# DIFFERENTIABLE DEPENDENCE OF EIGENVALUES OF OPERATORS IN BANACH SPACES

MANFRED MÖLLER AND ANTON ZETTL

Abstract. It is shown that the simple eigenvalues of operators in Banach spaces depend differentiably on the operator and the derivative is computed. This abstract result is applied to both ordinary and partial differential operators.

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## 1. Introduction

It is well known, see Kato [8, Theorem II.5.16 and p. 568] that simple eigenvalues of a (finite dimensional) matrix operator are differentiable functions of the coefficients. In [10] and [11] Kong and Zettl, see also Dauge and Helffer [2], showed that the simple eigenvalues of Sturm-Liouville problems are differentiable functions of the problem data and found the derivatives. This was extended to higher order boundary value problems by Kong, Wu and Zettl in [9]. Dauge and Helffer [3] also investigated a corresponding problem for partial differential operators.

In this paper we establish the differentiability of the simple eigenvalues of Fredholm operators in Banach spaces and find the derivatives. Our proof is based on the theory of these operators as expounded in the book of Mennicken and Möller [13]. The proofs in [9, 10, 11] are elementary in the sense that they depend primarily on the basic theory of differential equations.

To illustrate the wide range of applications of our abstract results we give examples from linear algebra, ordinary differential equations and partial differential equations. Since these are for illustrative purposes only no effort was made to make them very general. On the contrary, simple results were chosen to avoid technicalities. Our result covers: self-adjoint problems, non-self-adjoint problems (real and non-real eigenvalues), separated boundary conditions, coupled boundary conditions, $\lambda$-independent and $\lambda$-dependent boundary conditions, scalar problems, systems of differential equations, etc.

Finally we point out that in the applications we need relatively little information about eigenfunctions, fundamental systems, Green's functions etc. once we have established that the corresponding operator function is Fredholm valued. Of course, in order to show this, fundamental systems, Greens' functions etc. may be needed. But Fredholm properties of both ordinary and partial differential operators are well known and have been extensively investigated.

## 2. Operator functions and their eigenvalues

Let $E$ and $F$ be Banach spaces. We denote the space of bounded linear operators from $E$ to $F$ by $L(E,F)$ and the space of Fredholm operators in $L(E,F)$ by $\Phi(E,F)$. Recall that an operator $V \in L(E,F)$ is a Fredholm operator if and only if its null space $N(V)$ is finite dimensional and its range $R(V)$ is finite codimensional.

Let $\Omega$ be an open nonempty subset of $\mathbb{C}$. An operator function $S : \Omega \to L(E,F)$ is called holomorphic in $\Omega$ if for each $\lambda_0 \in \Omega$ it has a representation

$$S(\lambda) = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j S_j$$

which converges in the norm of $L(E,F)$ for $\lambda$ in a neighborhood of $\lambda_0$. Then $S$ is differentiable on $\Omega$, and its derivative is denoted by $S'$. The resolvent set of $S$ is defined by

$$\rho(S) := \{\lambda \in \Omega : S(\lambda)^{-1} \in L(F,E) \text{ exists}\}.$$

Its complement $\sigma(S) = \Omega \setminus \rho(S)$ is called the spectrum of $S$.

**Definition 2.1.** Suppose that $\lambda_0 \in \Omega$, $S(\lambda_0)$ is a Fredholm operator, $\dim N(S(\lambda_0)) = 1$, and $S(\lambda)$ is invertible in a pointed neighborhood of $\lambda_0$. Then $\lambda_0$ is called an *isolated simple eigenvalue of* $S$ if there are $u \in N(S(\lambda_0))$ and $v \in N(S(\lambda_0)^*)$ such that

$$\langle S'(\lambda_0)u, v\rangle = 1. \tag{2.1}$$

Here $\langle\, ,\, \rangle$ is the bilinear or sesquilinear form in the dual pair $(F, F^*)$. The pair $(u,v)$ is called a biorthogonal system of eigenvectors of $S$ and $S^*$ at $\lambda_0$.

**Proposition 2.2.** *Let $S : \Omega \to \Phi(E,F)$ be holomorphic in $\Omega$ and let $\lambda_0$ be an isolated point of $\sigma(S)$. Then $\lambda_0$ is an isolated simple eigenvalue of $S$ if and only if there is a a biorthogonal system of eigenvectors $(u,v)$ of $S$ and $S^*$ at $\lambda_0$ such that*

$$(S(\lambda))^{-1} - \frac{1}{\lambda - \lambda_0} u \otimes v \tag{2.2}$$

*is holomorphic in a neighborhood of $\lambda_0$, where*

$$(u \otimes v)(w) = \langle w, v\rangle u, \quad w \in F.$$

*Proof.* See [13, Section 1.7]. □

Throughout this paper we assume that the operator function $T : \Omega \to L(E,F)$ is holomorphic. For an operator $K \in L(E,F)$ we write

$$T(K)(\lambda) := T(\lambda) + K. \tag{2.3}$$

**Theorem 2.3.** *Let $K_0 \in L(E,F)$ and $\lambda(K_0) \in \Omega$ such that $T(K_0)(\lambda(K_0)) \in \Phi(E,F)$ and $\lambda(K_0)$ is an isolated simple eigenvalue of $T(K_0)$. Then there is a neighborhood $U$ of $K_0$ in $L(E,F)$ and a closed disk $\overline{B}(\lambda(K_0),\varepsilon)$ in $\Omega$ with center $\lambda(K_0)$ and radius $\varepsilon > 0$ such that for each $K_1 \in U$ there is exactly one point $\lambda(K_1)$ of $\sigma(T(K_1))$ inside the open disk $B(\lambda(K_0),\varepsilon)$, and $\sigma(T(K_1)) \cap \Gamma = \emptyset$, where $\Gamma$ is the boundary of $B(\lambda(K_0),\varepsilon)$. The map $\lambda : U \to \mathbb{C}$ is continuous.*

*Proof.* Since $T(K_0)(\lambda(K_0))$ is a Fredholm operator, there is a finite codimensional closed subspace $M \subset E$ and a finite dimensional subspace $N \subset F$ such that

$$E = M \dotplus N(T(K_0)(\lambda(K_0))), \quad F = R(T(K_0)(\lambda(K_0))) \dotplus N.$$

With respect to this decomposition we write

$$T(K_1)(\lambda) = \begin{pmatrix} T_{11}(K_1)(\lambda) & T_{12}(K_1)(\lambda) \\ T_{21}(K_1)(\lambda) & T_{22}(K_1)(\lambda) \end{pmatrix} : M \dot{+} N(T(K_0)(\lambda(K_0))) \to R(T(K_0)(\lambda(K_0))) \dot{+} N$$

for $\lambda \in \Omega$ and $K_1 \in L(E,F)$. Since $T_{11}(K_0)(\lambda(K_0))$ is invertible and $T_{11}(K_1)(\lambda)$ depends continuously on $K$ and $\lambda$, there are neighborhoods $U = U_\delta = \{K_1 \in L(E,F) : |K_1 - K_0| < \delta\}$ of $K_0$ in $L(E,F)$ and $\overline{B}(\lambda(K_0), \varepsilon)$ with $\delta > 0$, $\varepsilon > 0$, such that $T_{11}(K_1)(\lambda)$ is invertible for $K_1 \in U$ and $\lambda \in \overline{B}(\lambda(K_0), \varepsilon)$. Then

$$C(K_1, \lambda) := \begin{pmatrix} \mathrm{id}_{R(T(K_0)(\lambda(K_0)))} & 0 \\ T_{21}(K_1)(\lambda)(T_{11}(K_1)(\lambda))^{-1} & \mathrm{id}_N \end{pmatrix}$$

and

$$D(K_1, \lambda) = \begin{pmatrix} \mathrm{id}_M & (T_{11}(K_1)(\lambda))^{-1} T_{12}(K_1)(\lambda) \\ 0 & \mathrm{id}_{N(T(K_0)(\lambda(K_0)))} \end{pmatrix}$$

are invertible for $K_1 \in U$ and $\lambda \in \overline{B}(\lambda(K_0), \varepsilon)$, and we have

$$T(K_1)(\lambda) = C(K_1, \lambda) \begin{pmatrix} T_{11}(K_1)(\lambda) & 0 \\ 0 & S(K_1)(\lambda) \end{pmatrix} D(K_1, \lambda),$$

where

$$S(K_1)(\lambda) = T_{22}(K_1)(\lambda) - T_{21}(K_1)(\lambda)(T_{11}(K_1)(\lambda))^{-1} T_{12}(K_1)(\lambda).$$

Therefore it follows that $T(K_1)(\lambda)$ is invertible if and only if $S(K_1)(\lambda)$ is invertible.

Since $\lambda(K_0)$ is an isolated eigenvalue of $T(K_0)$, $\rho S(K_0) \neq \emptyset$. This implies that the dimensions of $N(T(K_0)(\lambda(K_0))$ and $N$ must coincide, and with respect to any bases of these spaces we can consider the determinant of $S(K_1)(\lambda)$. Therefore $T(K_1)(\lambda)$ is invertible if and only if $\det S(K_1)(\lambda) \neq 0$. Moreover, an eigenvalue $\lambda_0$ of $S(K_1)$, and therefore also of $T(K_1)$ via representations of the form (2.2), is simple if and only if $\det S(K_1)$ has a simple zero at $\lambda_0$, see [13, Section 1.8] or [5, Section XI.9]. Therefore the existence and uniqueness of $\lambda(K_1)$ is proved if we show that for each $K_1 \in U$ there is exactly one simple zero of $\det S(K_1)$ inside $\Gamma$ and no zero on $\Gamma$. By choosing a smaller $\varepsilon$, if necessary, we may assume that $\det S(K_0)(\lambda) \neq 0$ for all $\lambda \in \Gamma$. Then, choosing $\delta$ small enough, we may assume that $|\det S(K_1)(\lambda) - \det S(K_0)(\lambda)| < |\det S(K_0)(\lambda)|$ for all $\lambda \in \Gamma$. Since $\det S(K_0)$ has exactly one simple zero inside $\Gamma$, Rouché's theorem yields that $\det S(K_1)$ has exactly one simple zero inside $\Gamma$. Since for each sufficiently small $\varepsilon > 0$ we can choose $\delta > 0$ such that the above is true, we have

$$|\lambda(K_1) - \lambda(K_0)| < \varepsilon \text{ if } |K_1 - K_0| < \delta,$$

which proves the continuity. Repeating the above proof with $K_0$ replaced by an arbitrary $K_1 \in U$, the continuity of $\lambda$ on $U$ follows. □

**Theorem 2.4.** *Let the hypotheses and notation of Theorem 2.3 hold. The map $\lambda : U \to \mathbb{C}$ is differentiable at $K_0$, and the derivative is given by*

$$\lambda'(K_0)K = -\langle Ku, v \rangle, \ \ K \in L(E,F), \tag{2.4}$$

*where $(u, v)$ is a biorthogonal system of eigenvectors of $T(K_0)$ and $T(K_0)^*$ at $\lambda(K_0)$.*

*Proof.* Let $K_1 \in U$. From Proposition 2.2 and Theorem 2.3 we know that for $j = 0, 1$ there are operators $Q_j \in L(E,F)$ of rank 1 such that

$$(T(K_j)(\lambda))^{-1} - \frac{1}{\lambda - \lambda(K_j)} Q_j$$

is holomorphic inside and on $\Gamma$, where $\Gamma$ is defined in Theorem 2.3. From

$$\frac{1}{2\pi i}\oint_\Gamma (\lambda-\lambda(K_1))(T(K_j)(\lambda))^{-1}d\lambda = (\lambda(K_j)-\lambda(K_1))Q_j$$

and the resolvent identity

$$\begin{aligned}(T(K_1)(\lambda))^{-1}-(T(K_0)(\lambda))^{-1} &= (T(K_1)(\lambda))^{-1}[T(K_0)(\lambda)-T(K_1)(\lambda)](T(K_0)(\lambda))^{-1}\\ &= -(T(K_1)(\lambda))^{-1}[K_1-K_0](T(K_0)(\lambda))^{-1}\end{aligned}$$

we infer

$$\begin{aligned}(\lambda(K_1)-\lambda(K_0))Q_0 &= (\lambda(K_1)-\lambda(K_0))Q_0-(\lambda(K_1)-\lambda(K_1))Q_1\\ &= \frac{1}{2\pi i}\oint_\Gamma (\lambda(K_1)-\lambda)(T(K_1)(\lambda))^{-1}[K_1-K_0](T(K_0)(\lambda))^{-1}d\lambda.\end{aligned}$$

Now note that

$$D(K_1)(\lambda) := (\lambda(K_1)-\lambda)(T(K_1)(\lambda))^{-1}$$

depends holomorphically on $\lambda$ inside and on $\Gamma$ and converges on $\Gamma$ uniformly in norm to $D(K_0)(\lambda)$ as $K_1$ converges to $K_0$ because of the continuity of the map $\lambda$. Therefore

$$\begin{aligned}(\lambda(K_1)-\lambda(K_0))Q_0 &= \frac{1}{2\pi i}\oint_\Gamma D(K_0)(\lambda)[K_1-K_0](T(K_0)(\lambda))^{-1}d\lambda + o(|K_1-K_0|)\\ &= -Q_0[K_1-K_0]Q_0+o(|K_1-K_0|)\end{aligned}$$

since $D(K_0)(\lambda(K_0)) = -Q_0$. Now observe that

$$Q_0 = u\otimes v$$

by (2.2) and

$$Q_0T'(\lambda(K_0))u = \langle T'(\lambda(K_0))u, v\rangle u = u$$

by (2.1), which implies that

$$\langle T'(\lambda(K_0))Q_0T'(\lambda(K_0))u, v\rangle = 1.$$

Hence

$$\lambda(K_1)-\lambda(K_0) = -\langle T'(\lambda(K_0))Q_0(K_1-K_0)Q_0T'(\lambda(K_0))u, v\rangle + o(|K_1-K_0|),$$

which shows that $\lambda$ is differentiable at $K_0$ and that

$$\begin{aligned}\lambda'(K_0)K &= -\langle T'(\lambda(K_0))Q_0KQ_0T'(\lambda(K_0))u, v\rangle\\ &= -\langle Ku, v\rangle,\end{aligned}$$

since

$$Q_0^*T'(\lambda(K_0))^*v = \langle u, T'(\lambda(K_0))^*v\rangle v = v. \quad \square$$

We want to get a similar statement for the operator $Q(K)$ given by

$$(T(K)(\lambda))^{-1} = \frac{1}{\lambda-\lambda(K)}Q(K)+P(K)+O(\lambda-\lambda(K)). \tag{2.5}$$

**Theorem 2.5.** *Let the hypotheses and notation of Theorem 2.3 hold. The map $Q : U \to L(F, E)$ defined in (2.5) is differentiable at $K_0$, and with $Q_0 = Q(K_0)$ and $P_0 = P(K_0)$, its derivative is*

$$Q'(K_0)K = -Q_0KP_0 - P_0KQ_0,\ K\in L(E, F).$$

*Proof.* Arguing as above we get

$$\begin{aligned} Q(K_1) - Q(K_0) = Q_1 - Q_0 &= -\frac{1}{2\pi i}\oint_\Gamma (T(K_1)(\lambda))^{-1}[K_1 - K_0](T(K_0)(\lambda))^{-1} d\lambda \\ &= -\frac{1}{2\pi i}\oint_\Gamma (T(K_0)(\lambda))^{-1}[K_1 - K_0](T(K_0)(\lambda))^{-1} d\lambda + o(|K_1 - K_0|) \\ &= -Q_0(K_1 - K_0)P_0 - P_0(K_1 - K_0)Q_0 + o(|K_1 - K_0|). \end{aligned}$$

This proves the differentiability and the above formula for $Q'(K_0)$. □

Theorem 2.5 does not give much information about the derivative since, in general, we do not know much about $P_0$. Due to the fact that $Q_0$ is a tensor product of eigenvectors we could get some information about dependence of the eigenvectors. However, since eigenvectors are not unique, we would need some kind of normalization.

A more general question arises when the coefficients depend on $\lambda$. Let us consider polynomial dependence, i. e., we replace $K$ by $K(\lambda)$, where $K \in (L(E,F))^{k+1}$, $K = (K^0, \dots, K^k)$, and set

$$K(\lambda) := \sum_{j=0}^{k} \lambda^j K^j$$

and

$$T(K)(\lambda) = T(\lambda) + K(\lambda).$$

With similar notations as above we obtain

**Theorem 2.6.** *Let $K_0 \in (L(E,F))^{k+1}$ and $\lambda(K_0) \in \Omega$ such that $T(K_0)(\lambda(K_0)) \in \Phi(E,F)$ and $\lambda(K_0)$ is an isolated simple eigenvalue of $T(K_0)$. Then the map $\lambda : U \to \mathbb{C}$ is differentiable at $K_0$, and*

$$\lambda'(K_0)K = -\langle K(\lambda(K_0))u, v\rangle,\ K \in (L(E,F))^{k+1},$$

*where $(u,v)$ is a biorthogonal system of eigenvectors of $T(K_0)$ and $T(K_0)^*$ at $\lambda(K_0)$, i. e. particularly*

$$\langle T(K_0)'(\lambda(K_0))u, v\rangle = 1.$$

*Proof.* The difference from the proof of Theorem 2.4 is that we must replace $K_j$ by $K_j(\lambda)$. However, it is clear that in

$$K_1(\lambda) - K_0(\lambda) = \sum_{j=0}^{k} \lambda^j (K_1^j - K_0^j)$$

we can treat each term $\lambda^j(K_1^j - K_0^j)$ as before, and the result follows. □

**Example 2.7.** Let $E = F = \mathbb{C}^n$ and $A, B \in M_n(\mathbb{C})$, the set of $n \times n$ complex matrices. Assume that $\lambda(A)$ is an isolated simple eigenvalue of $A - \lambda B$. Then there is a neighborhood $U = \{C \in M_n(\mathbb{C}) : \|C - A\| < \varepsilon\}$ of $A$ in $M_n(\mathbb{C})$ and a neighborhood $V = \{\lambda \in \mathbb{C} : |\lambda - \lambda(A)| < \delta\}$ of $\lambda(A)$ in $\mathbb{C}$ such that for each $C \in U$ there is exactly one eigenvalue $\lambda(C)$ of $C - \lambda B$ in $V$, the map $\lambda : U \to \mathbb{C}$ is differentiable, and

$$\lambda'(A)D = v^\top D u,\ D \in M_n(\mathbb{C}),$$

where $(A - \lambda(A)B)u = 0$, $(A^\top - \lambda(A)B^\top)v = 0$, $v^\top B u = 1$.

## 3. Applications to systems of ordinary differential operators

Let $I = [a, b]$ be a compact interval and $n$ a positive integer. Let $W_1^1(a, b)$ be the Sobolev space of all absolutely continuous functions on $[a, b]$. We define

$$T(\lambda) = \begin{pmatrix} T^D(\lambda) \\ 0 \end{pmatrix} : E = (W_1^1(a,b))^n \to (L_1(a,b))^n \times \mathbb{C}^n = F$$

by

$$T^D(\lambda)y = y'$$

and

$$K(\lambda) = \begin{pmatrix} K^D(\lambda) \\ K^R \end{pmatrix} : (W_1^1(a,b))^n \to (L_1(a,b))^n \times \mathbb{C}^n$$

by

$$\begin{aligned} K^D(\lambda)y &= (G + \lambda H)y, \\ K^R y &= Ay(a) + By(b), \end{aligned}$$

where $G, H \in M_n(L_1(a, b))$ and $A, B \in M_n(\mathbb{C})$. As in the abstract case we define

$$T(K)(\lambda) = T(\lambda) + K(\lambda),$$

where $K := (G, H, A, B) \in (M_n(L_1(a, b)))^2 \times (M_n(\mathbb{C}))^2 =: \mathcal{L}$. Note that the latter space is identified with a closed subspace of $(L(E, F))^2$. It is well-known, see e. g. [13, Section 3.1] or [12, (4.16)], that $T(K)(\lambda)$ is a Fredholm operator with index 0.

**Theorem 3.1.** *Let $G_0, H_0 \in M_n(L_1(a, b))$, $A_0, B_0 \in M_n(\mathbb{C})$, and let $\lambda(K_0)$ be an isolated simple eigenvalue of the corresponding operator $T(K_0)$. Then there is a simple closed curve $\Gamma$ with $\lambda(K_0)$ in its interior and a neighborhood $U$ of $K_0 \in \mathcal{L}$ such that for each $K_1 \in U$ the operator function $T(K_1)$ has exactly one simple eigenvalue $\lambda(K_1)$ inside $\Gamma$. The map $\lambda : U \to \mathbb{C}$ is differentiable at $K_0$, and*

$$\lambda'(K_0)K = -\int_a^b v^\top [G + \lambda(K_0)H]u \,\mathrm{d}x - d^\top (Au(a) + Bu(b)), \ \ K = (G, H, A, B) \in \mathcal{L},$$

*where $u, v \in (W_1^1(a, b))^2$ are such that*

$$\begin{aligned} u' + (G_0 + \lambda(K_0)H_0)u &= 0, \\ A_0 u(a) + B_0 u(b) &= 0, \\ v' - (G_0^\top + \lambda(K_0)H_0^\top)v &= 0, \\ \int_a^b v^\top H_0 u \,\mathrm{d}x &= 1, \end{aligned}$$

*and where $d \in \mathbb{C}^n$ is such that*

$$v(a) = A_0^\top d, \quad v(b) = -B_0^\top d.$$

*Proof.* This follows immediately from Theorem 2.6 and the representation of the adjoint operator function $T(K)(\lambda)^*$, see [13, Sections 3.3 and 3.4] or [12, Section 4]. □

## 4. Sturm-Liouville problems depending on parameters

Here we apply Theorem 3.1 to the Sturm-Liouville problem

$$-(py')' + qy = \lambda wy \tag{4.1}$$

with two-point boundary conditions

$$AY(a) + BY(b) = 0, \tag{4.2}$$

where $-\infty < a < b < \infty$,

$$(1/p, q, w, A, B) =: \omega \in \Omega := (L_1(a,b))^3 \times (M_2(\mathbb{C}))^2,$$

and

$$Y = \begin{pmatrix} y \\ py' \end{pmatrix}.$$

Note that only $1/p \neq 0$ a. e. corresponds to (4.1), but in order to be able to differentiate, we do not require this condition for elements in $\Omega$. This is admissible if we work with the corresponding system considered below. When we write $(1/p, q, w, A, B) \in \Omega$, then we always understand this to include that $p$ is defined a. e., i. e. $1/p \neq 0$ a. e.

Since we are particularly interested in the selfadjoint case, we take here sesquilinear forms instead of bilinear forms for the dual pairs. We also need the adjoint differential equation

$$-(\overline{p}z')' + \overline{q}z = \overline{\lambda}\overline{w}z \tag{4.3}$$

and the corresponding solution vector

$$Z = \begin{pmatrix} z \\ \overline{p}z' \end{pmatrix}.$$

Together with (4.1), (4.2) we consider the operator

$$T(\omega)(\lambda) = \begin{pmatrix} T^D(\omega)(\lambda) \\ T^R(\omega) \end{pmatrix} : (W_1^1(a,b))^2 \to (L_1(a,b))^2 \times \mathbb{C}^2$$

given by

$$T^D(\omega)(\lambda)Y = Y' - (P - \lambda W)Y,$$
$$T^R(\omega)y = AY(a) + BY(b),$$

where

$$P = \begin{pmatrix} 0 & \frac{1}{p} \\ q & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}.$$

Note that, given an eigenvalue of (4.1), (4.2), its eigenfunction can be considered in either the space $L_1(a,b)$ or $L_2(a,b)$.

**Theorem 4.1.** *Let $(1/p_0, q_0, w_0, A_0, B_0) = \omega_0 \in \Omega$ and $\lambda(\omega_0)$ be an isolated simple eigenvalue of (4.1), (4.2). Then there is a simple closed curve $\Gamma$ with $\lambda(\omega_0)$ in its interior and a neighborhood $U$ of $\omega_0 \in \Omega$ such that for each $\omega_1 \in U$ the problem (4.1), (4.2) has exactly one simple eigenvalue $\lambda(\omega_1)$ inside $\Gamma$. The map $\lambda : U \to \mathbb{C}$ is differentiable at $\omega_0$, and*

$$\lambda'(\omega_0)\omega = \int_a^b \{-p_0^2\tilde{p}y'\overline{z}' + [q - \lambda(\omega_0)w]y\overline{z}\}\,\mathrm{d}x + d^*(AY(a) + BY(b)),$$

*where* $\omega = (\widetilde{p}, q, w, A, B) \in \Omega$, $y, z$ *are biorthogonal solutions of the given and the adjoint boundary value problem at* $\lambda(\omega_0)$, *i. e.*

$$-(p_0 y')' + q_0 y = \lambda(\omega_0) w_0 y,$$
$$A_0 Y(a) + B_0 Y(b) = 0,$$
$$-(\overline{p_0} z')' + \overline{q}_0 z = \overline{\lambda(\omega_0)}\, \overline{w_0} z,$$
$$\int_a^b w_0 y \overline{z}\, \mathrm{d}x = 1,$$

*and where* $d \in \mathbb{C}^n$ *is such that*

$$Z(a) = EA_0^* d, \quad Z(b) = -EB_0^* d$$

*with*

$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that for Theorem 4.1 no self-adjointness hypothesis is needed: the coefficients $p$, $q$ and the weight function $w$ may be complex-valued and the boundary conditions (4.2) need not be selfadjoint. The existence of infinitely many eigenvalues for non-self-adjoint Sturm-Liouville problems is well known for Birkhoff regular and Stone regular eigenvalue problems, see e. g. [1, p. 152] or [13, Sections 7.4 and 7.6].

*Proof.* For any skew-diagonal $2 \times 2$ matrix $K$ we have $EK^*E^* = -\overline{K}$, where $\overline{K}$ is obtained from $K$ by taking conjugate complex entries. Hence, if $V$ solves the adjoint system

$$V' + (P_0^* - \overline{\lambda(\omega_0)} W_0^*)V = 0,$$

then, for $Z = EV$, we have in view of $E^*E = I_2$ that

$$Z' - (\overline{P_0} - \overline{\lambda(\omega_0)}\, \overline{W_0})Z = EV' + E(P_0^* - \overline{\lambda(\omega_0)} W_0^*)V = 0,$$

i. e.

$$Z = \begin{pmatrix} z \\ \overline{p_0} z' \end{pmatrix},$$

where $z$ is a solution of (2.1) for $\omega_0$. Therefore we have in view of Theorem 3.1

$$\begin{aligned} \lambda'(\omega_0)\omega &= -\int_a^b Z^* E(P - \lambda(\omega_0) W) Y\, \mathrm{d}x + d^*(AY(a) + BY(b)) \\ &= \int_a^b \{-p_0^2 \widetilde{p} y' \overline{z}' + [q - \lambda(\omega_0) w] y \overline{z}\}\, \mathrm{d}x + d^*(AY(a) + BY(b)) \end{aligned}$$

if we observe that

$$V^* W_0 Y = Z^* E W_0 Y = -w_0 \overline{z} y. \quad \square$$

Next asssume that (4.1), (4.2) is selfadjoint at $\omega_0$, i. e., $p_0$, $q_0$, $w_0$, are real valued, $(A, B)$ has rank 2, and $AEA^* = BEB^*$.

**Corollary 4.2.** *Under the assumptions of Theorem* 4.1 *assume that* (4.1), (4.2) *is selfadjoint. Then*

$$\lambda'(\omega_0)\omega = \int_a^b \{-|p_0 y'|^2 \widetilde{p} + [q - \lambda(\omega_0) w]|y|^2\}\, \mathrm{d}x + d^*(AY(a) + BY(b)),$$

*where* $\omega = (\widetilde{p}, q, w, A, B) \in \Omega$, $y$ *is a normalized solution of the given boundary value problem at* $\lambda(\omega_0)$, *i. e.*

$$\begin{gathered} -(p_0 y')' + q_0 y = \lambda(\omega_0) w_0 y, \\ A_0 Y(a) + B_0 Y(b) = 0, \\ \int_a^b w_0 |y|^2 \, \mathrm{d}x = 1, \end{gathered}$$

*and where* $d \in \mathbb{C}^n$ *is such that*

$$Y(a) = EA_0^* d, \quad Y(b) = -EB_0^* d.$$

*Proof.* In the selfadjoint case it is obvious that $z$ satisfying (4.3) also satisfies (4.1) (note that eigenvalues are real). And

$$Z(a) = EA_0^* d, \quad Z(b) = -EB_0^* d$$

implies

$$A_0 Z(a) + B_0 Z(b) = (A_0 E A_0^* - B_0 E B_0^*) d = 0. \quad \square$$

By taking only one of $\widetilde{p}$, $q$, $w$, $A$, $B$ different from zero in Theorem 4.1 or Corollary 4.2, we obtain the formulas for the corresponding partial derivatives.

Taking in particular the case of dependence on $A$ let us assume that $A_0$ is invertible. Then

$$d^* = Z(a)^* E A_0^{-1},$$

and it follows that

$$\lambda'(A_0) A = Z(a)^* E A_0^{-1} A Y(a). \tag{4.4}$$

## 5. Separated boundary conditions

In this section we consider self-adjoint Sturm-Liouville problems with separated boundary conditions. In this case all eigenvalues are simple. Since the coefficient functions are not varied here, we omit the indices 0 at $p_0$, $q_0$, $w_0$. The normalized form of the separated boundary conditions is

$$A_0 = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{pmatrix}, \; B_0 = \begin{pmatrix} 0 & 0 \\ \cos\beta & -\sin\beta \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{R}$. Since

$$A_0 A_0^* + B_0 B_0^* = I_2 \text{ and } EA_0^* d = Y(a), \; EB_0^* d = -Y(b)$$

in the notations of Corollary 4.2, it follows that

$$d = A_0 E^* Y(a) - B_0 E^* Y(b).$$

Hence

$$\lambda'(A_0, B_0)(A, B) = (Y(a)^* E A_0^* - Y(b)^* E B_0^*)(AY(a) + BY(b)).$$

Now we fix $\beta$ and consider $A_0$ as a function of $\alpha$. Then $\lambda(\alpha) = \widetilde{\lambda}(\varphi(\alpha))$, where $\widetilde{\lambda}$ is $\lambda$ from above and

$$\varphi(\alpha) = (A_0(\alpha), B_0).$$

The chain rule gives

$$\begin{aligned} \lambda'(\alpha) &= (Y(a)^* E A_0^* - Y(b)^* E B_0^*) A_0'(\alpha) Y(a) \\ &= -(A_0'(\alpha) Y(a))^* A_0'(\alpha) Y(a) \\ &= -|(\sin\alpha \, y(a) + \cos\alpha \, (py')(a)|^2 \end{aligned}$$

since $B_0^* A_0'(\alpha) = 0$. Now note that $\cos\alpha\, y(a) = \sin\alpha\,(py')(a)$. Assuming $\sin\alpha \neq 0$ we get

$$(py')(a) = \cot\alpha\, y(a),$$

and therefore

$$\sin\alpha\, y(a) + \cos\alpha\,(py')(a) = \sin\alpha\, y(a) + \cos\alpha\cot\alpha\, y(a) = \frac{1}{\sin\alpha} y(a).$$

Hence

$$\begin{aligned}\lambda'(\alpha) &= -\frac{1}{\sin^2\alpha}|y(a)|^2 = -\Big(1+\cot^2\alpha\Big)|y(a)|^2\\ &= -|y(a)|^2 - |(py')(a)|^2,\end{aligned}$$

which is obviously also true in case $\sin\alpha = 0$.

We can get a differential inequality for $\lambda$ which does not contain any implicit terms if $p \geq 0$, $q \geq 0$ and $\beta \in [\frac{\pi}{2}, \pi]$. Using the normalization and the differential equation we have

$$\begin{aligned}\lambda(\alpha) &= \int_a^b \lambda(\alpha) w|y|^2\, dx = \int_a^b [-(py')' + qy]\overline{y}\, dx \\ &= \int_a^b [p|y'|^2 + q|y|^2]\, dx - [py'\overline{y}]_a^b \\ &= \int_a^b [p|y'|^2 + q|y|^2]\, dx - (py')(b)\overline{y}(b) + (py')(a)\overline{y}(a).\end{aligned} \tag{5.1}$$

If $\beta = \pi$ or $\beta = \frac{\pi}{2}$, then $(py')(b)\overline{y}(b) = 0$ in view of the boundary conditions at $b$. If $\beta \in (\frac{\pi}{2}, \pi)$, then

$$(py')(b)\overline{y}(b) = \cot\beta\, |y(b)|^2 \leq 0,$$

and

$$\lambda(\alpha) \geq (py')(a)\overline{y}(a) = \cot\alpha\, |y(a)|^2.$$

From

$$\lambda'(\alpha) = \frac{1}{\sin\alpha\cos\alpha}[-\cot\alpha\, |y(a)|^2]$$

it therefore follows that

$$\begin{aligned}\lambda'(\alpha) &\geq -\frac{2}{\sin(2\alpha)}\lambda(\alpha) \text{ if } \alpha \in (0, \frac{\pi}{2}),\\ \lambda'(\alpha) &\leq -\frac{2}{\sin(2\alpha)}\lambda(\alpha) \text{ if } \alpha \in (\frac{\pi}{2}, \pi).\end{aligned}$$

## 6. Coupled boundary conditions

In this section we consider self-adjoint Sturm-Liouville problems with coupled boundary conditions. A canonical form of these boundary conditions is $B = I_2$, $A = e^{i\varphi}K$ with $\varphi \in \mathbb{R}$ and $K \in M_2(\mathbb{R})$ such that $\det K = 1$. Then we have $Z(a) = Y(a)$ in (4.4). In the remainder of this section we will always assume that the chosen eigenvalues are simple.

We consider the particular case that

$$K(\alpha) = \begin{pmatrix} \gamma^{-1} & \alpha \\ 0 & \gamma \end{pmatrix},$$

where $\alpha$ varies in $\mathbb{R}$ and $\gamma$ is a fixed nonzero real number. An easy calculation yields

$$\lambda'(\alpha) = \gamma|(py')(a)|^2,$$

which shows that all simple eigenvalues increase with $\alpha$ if $\gamma > 0$. As in the previous section we can get an explicit differential inequality for $\lambda$. The bounday conditions yield

$$(py')(b)\overline{y}(b) = (py')(a)\overline{y}(a) + \alpha\gamma|(py')(a)|^2,$$

which together with (5.1) leads to

$$\lambda(\alpha) \geq -\alpha\gamma|(py')(a)|^2$$

if $p \geq 0$ and $q \geq 0$. This gives the differential inequality

$$\lambda(\alpha) \geq -\alpha\lambda'(\alpha).$$

## 7. Dependence on the endpoints of the interval

Now we consider the case that the eigenvalues depend on the endpoints of the interval $[a, b]$. Since varying the points $a$, $b$ changes the underlying spaces on which the operators act, we first transform the problem to one in which we have a fixed interval. Let $-\infty \leq a_1 < b_1 \leq \infty$ and set $U_0 = \{(a, b) \in \mathbb{R}^2 : a_1 < a < b < b_1\}$. For $(a, b) \in U_0$ we consider the eigenvalue problem

$$y' + (G + \lambda H)y = 0, \tag{7.1}$$
$$Ay(a) + By(b) = 0 \tag{7.2}$$

for $y \in (W_1^1(a, b))^n$, where $G, H \in M_n(L_1(a_1, b_1))$ and $A, B \in M_n(\mathbb{C})$ are fixed. Now fix some $(a_0, b_0) \in U_0$. Introducing the transformation

$$\eta(a, b)(x) = a_0 + \frac{b_0 - a_0}{b - a}(x - a),$$

which maps the interval $[a, b]$ onto the interval $[a_0, b_0]$, (7.1), (7.2) is equivalent to

$$(y \circ \eta(a, b))' + (G + \lambda H)(y \circ \eta(a, b)) = 0, \tag{7.3}$$
$$Ay(a_0) + By(b_0) = 0 \tag{7.4}$$

for $y \in (W_1^1(a_0, b_0))^n$. In view of $(y \circ \eta(a, b))' = \frac{b_0 - a_0}{b - a} y' \circ \eta(a, b)$, (7.3) can be rewritten as

$$y' + \frac{b - a}{b_0 - a_0}(G \circ \zeta(a, b) + \lambda H \circ \zeta(a, b))y = 0, \tag{7.5}$$

where $\zeta(a, b)$ is the inverse of $\eta(a, b)$:

$$\zeta(a, b)(x) = a + \frac{b - a}{b_0 - a_0}(x - a_0).$$

For later use let us note that $\zeta(a, b)$ is well-defined for all $(a, b) \in \mathbb{R}^2$.

We now define

$$T = \begin{pmatrix} T^D \\ 0 \end{pmatrix} : (W_1^1(a_0, b_0))^n \to (L_1(a_0, b_0))^n \times \mathbb{C}^n$$

by

$$T^D y = y'$$

and

$$K(a, b)(\lambda) = \begin{pmatrix} K^D(a, b)(\lambda) \\ K^R \end{pmatrix} : (W_1^1(a_0, b_0))^n \to (L_1(a_0, b_0))^n \times \mathbb{C}^n$$

by

$$K^D(a,b)(\lambda)y = \frac{b-a}{b_0-a_0}[G\circ\zeta(a,b) + \lambda H\circ\zeta(a,b)]y,$$
$$K^R y = Ay(a_0) + By(b_0).$$

Finally we set

$$T(a,b)(\lambda) = T + K(a,b)(\lambda).$$

The Sobolev space $W^1_\infty(a,b)$ is the set of all absolutely continuous functions on $(a,b)$ with essentially bounded derivatives.

**Theorem 7.1.** *Let $G,H \in M_n(W^1_\infty(a_1,b_1))$. Let $(a_0,b_0)\in U_0$ and $\lambda(a_0,b_0)$ be an isolated simple eigenvalue of the operator $T(a_0,b_0)$. Then there is a simple closed curve $\Gamma$ with $\lambda(a_0,b_0)$ in its interior and a neighborhood $U\subset U_0$ of $(a_0,b_0)$ such that for each $(a,b)\in U$ the operator function $T(a,b)$ has exactly one simple eigenvalue $\lambda(a,b)$ inside $\Gamma$. The map $\lambda: U\to\mathbb{C}$ is differentiable at $(a_0,b_0)$, and*

$$\lambda'(a_0,b_0)(\alpha,\beta) = \alpha v(a_0)^\top G_0(a_0)u(a_0) - \beta v(b_0)^\top G_0(b_0)u(b_0)$$

*for $(\alpha,\beta)\in\mathbb{R}^2$, where $G_0 = G+\lambda(a_0,b_0)H$, $u,v\in(W^1_1(a_0,b_0))^2$ are such that*

$$u' + G_0 u = 0,$$
$$Au(a_0) + Bu(b_0) = 0,$$
$$v' - G_0^\top v = 0,$$
$$\int_{a_0}^{b_0} v^\top H u\,\mathrm{d}x = 1,$$

*and there is some $d\in\mathbb{C}^n$ is such that*

$$v(a) = A_0^\top d, \quad v(b) = -B_0^\top d.$$

Before proving this theorem we state and prove an auxiliary result. Let $I_0\subset\mathbb{R}$ be a compact interval, $I_1\subset\mathbb{R}$ an open interval, and $f\in W^1_\infty(I_1)$. We set

$$V := \{\zeta : I_0\to I_1 : \zeta \text{ is continuous}\}$$

and

$$V_0 := \{\zeta\in V\cap C^1(I_0) : \min_{x\in I_0}|\zeta'(x)| > 0\}.$$

Then $V$ is an open subset of the Banach space $C_{\mathbb{R}}(I_0)$ of real-valued continuous functions on $I_0$. For $\zeta\in V$ define

$$\Psi(\zeta) := f\circ\zeta.$$

Clearly, $\Psi(\zeta)\in C(I_0)$ for each $\zeta\in V$. In the lemma below, the set $V_0$ could be made larger if we would know more about $f$. For example, if $f\in C^1(I_1)$, then we could simply replace $V_0$ by $V$.

**Lemma 7.2.** $\Psi: V\to L_1(I_0)$ *is differentiable at each $\zeta\in V_0$, and*

$$\Psi'(\zeta)h = (f'\circ\zeta)h,\ \ h\in C_{\mathbb{R}}(I_0).$$

*Proof.* Let $h \in C_{\mathbb{R}}(I_0)$ such that $\zeta + h \in V$. Then, for $x \in I_0$,

$$\begin{aligned}\Psi(\zeta+h)(x) - \Psi(\zeta)(x) &= f(\zeta(x)+h(x)) - f(\zeta(x)) \\ &= \int_{\zeta(x)}^{\zeta(x)+h(x)} f'(t)\,\mathrm{d}t \\ &= f'(\zeta(x))h(x) + \int_{\zeta(x)}^{\zeta(x)+h(x)} [f'(t) - f'(\zeta(x))]\,\mathrm{d}t.\end{aligned}$$

Since $\zeta \in V_0$, the theorem on integration by substitution, see [6, (20.5)], gives that $h \mapsto (f' \circ \zeta)h$ is bounded from $C(I_0)$ to $L_1(I_0)$. By Lebesgue's lemma, see [6, (18.4)], there is a subset $N$ of $I_1$ of Lebesgue measure zero such that

$$g_h(x) = \int_{\zeta(x)}^{\zeta(x)+h(x)} [f'(t) - f'(\zeta(x))]\,\mathrm{d}t = o(|h(x)|)$$

whenever $\zeta(x) \notin N$. Since $\zeta \in V_0$, the set $N_0 := \{x \in I_0 : \zeta(x) \in N\}$ has Lebesgue measure zero. From

$$|g_h(x)| \le 2\operatorname{ess\,sup}_{t \in I_1}|f'(t)||h(x)|$$

it follows by Lebesgue's dominated convergence theorem that

$$g_h = o(\|h\|) \text{ in } L_1(I_0)$$

as $h \to 0$ in $C(I_0)$. □

*Proof of Theorem* 7.1. We are going to apply Theorem 3.1. In order to distinguish between the maps $\lambda$ in Theorems 3.1 and 7.1, we will denote the map $\lambda$ from Theorem 3.1 by $\lambda_1$. Then $\lambda(a,b) = \lambda_1(\kappa(a,b))$, where $\kappa : U_0 \to \mathcal{L}$ is given by

$$\kappa(a,b) = \Big(\frac{b-a}{b_0-a_0} G \circ \zeta(a,b), \frac{b-a}{b_0-a_0} H \circ \zeta(a,b), A, B\Big).$$

We first show that $\kappa$ is differentiable at $(a_0,b_0)$ and find its derivative. Since $\zeta(a,b)$ is linear in $(a,b)$, we get

$$\frac{d}{d(a,b)}\zeta(a,b)\big|_{(a,b)=(a_0,b_0)}(\alpha,\beta) = \zeta(\alpha,\beta)$$

for $(\alpha,\beta) \in \mathbb{R}^2$. Therefore the product rule, the chain rule and Lemma 7.2 show that $\kappa$ is differentiable at $(a_0,b_0)$ and that

$$\kappa'(a_0,b_0)(\alpha,\beta) = (\mu_G(\alpha,\beta), \mu_H(\alpha,\beta), 0, 0),$$

where

$$\begin{aligned}\mu_G(\alpha,\beta) &= \frac{d}{d(a,b)}\Big[\frac{b-a}{b_0-a_0} G \circ \zeta(a,b)\Big]_{(a,b)=(a_0,b_0)}(\alpha,\beta) \\ &= \frac{\beta-\alpha}{b_0-a_0} G + G'\zeta(\alpha,\beta) = (\zeta(\alpha,\beta)G)'\end{aligned}$$

in view of $\zeta(a_0,b_0)(x) = x$. Finally, the chain rule shows that $\lambda$ is differentiable at $(a_0,b_0)$, and from the above calculations and Theorem 3.1 it follows that

$$\begin{aligned}\lambda'(a_0,b_0)(\alpha,\beta) &= \lambda_1'(\kappa(a_0,b_0))\kappa'(a_0,b_0)(\alpha,\beta) \\ &= -\int_{a_0}^{b_0} v^\top[\mu_G(\alpha,\beta) + \lambda(a_0,b_0)\mu_H(\alpha,\beta)]u\,\mathrm{d}x \\ &= -\int_{a_0}^{b_0} v^\top(\zeta(\alpha,\beta)G_0)'u\,\mathrm{d}x.\end{aligned}$$

An integration by parts leads to

$$\lambda'(a_0,b_0)(\alpha,\beta) = \int_{a_0}^{b_0} \zeta(\alpha,\beta) v'^\top G_0 u \,\mathrm{d}x + \int_{a_0}^{b_0} \zeta(\alpha,\beta) v^\top G_0 u' \,\mathrm{d}x - [\zeta(\alpha,\beta) v^\top G_0 u]_{a_0}^{b_0},$$

which completes the proof since $v' = G_0^\top v$ and $u' = -G_0 u$. □

## 8. An application to the Schrödinger operator

In this section, we consider the Schrödinger operator on the unit disk $\Omega$ in $\mathbb{R}^2$ with Dirichlet boundary conditions and a potential $q \in L_2(\Omega)$. That means, we consider the differential operator function $T(q)(\lambda) = T(\lambda) + K(q)$, where

$$T(\lambda) = \begin{pmatrix} T^D(\lambda) \\ T^R \end{pmatrix} : H_2(\Omega) \to L_2(\Omega) \times H_{3/2}(\partial\Omega)$$

is given by

$$T^D(\lambda)y = -\Delta y - \lambda y,$$
$$T^R y = y|_{\partial\Omega},$$

and

$$K(q) = \begin{pmatrix} K^D(q) \\ 0 \end{pmatrix} : H_2(\Omega) \to L_2(\Omega) \times H_{3/2}(\partial\Omega)$$

is given by

$$K^D(q)y = qy.$$

For the definition and properties of the Sobolev spaces $H_s(\Omega)$ and $H_s(\partial\Omega)$ see e. g. [7, Appendix B]. It is well known, see e. g. [7, Theorem 20.1.2] that $T(\lambda)$ is a Fredholm operator. From the fact that the map $y \mapsto y|_{\partial\Omega}$ from $H_2(\Omega)$ to $H_{3/2}(\partial\Omega)$ is onto, see [7, Theorem B.1.9], and from [4, Theorem 8.12] it follows that $T(0)$ is bijective. Since the embedding from $H_2(\Omega)$ to $C(\Omega)$ is compact, $K$ is a compact operator. This shows that $T(q)(\lambda)$ is a Fredholm operator for all $q \in L_2(\Omega)$ and all $\lambda \in \mathbb{C}$.

**Theorem 8.1.** *Let $q_0 \in L_2(\Omega)$ and $\lambda(q_0)$ be an isolated simple eigenvalue of the operator function $T(q_0)$. Then there is a simple closed curve $\Gamma$ with $\lambda(q_0)$ in its interior and a neighborhood $U$ of $q_0$ in $L_2(\Omega)$ such that for each $q_1 \in U$ the operator function $T(q_1)$ has exactly one simple eigenvalue $\lambda(q_1)$ inside $\Gamma$. The map $\lambda : U \to \mathbb{C}$ is differentiable at $q_0$, and*

$$\lambda'(q_0)q = \int_\Omega q u^2 \,\mathrm{d}x$$

*for $q \in L_2(\Omega)$, where $u \in H_2(\Omega)$ is a solution of $-\Delta u + q_0 u = \lambda(q_0) u$ which satisfies $u|_{\partial\Omega} = 0$ and*

$$\int_\Omega u^2 \,\mathrm{d}x = 1.$$

*Proof.* In view of Theorem 2.4 we have

$$\lambda'(q_0)q = \int_\Omega quv \,\mathrm{d}x,$$

where $u$ is an eigenfunction of $T(q_0)$ at $\lambda(q_0)$ and $(v,w) \in L_2(\Omega) \times H_{3/2}(\partial\Omega)^*$ is an eigenfunction of $T(q_0)^*$ at $\lambda(q_0)$ such that

$$\int_\Omega uv \,\mathrm{d}x = 1.$$

We still have to show that we can choose $u = v$. Indeed, we are going to show that

$$T(q_0)(\lambda(q_0))^*(u, -\nu \cdot \nabla u) = 0,$$

where $\nu$ is the outer normal on $\partial\Omega$. For this let $f \in H_2(\Omega)$. Then

$$\begin{aligned}\langle f, T(q_0)(\lambda(q_0))^*(u, -\nu \cdot \nabla u)\rangle &= \langle T(q_0)(\lambda(q_0))f, (u, -\nu \cdot \nabla u)\rangle \\ &= \int_\Omega (-\Delta + q_0 - \lambda(q_0)) f u \, \mathrm{d}x - \int_{\partial\Omega} f \nu \cdot \nabla u \, \mathrm{d}\sigma,\end{aligned}$$

where $\sigma$ is the Lebesgue measure on $\partial\Omega$. By Green's formula,

$$\begin{aligned}\int_\Omega (-\Delta + q_0 - \lambda(q_0)) f u \, \mathrm{d}x &= \int_\Omega f(-\Delta + q_0 - \lambda(q_0)) u \, \mathrm{d}x \\ &\quad + \int_{\partial\Omega} f \nu \cdot \nabla u \, \mathrm{d}\sigma - \int_{\partial\Omega} u \nu \cdot \nabla f \, \mathrm{d}\sigma.\end{aligned}$$

Since $T(q_0)(\lambda(q_0))u = 0$, we have $(-\Delta + q_0 - \lambda(q_0))u = 0$ and $u|_{\partial\Omega} = 0$. This implies

$$\langle f, T(q_0)(\lambda(q_0))^*(u, -\nu \cdot \nabla u)\rangle = 0$$

for all $f \in H_2(\Omega)$, and hence $T(q_0)(\lambda(q_0))^*(u, -\nu \cdot \nabla u) = 0$. $\square$

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Department of Mathematics, University of the Witwatersrand, WITS, 2050, South Africa

Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115, U.S.A.