3.1 and Theorems 3.1, 3.2 will be based on several more lemmas. In (3.8) how do A, B change when the boundary condition bases are changed ? To help answer this question we first establish a lemma, see Fulton [14], Littlejohn and Krall [21]. This Lemma will be used repeatedly below.

Lemma 4.1. *Let $y, z, u, v \in D_{\max}$. If $[v, u](a) = 1$ then*

$$[y, z](a) = [y, \overline{v}](a)[\overline{z}, u](a) - [y, u](a)[\overline{z}, \overline{v}](a). \tag{4.1}$$

Similarly, if $[v, u](b) = 1$ then

$$[y, z](b) = [y, \overline{v}](b)[\overline{z}, u](b) - [y, u](b)[\overline{z}, \overline{v}](b).$$

Proof. We prove (4.1), the proof for the endpoint b is similar. Note that

$$\begin{aligned}
[y,z] &= (\overline{z}, p\overline{z}')\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y \\ py' \end{pmatrix} \\
&= (\overline{z}, p\overline{z}')\begin{pmatrix} (pv') & (p\overline{u}') \\ -v & -\overline{u} \end{pmatrix}\begin{pmatrix} -(p\overline{u}') & \overline{u} \\ (pv') & -v \end{pmatrix}\begin{pmatrix} y \\ py' \end{pmatrix} \\
&= (-[v,z], -[\overline{u},z])\begin{pmatrix} -[y,u] \\ [y,\overline{v}] \end{pmatrix} = [y,\overline{v}][\overline{z},u] - [y,u][\overline{z},\overline{v}] \qquad (4.2)
\end{aligned}$$

holds for each t in some neighborhood of a. Take the limit as $t \to a$ on both sides of (4.2) to get (4.1). □

The next result is the 'change of BC bases theorem', it describes how the BC change when the bases change, see Theorem 3.3 in [17].

Theorem 4.1. *Let the notation and hypotheses of Proposition 3.1 hold. Assume that $\{f_1, g_1\}$ and $\{h_1, k_1\}$ are other BC bases at a and b, respectively. Let*

$$A_1 = AC,\ C = \begin{pmatrix} -[f,g_1](a) & [f,f_1](a) \\ -[g,g_1](a) & [g,f_1](a) \end{pmatrix},\ B_1 = BD,\ D = \begin{pmatrix} -[h,k_1](b) & [h,h_1](b) \\ -[k,k_1](b) & [k,h_1](b) \end{pmatrix}. \qquad (4.3)$$

Then (3.8) is equivalent to

$$A_1 \begin{pmatrix} [y,f_1](a) \\ [y,g_1](a) \end{pmatrix} + B_1 \begin{pmatrix} [y,h_1](b) \\ [y,k_1](b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (4.4)$$

Proof. This follows from a direct computation using Lemma 4.1. □

Proof of Proposition 3.1. The special case when $f = h$ and $g = k$ and f, g are real-valued solutions on J for some real λ follows from [20]. The general case then follows from this special case and Theorem 4.1.

To prepare for the proofs of Lemma 3.1 and Theorems 3.1, 3.2 we show that the SLP (3.1), (3.8) can be represented as a boundary value problem for a regular system.

Theorem 4.2. *Let (3.2), (3.3) hold, let $r \in \mathbb{R}$ and let u, v be real-valued linearly independent solutions of (3.1) with $\lambda = r$, normalized to make their Wronskian $[v,u] = 1$. Let*

$$U = \begin{bmatrix} v & u \\ pv' & pu' \end{bmatrix},\ G = U^{-1}WU = \begin{bmatrix} -v\,u\,w & -u^2\,w \\ v^2\,w & v\,u\,w \end{bmatrix}. \qquad (4.5)$$

For $\lambda \in \mathbb{C}$, consider the first order system

$$Z' = (r-\lambda)GZ \quad \text{on } J. \qquad (4.6)$$

Then

1. *The system (4.6) is regular (and consequently $Z(a,\lambda)$ and $Z(b,\lambda)$ exist).*
2. *For each $\lambda \in \mathbb{C}$, $Z(t,\lambda)$ is a (vector or matrix) solution of (4.6) if and only if*

$$Y(t,\lambda) = U(t)\,Z(t,\lambda),\ a < t < b, \qquad (4.7)$$

is a (vector or matrix) solution of (2.6).

3. *Let (4.7) hold with $Y = \begin{pmatrix} y \\ py' \end{pmatrix}$, $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Then y is a solution of (3.1) which satisfies the (singular) boundary condition (3.8) if and only if Z satisfies (4.6) and the regular BC*

$$A_r\,Z(a) + B_r\,Z(b) = 0, \qquad (4.8)$$

where

(4.9)
$$A_r = -AC(a),\ C(a) = \begin{pmatrix} [f,v](a) & [f,u](a) \\ [g,v](a) & [g,u](a) \end{pmatrix},\ B_r = -BD(b),\ D(b) = \begin{pmatrix} [h,v](b) & [h,u](b) \\ [k,v](b) & [k,u](b) \end{pmatrix}.$$

Proof. A direct computation establishes (1). The Schwartz inequality and the hypothesis (3.3) imply that each component of G is in $L^1(J)$, proving (2). Hence the BC (4.8) is well-defined. To prove part (3), let $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ be a vector solution of (4.6), apply Cramer's rule to (4.7) to get

$$z_1(t,\lambda) = [y,u](t,\lambda),\ z_2(t,\lambda) = -[y,v](t), a \le t \le b. \tag{4.10}$$

The equivalence of (3.8) with (4.8) then follows from Lemma 4.1, and (4.9). □

Proof of Lemma 3.1. From Lemma 4.1 we get, for any $y \in D_{\max}$, and, in particular, for any solution y of (3.1) for any $\lambda \in \mathbb{C}$

$$\begin{pmatrix} [y,f](a,\lambda) \\ [y,g](a,\lambda) \end{pmatrix} = \begin{pmatrix} [f,v](a) & [f,u](a) \\ [g,v](a) & [g,u](a) \end{pmatrix} \begin{pmatrix} [y,u](a,\lambda) \\ [y,v](a,\lambda) \end{pmatrix} = C(a) \begin{pmatrix} [y,u](a,\lambda) \\ [y,v](a,\lambda) \end{pmatrix}, \tag{4.11}$$

and $\det C(a) = 1$. Note that f, g, u, v and hence $C(a)$ do not depend on λ. From the theory of regular systems it follows that for any $c, d \in \mathbb{C}$, $z_1(a,\lambda) = c$, $z_2(a,\lambda) = d$ determines a unique solution $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ of (4.6) on (a,b) for any $\lambda \in \mathbb{C}$ and $z_1(b,\lambda)$, $z_2(b,\lambda)$ are entire functions of λ. Hence from (4.10) we may conclude that for any $c, d \in \mathbb{C}$, the 'singular initial condition' $[y,u](a,\lambda) = c$, $[y,v](a,\lambda) = d$ determines a unique solution $Y = \begin{pmatrix} y \\ py' \end{pmatrix}$ on (a,b) for any $\lambda \in \mathbb{C}$ and $[y,u](b,\lambda)$, $[y,v](b,\lambda)$ are entire functions of λ. It follows from (4.11) that the same result holds for $[y,f](a,\lambda) = c$, $[y,g](a,\lambda)$. There is a similar argument for 'initial conditions' at the endpoint b. This concludes the proof of Lemma 3.1.

Proof of Theorem 3.1. The existence of solutions ψ_1, ψ_2 determined by the singular 'initial conditions' (3.10) and the entire dependence of Δ on λ follows from Lemma 3.1. Let $y = c\psi_1 + d\psi_2$ and consider

$$\begin{aligned} & A \begin{pmatrix} [y,f](a,\lambda) \\ [y,g](a.\lambda) \end{pmatrix} + B \begin{pmatrix} [y,h](b,\lambda) \\ [y,k](b,\lambda) \end{pmatrix} \\ = \ & A \begin{pmatrix} [c\psi_1 + d\psi_2, f](a,\lambda) \\ [c\psi_1 + d\psi_2, g](a,\lambda) \end{pmatrix} + B \begin{pmatrix} [c\psi_1 + d\psi_2, h](b,\lambda) \\ [c\psi_1 + d\psi_2, k](b,\lambda) \end{pmatrix} \\ = \ & \left(A \begin{pmatrix} [\psi_1,f](a,\lambda), [\psi_2,f](a,\lambda) \\ [\psi_1,g](a,\lambda), [\psi_2,g](a,\lambda) \end{pmatrix} + B \begin{pmatrix} [\psi_1,f](b,\lambda), [\psi_2,f](b,\lambda) \\ [\psi_1,g](b,\lambda), [\psi_2,g](b,\lambda) \end{pmatrix} \right) \begin{pmatrix} c \\ d \end{pmatrix} \\ = \ & \left(A + B \begin{pmatrix} [\psi_1,f](b,\lambda), [\psi_2,f](b,\lambda) \\ [\psi_1,g](b,\lambda), [\psi_2,g](b,\lambda) \end{pmatrix} \right) \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

This algebraic system has a nontrivial solution for c, d if and only if $\Delta(\lambda) = 0$.

The proof of Theorem 3.2 is based on a representation of the characteristic matrix $\Delta(\lambda)$ of the singular problem (3.1), (3.8) in terms of a characteristic function of the regular system (4.6), (4.8).

Remark 4.1. *Note that the system (4.6) does not reduce to a scalar Sturm-Liouville equation (3.1) because the coefficient matrix G does not have the same form as P in (2.6). In particular g_{11} and g_{22} are not the zero function. Nevertheless, it follows from Theorem 4.2 that regular boundary value problems for the system (4.6) are equivalent to singular problems for the scalar Sturm-Liouville* *equation (3.1). Equivalent in the sense that they have the same eigenvalues and their eigenfunctions are related as shown by Theorem 4.2.*

Definition 4.1. *Let the hypotheses and notation of Theorem 4.2 hold. For each $\lambda \in \mathbb{C}$, and every s, $a \le s \le b$, let $\Phi_r(t,s,\lambda)$ be the fundamental matrix solution of (4.6) determined by the initial condition*

$$\Phi_r(s,s,\lambda) = I. \tag{4.12}$$

Thus, $\Phi_r(t,s,\lambda)$ is defined for all t, $a \le t, s \le b$. For any $A, B \in M_2(\mathbb{C})$, let A_r, B_r be given by (4.9) and define

$$\Delta_r(\lambda) = \Delta_r(\lambda, A_r, B_r) = \det[A_r + B_r\,\Phi_r(b,a,\lambda)],\ \lambda \in \mathbb{C}. \tag{4.13}$$

This function $\Delta_r(\lambda)$ is called a characteristic function of the system boundary value problem (4.6), (4.8); we write $\Delta_r(\lambda, A_r, B_r)$ to indicate the dependence of Δ on A, B and r.

Remark 4.2. *The function $\Delta_r(\lambda)$ given by (4.13) is not to be confused with $\Delta(\lambda)$ given by (3.11). The relationships between these functions will be established below.*

Lemma 4.2. *Let the hypotheses and notation of Theorem 4.2 hold and let $\Delta_r(\lambda)$ be defined by (4.13). Then*

1. *$\Delta_r(\lambda)$ is an entire function of λ.*
2. *λ is an eigenvalue of the regular boundary value problem (4.6), (4.8) if and only if $\Delta_r(\lambda) = 0$.*

Proof. Part (1) follows from the well known theory of regular boundary value problems and (2) follows from a direct computation. □

Theorem 4.3. *Let the notation and hypotheses of Theorems 3.1 and 4.2 hold; let matrices $A, B \in M_2(\mathbb{C})$ satisfy (2.3), (2.4), and let A_r, B_r given by (4.9). Let $\Delta(\lambda) = \Delta(\lambda, A, B)$ be given by (3.11), and let $\Delta_r(\lambda) = \Delta_r(\lambda, A_r, B_r)$ be given by (4.13).. Then A_r, B_r satisfy (2.3), (2.4) and*

$$\Delta(\lambda, A, B) = -\Delta_r(\lambda, A_r, B_r). \tag{4.14}$$

Proof.

$$\begin{aligned}
\Delta(\lambda) &= \det\left(A + B\begin{pmatrix} [\psi_1,h] & [\psi_2,h] \\ [\psi_1,k] & [\psi_2,k] \end{pmatrix}(b,\lambda)\right) \\
&= \det\left(A\begin{pmatrix} [\psi_1,f] & [\psi_2,f] \\ [\psi_1,g] & [\psi_2,g] \end{pmatrix}(a,\lambda) + B\begin{pmatrix} [\psi_1,h] & [\psi_2,h] \\ [\psi_1,k] & [\psi_2,k] \end{pmatrix}(b,\lambda)\right) \\
&= \det\Bigg(A\begin{pmatrix} [f,v] & [f,u] \\ [g,v] & [g,u] \end{pmatrix}(a)\begin{pmatrix} [\psi_1,u] & [\psi_2,u] \\ [\psi_1,v] & [\psi_2,v] \end{pmatrix}(a,\lambda) \\
&\qquad + B\begin{pmatrix} [h,v] & [h,u] \\ [k,v] & [k,u] \end{pmatrix}(b)\begin{pmatrix} [\psi_1,u] & [\psi_2,u] \\ [\psi_1,v] & [\psi_2,v] \end{pmatrix}(b,\lambda)\Bigg) \\
&= \det\left(A\,C(a)\begin{pmatrix} [\psi_1,u] & [\psi_2,u] \\ [\psi_1,v] & [\psi_2,v] \end{pmatrix}(a,\lambda) + B\,C(b)\begin{pmatrix} [\psi_1,u] & [\psi_2,u] \\ [\psi_1,v] & [\psi_2,v] \end{pmatrix}(b,\lambda).\right) \qquad (4.15)
\end{aligned}$$

Let

$$D = \begin{pmatrix} [\psi_1,u] & [\psi_2,u] \\ [\psi_1,v] & [\psi_2,v] \end{pmatrix}(a,\lambda)$$

and let Z be the solution of (4.6) determined by the initial condition $Z(a, \lambda) = D$ and note that $Z(t, \lambda) = \Phi(t, a, \lambda)\, D$, $a \leq t \leq b$. Hence we get from (4.15) and (4.11)

$$\begin{aligned} \Delta(\lambda, A, B) &= -\det(A_r \Phi_r(a, a, \lambda)\, D + B_r \Phi_r(b, a, \lambda)\, D) = -\det(A_r\, \Phi_r(a, a, \lambda) + B_r\, \Phi_r(b, a, \lambda)) \\ &= -\Delta_r(\lambda, A_r, B_r). \end{aligned}$$

In the penultimate step we used $\det D = 1$ which follows from Lemma 4.1.

It remains to show that A_r, B_r satisfy (2.3), (2.4). From Lemma 4.1 it follows that $\det C(a) = 1 = \det C(b)$ and that

$$C(a)EC^*(a) = E = C(b)EC^*(b)$$

Hence the self-adjointness properties (2.3), (2.4) are preserved i.e. $rank(A|B) = rank(A_r|B_r)$ and $A_r E A_r^* = B_r E B_r^*$. This completes the proof of Theorem 4.3. □

Remark 4.3. *We comment on the remarkable identity (4.14). Note that the left hand side is independent of r and of the fundamental matrix U associated with r. But the matrices A_r, B_r depend on U and hence on r as we have indicated with the notation. Thus as r is changed the identity (4.14) holds provided the matrices A_r, B_r are chosen according to (4.11). The self-adjointness conditions (2.3), (2.4) are preserved.*

We can now proceed with the proof of Theorem 3.2 which is based on the representation (4.14).

Lemma 4.3. *Let the notation and hypotheses of Theorem 4.2 hold and let $A, B \in M_2(\mathbb{C})$ satisfy (2.3), (2.4). The algebraic multiplicity of any eigenvalue of (4.6), (4.8) is greater than or equal to its geometric multiplicity.*

Proof. Assume ρ is an eigenvalue of (4.6), (4.8) on (a, b). If the geometric multiplicity of ρ is one then its algebraic multiplicity is at least one since it is a root of the characteristic equation. Suppose ρ has geometric multiplicity two. Then by the proof of Theorem 4.1 of [18] the boundary condition (4.8) is equivalent to

$$A_r = \Phi(b, a, \lambda),\ B_r = -I. \tag{4.16}$$

For $a < c < d < b$, let

$$A(c) = \Phi(d, c, \rho)\,,\ B(d) = -I, \tag{4.17}$$

and consider the BC

$$A(c)Y(c) + B(d)Y(d) = 0 \text{ on } (c, d). \tag{4.18}$$

Note that

$$\Phi(t, c, \rho)\ = \Phi(t, a, \rho)\, \Phi^{-1}(c, a, \rho)\ c \leq t \leq d \tag{4.19}$$

is the fundamental matrix of (4.6) determined by the initial condition $\Phi(c, c, \rho) = I$. Hence the characteristic function of (4.6), (4.18) on (c, d) is given by

$$\delta(\lambda) = \delta(\lambda, (c, d), A(c), B(d)) = \det[A(c) - \Phi(d, c, \lambda)], \tag{4.20}$$

and ρ is a geometrically double eigenvalue of this problem on (c, d). But on (c, d) this problem is equivalent to a regular SLP and hence by Theorem 2.1 the algebraic multiplicity of ρ is also two. Therefore $\delta(\rho) = \delta'(\rho) = 0$. From the continuity of $\Phi(t, s, \lambda)$ it follows that as $c \to a,\ d \to b$ we have

$$\delta(\lambda, (c, d), A(c), B(d)) \to \Delta_r(\lambda, (a, b), A_r, B_r). \tag{4.21}$$

Consequently $\Delta_r(\rho) = \Delta_r'(\rho) = 0$ and we may conclude that the algebraic multiplicity of ρ as an eigenvalue of the regular system (4.6), (4.8) on (a, b) is at least two. □

Remark 4.4. *It is interesting to note that the proof of Lemma 4.3 shows that if ρ is an eigenvalue of geometric multiplicity two for the system (4.6), (4.8) satisfying the self-adjointness conditions (2.3), (2.4) on the interval (a,b), then ρ is also an eigenvalue of geometric and algebraic multiplicity two on all truncated intervals (c,d) provided the boundary condition on (c,d) is given by (4.17), (4.18).*

Proof of Theorem 3.2. By Lemma 4.3 it suffices to prove that if the algebraic multiplicity of an eigenvalue is at least two, then its geometric multiplicity is two. Assume that ρ is an eigenvalue of (4.6), (4.8) on (a,b) with algebraic multiplicity two. By the "Continuation Principle" [18] all nearby problems have two eigenvalues, counting multiplicity. In particular, for $a < c < d < b$ and c sufficiently close to a, d sufficiently close to b the "inherited problem" consisting of (4.8) with the boundary condition

$$A_r Z(c) + B_r Z(d) = 0 \text{ on } (c,d), \tag{4.22}$$

has two (not necessarily distinct) eigenvalues, counting multiplicity, say $\rho_1(c,d)$, $\rho_2(c,d)$ such that

$$\rho_j(c,d) \to \rho,\ j = 1,2;\ \text{as } c \to a,\ d \to b. \tag{4.23}$$

On (c,d) each problem (4.6), (4.22) is equivalent to a self-adjoint regular SLP. Let $y_1 = y_1(c,d)$, $y_2 = y_2(c,d)$ be eigenfunctions with eigenvalues $\rho_1(c,d)$, $\rho_2(c,d)$ of this regular problem satisfying (4.23). These eigenvalues may or may not be distinct. By the well known Sturm-Liouville theory for regular problems, if they are distinct then their eigenfunctions are orthogonal; if they are not distinct their eigenfunctions can be chosen to be orthogonal. Thus, in either case, we have

$$\int_c^d y_1\, \overline{y_2}\, w = 0. \tag{4.24}$$

We normalize these eigenfunctions by choosing a fixed h, $a < c < h < d < b$, letting $Y_j = \begin{bmatrix} y_j \\ py_j' \end{bmatrix}$, $Z_j = U^{-1} Y_j$ and requiring that

$$||Z_j(h,(c,d))||_2 = 1,\ j = 1,2. \tag{4.25}$$

Note that this normalization is with respect to the Euclidean $2-norm$, and y_j, Y_j, Z_j depend on the interval (c,d) but we sometimes omit this interval in the notation for simplicity. There exist sequences c_n, d_n and vectors K_j such that

$$c_n \to a,\ d_n \to b,\ Z_j(h,(c_n,d_n)) \to K_j,\ \text{and } ||K_j||_2 = 1,\ j = 1,2. \tag{4.26}$$

Let Z_1, Z_2 be solutions of (4.6) determined by the initial condition

$$Z_j^*(h) = K_j,\ j = 1,2. \tag{4.27}$$

Each Z_j can be extended to $[a,b]$ as a solution of (4.6) and as a consequence of the continuous dependence of solutions of regular systems on initial conditions, it follows that

$$Z_j(c_n,d_n) \to Z_j^* \text{ on } [a,b], \tag{4.28}$$

and this convergence is uniform. Here we use the notation $[a,b]$ even when a or b may be infinite and at a finite or infinite endpoint the solutions are defined as a limit. For a proof of the uniform convergence of (4.28) on bounded or unbounded intervals, see [24] Theorem 2.12. It follows that Z_j^*, $j = 1,2$ satisfies the boundary condition (4.8). It remains to show that Z_1^*, Z_2^* are linearly independent. Let y_1^*, y_2^* be eigenfunctions satisfying (4.9) and let

$$Y_j^* = \begin{bmatrix} y_j^* \\ py_j^{*\prime} \end{bmatrix},\ Y_j^* = UZ_j^*,\ j = 1,2,\ Z_1^* = \begin{bmatrix} z_{11}^* \\ z_{21}^* \end{bmatrix},\ Z_2^* = \begin{bmatrix} z_{12}^* \\ z_{22}^* \end{bmatrix}. \tag{4.29}$$

From (4.24), (4.29) we have

$$\int_{c_n}^{d_n} [v\, z_{11} + u\, z_{21}][v\, z_{12} + u\, z_{22}]w = 0. \tag{4.30}$$

From (4.30), the uniform convergence of (4.28) on $[a, b]$ and the fact that $u, v \in L^2(J, w)$ we conclude that

$$\int_a^b y_1^* \, \overline{y_2^*}\, w = \int_a^b [v\, z_{11}^* + u\, z_{21}^*][\overline{v\, z_{12}^* + u\, z_{22}^*}]\, w = 0. \tag{4.31}$$

Therefore $y_{1,}^* \, y_2^*$ and consequently Z_1^*, Z_2^* are linearly independent. The algebraic multiplicity of any eigenvalue cannot be greater than two since this would imply, by a similar argument, that its geometric multiplicity is greater than two which is impossible. This completes the proof of Theorem 3.2.

5. Canonical Boundary Conditions

Next we give a canonical form of the coupled LC boundary conditions; this is then used to define an alternate version of the characteristic function. This alternate version has been used in [3] and parallels the version used in Floquet theory in the regular case.

Just as in the regular case the singular self-adjoint boundary conditions (3.8) fall into two disjoint classes: the separated conditions and the coupled ones. And there is a canonical representation for each of these classes analogous to the regular case. The separated conditions

$$\begin{aligned} A_1[y, f](a) + A_2[y, g](a) &= 0,\ A_1, A_2 \in \mathbb{R},\ (A_1, A_2) \neq (0, 0); \\ B_1[y, h](b) + B_2[y, k](b) &= 0,\ \ B_1, B_2 \in \mathbb{R},\ (B_1, B_2) \neq (0, 0). \end{aligned} \tag{5.1}$$

have the canonical representation

$$\begin{aligned} \cos(\alpha)[y, f](a) - \sin(\alpha)[y, g](a) &= 0,\ 0 \leq \alpha < \pi; \\ \cos(\beta)[y, h](b) - \sin(\beta)[y, k](b) &= 0,\ 0 < \beta \leq \pi. \end{aligned} \tag{5.2}$$

And the canonical representation of the coupled BC is given by:

$$\begin{pmatrix} [y, h](b) \\ [y, k](b) \end{pmatrix} = e^{i\gamma} K \begin{pmatrix} [y, f](a) \\ [y, g](a) \end{pmatrix},\ \ -\pi < \gamma \leq \pi,\ K = (k_{ij}),\ k_{ij} \in \mathbb{R}, \det K = 1. \tag{5.3}$$

As in the regular case, it follows directly from these representations of the BC that S is a *real* self-adjoint extension of the minimal operator $S_{\min}$ if and only if it is determined by either the separated BC or the coupled BC with $\gamma = 0$. (Note that $\gamma = \pi$ corresponds to replacing K by $-K$.)

How does the representation of the coupled BC change when the BC basis f, g at a changes ?

Lemma 5.1. *Let $\{f, g\}$ be a BC basis at a and $\{h, k\}$ a BC basis at b. If f_1, g_1 is another boundary condition basis at a, then (5.3) is equivalent with*

$$\begin{pmatrix} [y, h](b) \\ [y, k](b) \end{pmatrix} = e^{i\gamma} K \begin{pmatrix} -[f, g_1](a) & [f, f_1](a) \\ -[g, g_1](a) & [g, f_1](a) \end{pmatrix} \begin{pmatrix} [y, f_1](a) \\ [y, g_1](a) \end{pmatrix}. \tag{5.4}$$

There is a similar result when the BC basis at b changes.

Proof. This follows directly from Lemma 4.1. Let

$$C = \begin{pmatrix} -[f, g_1](a) & [f, f_1](a) \\ -[g, g_1](a) & [g, f_1](a) \end{pmatrix} \tag{5.5}$$

and note that Lemma 4.1 and the normalization of f, g imply $\det C = 1$ so that for $K_1 = KC$ we have $\det K_1 = 1$ consistent with (5.3). ☐

Theorem 5.1. *Let the notation and the hypotheses of Theorem 4.2 hold. Assume that* $B = -I$ *and* $A = e^{i\gamma}K$, $-\pi < \gamma \leq \pi$, $K \in SL(2, \mathbb{R})$. *Let* $K = (k_{ij})$ *and for each* $\lambda \in \mathbb{C}$, *let* $\Phi(t, s, \lambda)$ *be the fundamental matrix solution of (4.6) determined by the initial condition* $\Phi(s, s, \lambda) = I,\ a \leq s \leq b$.
Define

$$D(\lambda, K) = k_{11}\phi_{22}(b, a, \lambda) + k_{22}\phi_{11}(b, a, \lambda) - k_{12}\phi_{21}(b, a, \lambda) - k_{21}\phi_{12}(b, a, \lambda). \tag{5.6}$$

Then λ *is an eigenvalue of (3.1) with boundary condition*

$$\begin{pmatrix} [y, u](b) \\ [y, v](b) \end{pmatrix} = e^{i\gamma} K \begin{pmatrix} [y, u](a) \\ [y, v](a) \end{pmatrix}, \tag{5.7}$$

if and only if

$$D(\lambda, K) = 2\cos(\gamma). \tag{5.8}$$

Proof. Note that $\det\ \Phi(b, a, \lambda) = 1$ by Abel's Theorem since $trace(G) = 0$. Expanding (5.4) and letting $\Phi = (\phi_{ij})$ we get

$$\begin{aligned}
\Delta(\lambda) &= \det[e^{i\gamma}K - \Phi(b, a, \lambda)] \\
&= \begin{vmatrix} e^{i\gamma}k_{11} - \phi_{11} & e^{i\gamma}k_{12} - \phi_{12} \\ e^{i\gamma}k_{21} - \phi_{21} & e^{i\gamma}k_{22} - \phi_{22} \end{vmatrix}(b, a, \lambda) \\
&= [(e^{i\gamma}k_{11} - \phi_{11})(e^{i\gamma}k_{22} - \phi_{22}) - (e^{i\gamma}k_{12} - \phi_{12})(e^{i\gamma}k_{21} - \phi_{21})](b, a, \lambda) \\
&= [e^{2i\gamma}[k_{11}k_{22} - k_{12}k_{21}] - e^{i\gamma}[k_{11}\phi_{22} + k_{22}\phi_{11} - k_{12}\phi_{21} - k_{21}\phi_{12}] + \phi_{11}\phi_{22} - \phi_{12}\phi_{21}](b, a, \lambda) \\
&= e^{2i\gamma} - e^{i\gamma}D(\lambda, K) + 1.
\end{aligned}$$

Now dividing by $e^{i\gamma}$ we get

$$e^{-i\gamma}\Delta(\lambda) = e^{i\gamma} + e^{-i\gamma} - D(\lambda, K).$$

Hence $\Delta(\lambda) = 0$ if and only if $D(\lambda, K) = e^{i\gamma} + e^{-i\gamma} = 2\cos(\gamma)$. ☐

Remark 5.1. *The characterization of the eigenvalues for coupled BC given by (5.8) was proved in Bailey, Everitt and Zettl* [3] *for the case when* $p > 0$ *and used by Everitt and Nasri-Roudsari* [12] *to define the algebraic multiplicity. These authors did not consider the equivalence between the algebraic and geometric multiplicities. Theorem 5.1 shows that our definition of algebraic multiplicity is equivalent with the one given in* [3].

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