STURM-LIOUVILLE PROBLEMS

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Dedicated to the memory of John Barrett.

ABSTRACT. Regular and singular Sturm-Liouville problems (SLP) are studied including the continuous and differentiable dependence of eigenvalues on the problem. Also initial value problems (IVP) are considered for the SL equation and for general first order systems.

1 INTRODUCTION

The purpose of this paper is to survey some basic properties of Sturm-Liouville problems (SLP). These problems originated in a series of papers by these two authors in 1836-1837. Many thousands of papers, by Mathematicians and by others, have been published on this topic since then. Yet, remarkably, this subject is an intensely active field of research today. Dozens of papers are written on SLP every year. We have made a serious effort to try to make this article useful to both researchers as well as students.

There are major and deep theorems in the theory of nonlinear ordinary differential equations about the continuous and differentiable dependence of solutions on parameters. When specialized to the linear case these results are often not best possible and, even when they are, there frequently exist simpler proofs for the linear case.

One of the main goals of this paper is to give the linear results under minimal hypotheses and with elementary proofs whenever possible. Following this introductory section we take up initial value problems (IVP) for first order systems of arbitrary dimension in Section 2; for scalar second order problems in Section 3. In Section 4 we study eigenvalues and eigenfunctions of regular SLP; in Section 5 we take up the singular case.

Readers not particularly interested in initial value problems are advised to start in Section 4 and refer back to Sections 2 and 3 as needed.

Each section, except the introduction, ends with comments. Here we make some historical remarks, mention the names of some of the major contributors, give references for further reading, state some problems, etc. These problems reflect the interests and knowledge, or lack thereof, of the author. No effort has been made to classify these problems by difficulty, some may be routine, others intractible.

Although the subject of Sturm-Liouville problems is over 160 years old a surprising number of the results surveyed here are of recent origin, some were published within the last couple of years and a few are not in print at the time of this writing.

Instructions for downloading the SLEIGN2 package, including a FORTRAN code to compute eigenvalues and eigenfunctions for regular and singular SLP, from the internet are given in Subsectiom 5.3. This package consists of 8 files and a number of associated papers, all on SLP. The "readme" file has more detailed instructions and some additional information. The code "sleign2" comes with a user friendly interface and can be used by novices and experts alike. It can also be used to approximate the continuous (essential) spectrum of singular problems when combined with some theoretical results. These can be found in the associated papers which can also be downloaded.

I take this opportunity to thank Don Hinton and the Mathematics Department of the University of Tennessee, Knoxville, particularly Phil Schaefer and John Conway, for the invitation to give the Barrett Lectures of 1996. This forced me to try to organize my notes and my thoughts on SLP. Also Don Hinton has been one of my mathematical heroes from the time we were graduate students together until now.

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Also I wish to thank my colleague Qingkai Kong for finding and correcting my errors. His criticisms have significantly improved the final version of the paper in several respects. But I am solely responsible for any remaining errors.

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The world of Mathematics is full of wonders and of mysteries, at least as much so as the physical world.

2 FIRST ORDER SYSTEMS

2.1 Introduction.

This section is devoted to the study of basic properties of first order systems.

NOTATION. An open interval is denoted by (a,b) with $-\infty \leq a < b \leq \infty$; $[a,b]$ always denotes the compact interval with (finite) left endpoint a and (finite) right endpoint b. Let $\mathbb{R}$ denote the reals, $\mathbb{C}$ the complex numbers, and

$$\mathbb{N} = \{1,2,3,\dots\},\ \mathbb{N}_0 = \{0,1,2,\dots\},\ \mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}.$$

For any interval J of the real line, open, closed, half open, bounded or unbounded, by $L(J)$ we denote the linear manifold of complex valued Lebesgue measurable functions y defined on J for which

$$\int_a^b |y(t)|\,dt \equiv \int_J |y(t)|\,dt \equiv \int_J |y| < \infty.$$

The notation $L_{loc}(J)$ is used to denote the linear manifold of functions y satisfying $y \in L([\alpha,\beta])$ for all compact intervals $[\alpha,\beta] \subseteq J$. If $J = [a,b]$, then $L_{loc}(J) = L(J)$. Also, we denote by $AC_{loc}(J)$ the collection of complex-valued functions y which are absolutely continuous on all compact intervals $[\alpha,\beta] \subseteq J$.

For a given set S, $M_{n,m}(S)$ denotes the set of $n\times m$ matrices with entries from S. If $n = m$ we write $M_n(S)$; also if $m = 1$ we sometimes write S^n for $M_{n,1}(S)$. The norm of a constant matrix as well as the norm of a matrix function P is denoted by $|P|$. This may be taken as

$$|P| = \sum |p_{ij}|.$$

2.2 Existence and uniqueness of solutions.

DEFINITION 2.1 (Solution)**.** Let J be any interval, open, closed, half open, bounded or unbounded; let $n, m \in \mathbb{N}$, let $P : J \to M_n(\mathbb{C})$, $F : J \to M_{n,m}(\mathbb{C})$. By a solution of the equation $Y' = PY + F$ on J we mean a function Y from J into $M_{n,m}(\mathbb{C})$ which is absolutely continuous on all compact subintervals of J and satisfies the equation a.e. on J. A matrix function is absolutely continuous if each of its components is absolutely continuous.

THEOREM 2.2. Let J be any interval, open, closed, half open, bounded or unbounded; let $n, m \in \mathbb{N}$. If

$$P \in M_n(L_{loc}(J)) \tag{2.1}$$

and

$$F \in M_{n,m}(L_{loc}(J)) \tag{2.2}$$

then every initial value problem (IVP)

$$Y' = PY + F \tag{2.3}$$

$$(2.4) \qquad Y(u) = C,\ u \in J,\ C \in M_{n,m}(\mathbb{C})$$

has a unique solution defined on all of J. Furthermore, if C, P, F, are all real valued, then the solution is also real valued.

PROOF: We give two proofs of this important theorem; the second one is the standard succesive approximations proof. As we will see later the analytic dependence of solutions on the spectral parameter λ follows more readily from the second proof than the first.

For both proofs we note that if Y is a solution of the IVP (2.3), (2.4) then an integration yields

$$(2.5) \qquad Y(t) = C + \int_u^t (PY + F),\ t \in J.$$

Conversely, every solution of the integral equation (2.5) is also a solution of the IVP (2.3) (2.4).

Choose c in J, $c \neq u$. We show that (2.3), (2.4) has a unique solution on $[u, c]$ if $c > u$ and on $[c, u]$ if $c < u$. Assume $c > u$. Let

$$B = \{Y : [u, c] \to M_{n,m}(\mathbb{C}),\ Y \in C[u, c]\}.$$

Following Bielecki [12] we define the norm of any function $Y \in B$ to be

$$(2.6) \qquad \|Y\| = \sup\{e^{-K\int_u^t |P(s)|ds}|Y(t)|,\ t \in [u, c]\},$$

where K is a fixed positive constant $K > 1$. It is easy to see that with this norm B is a Banach space. Let the operator $T: B \to B$ be defined by

$$(2.7) \qquad (TY)(t) = C + \int_u^t (PY + F)(s)\, ds,\ t \in [u, c],\ Y \in B.$$

Then for $Y, Z \in B$ we have

$$|(TY)(t) - (TZ)(t)| \le \int_u^t |P(s)||Y(s) - Z(s)|ds$$

and hence

$$\begin{aligned} e^{-K\int_u^t |P(s)|ds}|(TY)(t) - (TZ)(t)| &\le \|Y - Z\| \int_u^t |P(s)|e^{-K\int_s^t |P(r)|dr}ds \\ &\le \frac{1}{K}\|Y - Z\|. \end{aligned}$$

Therefore

$$\|TY - TZ\| \leq \frac{1}{K}\,\|Y - Z\|.$$

¿From the contraction mapping principle in Banach space it follows that the map T has a unique fixed point and therefore the IVP (2.3), (2.4) has a unique solution on $[u, c]$. The proof for the case $c < u$ is similar; in this case the norm of B is modified to

$$\|Y\| = \sup\{e^{K\int_u^t |P(s)|ds}\,|Y(t)|,\ t \in [c, u]\}.$$

Since there is a unique solution on every compact subinterval $[u, c]$ and $[c, u]$ for $c \in J,\ c \neq u$ it follows that there is a unique solution on J. To establish the furthermore part take the Banach space of real-valued functions and proceed similarly. This completes the first proof.

For the second proof we constuct a solution of (2.5) by successive approximations. Define

$$\text{(2.8)} \qquad Y_0(t) = C,\ Y_{n+1}(t) = C + \int_u^t (PY_n + F),\ t \in J,\ n = 0, 1, 2, \ldots$$

Then Y_n is a continuous function on J for each $n \in N_0$. We show that the sequence $\{Y_n : n \in N_0\}$ converges to a function Y uniformly on each compact subinterval of J and that the limit function Y is the unique solution of the integral equation and hence also of the IVP. Choose $b \in J$, $b > u$ and define

$$\text{(2.9)}\ \ p(t) = \int_u^t |P(s)|\,ds,\ t \in J;\ B_n(t) = max_{u \leq s \leq t}\,|Y_{n+1}(s) - Y_n(s)|,\ u \leq t \leq b.$$

Then

$$\text{(2.10)} \qquad Y_{n+1}(t) - Y_n(t) = \int_u^t P(s)[Y_n(s) - Y_{n-1}(s)]\,ds,\ t \in J,\ n \in \mathbb{N}.$$

From this we get

$$\text{(2.11)} \quad |Y_2(t) - Y_1(t)| \leq B_0(t)\int_u^t |P(s)|\,ds = B_0(t)\,p(t) \leq B_0(b)\,p(b),\ u \leq t \leq b.$$

$$\begin{aligned}|Y_3(t)-Y_2(t)| &\le \int_u^t |P(s)|\,|Y_2(s)-Y_1(s)|\,ds \le \int_u^t |P(s)|\,B_0(s)\,p(s)\,ds\\ &\le B_0(t)\int_u^t |P(s)|\,p(s)\,ds \le B_0(b)\,\frac{p^2(t)}{2!}\\ &\le B_0(b)\,\frac{p^2(b)}{2!}\ ,\ u\le t\le b.\end{aligned}$$

¿From this and mathematical induction we get

$$|Y_{n+1}(t)-Y_n(t)| \le B_0(b)\,\frac{p^n(b)}{n!}\ ,\ u\le t\le b.$$

Hence for any $k \in N$

$$\begin{aligned}|Y_{n+k+1}(t)-Y_n(t)| &\le |Y_{n+k+1}(t)-Y_{n+k}(t)|+|Y_{n+k}(t)-Y_{n+k-1}(t)|+\\ &\quad \ldots+|Y_{n+1}(t)-Y_n(t)|\\ &\le B_0(b)\,\frac{p^n(b)}{n!}[1+\frac{p(b)}{n+1}+\frac{p^2(b)}{(n+2)(n+1)}+\ldots]\end{aligned}$$

Choose m large enough so that $\frac{p(b)}{n+1}\le\frac{1}{2}$ then $\frac{p^2(b)}{(n+2)(n+1)}\le\frac{1}{4}$, etc. when $n>m$ and the term in brackets is bounded above by 2. It follows that the sequence $\{Y_n : n\in N_0\}$ converges uniformly, say to Y, on $[u,b]$. From this it follows that Y satisfies the integral equation (2.5) and hence also the IVP (2.3), (2.4) on $[u,b]$.

To show that Y is the unique solution assume Z is another one; then Z is continuous and therefore $|Y-Z|$ is bounded, say by $M>0$ on $[u,b]$. Then

$$|Y(t)-Z(t)| = \left|\int_u^t P(s)[Y(s)-Z(s)]\,ds\right| \le M\int_u^t P(s)\,ds \le M\,p(t),\ u\le t\le b.$$

Now proceeding as above we get

$$|Y(t)-Z(t)| \le M\,\frac{p^n(t)}{n!} \le M\,\frac{p^n(b)}{n!}\ ,\ u\le t\le b,\ n\in\mathbb{N}.$$

Therefore $Y=Z$ on $[u,b]$. There is a similar proof for the case when $b<u$. This completes the second proof. □

It is interesting to observe that the initial approximation $Y_0(t)=C$ can be replaced with $Y_0(t)=G(t)$ for any continuous function G without any essential change in the proof.

Let J be an interval. For each $P\in M_n(L_{loc}(J))$, each $F\in M_{n,m}(L_{loc}(J))$, each $u\in J$ and each $C\in M_{n,m}(\mathbb{C})$ there is, according to Theorem 2.2, a unique $Y\in M_{n,m}(AC_{loc}(J))$ such that $Y'=PY+F$, $Y(u)=C$. We use the notation

$$Y = Y(\cdot, u, C, P, F) \tag{2.12}$$

to indicate the dependence of the unique solution Y on these quantities. Below, if the variation of Y with respect to some of the variables u, C, P, F is studied while the others remain fixed we abbreviate the notation (2.12) by dropping those quantities which remain fixed. Thus we may use $Y(t)$ for the value of the solution at $t \in J$ when u, C, P, F are fixed or $Y(\cdot, u)$ to study the variation of the solution function Y with respect to $u, Y(\cdot, P)$ to study Y as a function of P, etc.

THEOREM 2.3. Let $J = (a, b)$, and assume that $P \in M_n(L_{loc}(J))$. If Y is an $n \times m$ matrix solution of

$$Y' = PY \ on \ J, \tag{2.13}$$

then we have

$$rank\, Y(t) = rank\, Y(u), \quad t, u \in J. \tag{2.14}$$

Moreover, if $m = n$ and $u \in J$, then

$$(\det Y)(t) = (\det Y)(u) exp\left(\int_u^t traceP(s)ds\right), \ t \in J. \tag{2.15}$$

PROOF: The formula (2.15) follows from the fact that $y = detY$ satisfies the first order scalar equation $y' + py = 0$ where $p = traceP$. To prove the general case let $Y(u) = C$ and let $rank\, C = r$. If $r = 0$, then $Y(t) = 0$ for all t by Theorem 2.2. For $r > 0$ let $C_i, i = 1, \ldots r$ be linearly independent columns of C and construct a nonsingular $n \times n$ matrix D by adding $n - r$ appropriate constant vectors to $C_i, \ i = 1, \ldots, r$. Denote by Z the solution of (2.13) satisfying the initial condition $Z(u) = D$. Then by (2.15) $rank\, Z(t) = n$, for $t \in J$. Hence the first r columns of $Z(t)$, $Z_1(t), ..., Z_r(t)$ are linearly independent. From this and the uniqueness part of Theorem 2.2 the (constant) n-vectors $Y_1(t), Y_2(t), \ldots, Y_r(t)$ are linearly independent since $Z_j = Y_j$ on J. Hence $rank\, Y(t) \geq r$, for $t \in J$. Now suppose that $rank\, Y(c) > r$ for some c in J. Then by repeating the above argument with u replaced by c we reach the conclusion that $rank\, Y(t) > r$ for all $t \in J$. But this contradicts $rank\, Y(u) = r$ and concludes the proof. □

THEOREM 2.4. Let $P : J \to M_n(C)$ and $F : J \to M_{n,1}(C)$, $J = (a, b)$, $-\infty \leq a < b \leq \infty$. If for any $u \in J$ and any linearly independent constant vectors $C_1, \ldots, C_n$ each initial value problem

$$Y' = PY + F, \quad Y(u) = C_i, \quad i = 1, \ldots, n,$$

has a (vector) solution Y_i on J, then $P \in M_n(L_{loc}(J))$ and $F \in M_{n,1}(L_{loc}(J))$. Furthermore, if each Y_i is a C^1 solution, then there exist such P and F which are continuous.

PROOF: We first prove the special case when $F = 0$ on J. Let Y_i be a vector solution satisfying $Y_i(u) = C_i$ and let Y be the matrix whose $i-th$ column is $Y_i, i = 1, \ldots, n$. By Theorem 2.3 the solution Y is nonsingular at each point of J. Choose

$$P = Y' Y^{-1}.$$

Let K be a compact subinterval of J. Then since Y is continuous and invertible on J it follows that Y^{-1} is continuous and hence bounded on K. Also, Y' is integrable on K since Y is absolutely continuous on K by virtue of the fact that it is a solution on J. Therefore $P \in M_n(L_{loc}(J))$.

To establish the case when F is not identically zero on J let Y be a matrix solution of $Y' = PY + F$ satisfying $Y(u) = 0$ and choose a solution Z of this equation such that $Z(u) = C$ and let $V = Z - Y$. Then $V' = PV$ and $V(u) = C$. Since this holds for arbitrary C we may conclude from the special case established above that $P \in L_{loc}(J)$. Hence $F = V' - PV \in L_{loc}(J)$. The furthermore statement is clear from the proof. □

2.3 Variation of parameters.

Let $P \in M_n(L_{loc}(J))$. From Theorem 2.2 we know that for each point u of J there is exactly one matrix solution X of (2.13) satisfying $X(u) = I_n$ where I_n denotes the $n \times n$ identity matrix.

DEFINITION 2.5 (The Fundamental Matrix Φ)**.** For each fixed $u \in J$ let $\Phi(\cdot, u)$ be the fundamental matrix of (2.13) satisfying

$$\Phi(u, u) = I_n \, .$$

Note that for each fixed u in J, $\Phi(\cdot, u)$ belongs to $M_n(AC_{loc}(J))$. Furthermore, if J is compact and $P \in M_n(L(J))$, then u can be an endpoint of J and $\Phi(\cdot, u)$ belongs to $M_n(AC(J))$. By Theorem 2.3, $\Phi(t, u)$ is invertible for each $t, u \in J$ and we note that

$$\Phi(t, u) = Y(t)Y^{-1}(u) \tag{2.16}$$

for any fundamental matrix Y of (2.13).

We also write

$$\Phi = \Phi(P) = (\Phi_{rs})_{r,s=1}^{n}\ ,\quad \Phi(P)(t,u) = \Phi(t,u,P). \tag{2.17}$$

Observe that for any constant $n \times m$ matrix C, ΦC is also a solution of (2.13). If C is a constant nonsingular $n \times n$ matrix then ΦC is a fundamental matrix solution and every fundamental matrix solution has this form.

The next result is called the variation of parameters formula and is fundamental in the theory of linear differential equations.

THEOREM 2.6 (Variation of Parameters Formula)**.** Let J be any interval, let $P \in M_n(L_{loc}(J))$ and let $\Phi = \Phi(\cdot, u, P)$ be the fundamental matrix of (2.13) defined above. Let $F \in M_{n,m}(L_{loc}(J))$, $u \in J$ and $C \in M_{n,m}(\mathbb{C})$. Then

$$Y(t) = \Phi(t,u,P)\,C + \int_u^t \Phi(t,s,P)\,F(s)\,ds, \quad t \in J \tag{2.18}$$

is the solution of (2.3), (2.4). Note that if J is compact and $P \in M_n(L(J))$, $F \in M_{n,m}(L(J))$, then $Y \in M_{n,m}(AC(J))$, and u can be an endpoint or an interior point of J.

PROOF: Clearly $Y(u) = C$. Differentiate (2.18) and substitute into the equation (2.3). $\square$

2.4 The Gronwall Inequality.

Since we need a Gronwall inequality which is more general than the one usually found in the literature we state and proof it here.

THEOREM 2.7. (**The Gronwall Inequality**)

(i) (The "right" Gronwall inequality) Let $J = [a,b]$. Assume g in $L(J)$ with $g \geq 0$ *a.e.* , f real valued and continuous on J. If y is continuous, real valued, and satisfies

$$y(t) \ \leq f(t) \ + \int_a^t g(s)\,y(s)ds\ , \quad a \leq t \leq b, \tag{2.19}$$

then

$$y(t) \ \leq \ f(t) \ + \left(\int_a^t f(s)\,g(s)\,exp(\int_s^t g(u)\,du\,)\,ds\right),\ a \leq t \leq b. \tag{2.20}$$

For the special case when $f(t) = c$, *a constant*, we get

$$y(t) \leq c\; exp\left(\int_a^t g(s)ds\right), \quad t \in J. \tag{2.21}$$

For the special case when f is nondecreasing on $[a,b]$ we get

$$y(t) \leq f(t)\, \exp(\int_a^t g(s)ds),\; a \leq t \leq b. \tag{2.22}$$

(ii) (The "left" Gronwall inequality) Let $J = [a,b]$. Assume g in $L(J)$ with $g \geq 0$ *a.e.* f real valued and continuous on J. If y is continuous, real valued, and satisfies

$$y(t) \leq f(t) + \int_t^b g(s)\, y(s)ds\;,\; a \leq t \leq b, \tag{2.23}$$

then

$$y(t) \leq f(t) + \left(\int_t^b f(s)\, g(s)\, exp(\int_t^s g(u)\, du\,)\, ds\right),\; a \leq t \leq b. \tag{2.24}$$

For the special case when $f(t) = c$, a constant, we get

$$y(t) \leq c\, exp\left(\int_t^b g(s)ds\right),\; a \leq t \leq b. \tag{2.25}$$

For the special case when f is nondecreasing on $[a,b]$ we have

$$y(t) \leq f(t)\; exp\left(\int_t^b g(s)ds\right),\; a \leq t \leq b. \tag{2.26}$$

PROOF: For part (i) let $z(t) = \int_a^t g\, y,\; t \in J$ and note that

$$z' = gy \leq gf + gz\; a.e.$$

Hence with $G = \exp(\int_a^t g)$ we have

$$(z\, \exp(-G))' \leq g\, f\, \exp(-G)$$

and

$$y(t) \le f(t) + z(t) \le c\, \exp(G) + \exp(G) \int_a^t (g\, f\, G)$$

from which the conclusion follows. For part (ii) let $z(t) = \int_t^b g\, y$ and note that

$$z' + gy \ge -g\, f,$$

then proceed as in part (i). □

2.5 Bounds and extensions to the endpoints.

THEOREM 2.8. Let $J = (a, b),\ -\infty \le a < b \le \infty$, let $n, m \in N$. Suppose that $P \in M_n(L(J))$; $F \in M_{n,m}(L(J))$. Assume that for some $u \in J$, $C \in M_{n,m}(\mathbb{C})$, we have

$$Y' = PY + F \ \ on\ J, \quad Y(u) = C. \tag{2.27}$$

Then

$$|Y(t)| \le \left(|C| + \int_a^b |F|\right) \exp\left(\int_a^b |P|\right), \quad a < t < b. \tag{2.28}$$

PROOF: Note that (2.27) is equivalent to

$$Y(t) = C + \int_u^t (P(s)\, Y(s) + F(s)\,)\, ds,\ a < t < b. \tag{2.29}$$

Case 1. $u \le t < b$. From (2.29) we get

$$\begin{aligned} |Y(t)| &\le |C| + |\int_u^t (PY + F)\,| \le |C| + \int_u^t (|P|\,|Y| + |F|\,) \\ &\le \left(|C| + \int_u^b |F|\right) + \int_u^t (|P||Y|)\,,\ u \le t < b. \end{aligned}$$

¿From this and Gronwall's inequality we get

$$|Y(t)| \le \left(|C| + \int_u^b |F|\right) e^{\left(\int_u^t |P|\right)} \le \left(|C| + \int_u^b |F|\right) e^{\left(\int_u^b |P|\right)},\ u \le t < b.$$

Case 2. $a < t \le u$. From (2.29)

$$\begin{aligned} |Y(t)| &\le |C| + \left| \int_u^t (PY+F) \right| \le |C| + \int_t^u (|P|\,|Y| + |F|) \\ &\le \left(|C| + \int_a^u |F| \right) + \int_t^u (|P||Y|)\,,\ a \le t < u. \end{aligned}$$

¿From this and the "left" Gronwall inequality we get

$$|Y(t)| \le \left(|C| + \int_a^u |F| \right) e^{\left(\int_t^u |P|\right)} \le \left(|C| + \int_a^u |F| \right) e^{\left(\int_a^u |P|\right)},\ a < t \le u.$$

Combining the two cases we conclude that (2.28) holds. □

Below we will show that, under the conditions of Theorem 2.8, $a < t < b$ can be replaced with $a \le t \le b$ in (2.28). For this $Y(a)$ and $Y(b)$ are defined as limits. This holds for both finite and infinite endpoints a, b.

THEOREM 2.9. Let $J = (a,b),\ -\infty \le a < b \le \infty$. Assume that

$$P \in M_n(L_{loc}(a,b)); \quad F \in M_{n,m}(L_{loc}(a,b)) \tag{2.30}$$

i) Suppose, in addition to (2.30), that

$$P \in M_n(L(a,c)); \quad F \in M_{n,m}(L(a,c)) \tag{2.31}$$

for some $c \in (a,b)$. For some $u \in J$ and $C \in M_{n,m}(\mathbb{C})$, let Y be the solution of the IVP (2.3), (2.4) on J. Then

$$Y(a) = \lim_{t \to a^+} Y(t) \tag{2.32}$$

exists and is finite.

ii) Suppose that, in addition to (2.30), P, F satisfy

$$P \in M_n(L(c,b)); \quad F \in M_{n,m}(L(c,b)) \tag{2.33}$$

for some $c \in (a,b)$. For some $u \in J$ and $C \in M_{n,m}(\mathbb{C})$, let Y be the solution of the IVP (2.3), (2.4) on J. Then

$$Y(b) = \lim_{t \to b^-} Y(t) \tag{2.34}$$

exists and is finite.

PROOF: We establish Theorem 2.9 for b; the proof for the endpoint a is similar and hence omitted. It follows from (2.28) that $|Y|$ is bounded on $[c, b)$ for $c \in J$, say by B. Let $\{b_i\}$ be any strictly increasing sequence converging to b. Then for $j > i$ we have

$$|Y(b_j) - Y(b_i)| = \left| \int_{b_i}^{b_j} PY \right| \leq \; B \int_{b_i}^{b_j} |P|.$$

¿From this and the absolute continuity of the Lebesgue integral it follows that $\{Y(b_i) : i \in N\}$ is a Cauchy sequence and hence converges to a finite limit. □

THEOREM 2.10. Let $J = (a, b), \; -\infty \leq a < b \leq \infty$. Assume that

$$P \in M_n(L_{loc}(a, b)). \tag{2.35}$$

i) Suppose, in addition to (2.35), that

$$P \in M_n(L(a, c)) \tag{2.36}$$

for some $c \in (a, b)$. Let, for some $u \in J$ and $C \in M_{n,m}(\mathbb{C})$, Y be the solution of the IVP (2.3), (2.4) with $F = 0$ on J. Then

$$rank\, Y(a) = rank\, Y(u) \tag{2.37}$$

where $Y(a)$ is given by (2.32). Moreover, given any $C \in M_{n,m}(\mathbb{C})$ there exists a unique solution Y of the "endpoint" value problem:

$$Y' = PY, \;\; Y(a) = C. \tag{2.38}$$

ii) Suppose, in addition to (2.35), that

$$P \in M_n(L(c, b)) \tag{2.39}$$

for some $c \in (a, b)$. Let, for some $u \in J$ and $C \in M_{n,m}(\mathbb{C})$, Y be the solution of the IVP (2.3), (2.4) with $F = 0$ on J. Then

$$rank\, Y(b) = rank\, Y(u) \tag{2.40}$$

where $Y(b)$ is given by (2.34). Moreover, given any $C \in M_{n,m}(\mathbb{C})$ there exists a unique solution Y of the "endpoint" value problem:

$$(2.41) \qquad Y' = PY, \;\; Y(b) = C.$$

Note that the endpoints a and b in Theorem 2.10 may be finite or infinite.

PROOF: Note that (2.37) or (2.40) do not follow directly from (2.14) and (2.32) or (2.34) since the rank of a matrix is not a continuous function of the matrix. We argue as follows: Let $Y(u) = C$, $rank\, C = r$. If $r = 0$, then $Y(t) = 0$ for all $t \in J$ and $Y(b) = 0$ by (2.34). If $r > 0$, let $C_1, \dots, C_r$ be linearly independent columns of $Y(u)$ and construct a nonsingular $n \times n$ matrix D by adding $n - r$ appropriate columns to C_j, $j = 1, ..., r$. Let Z denote the solution of (2.13) determined by the initial condition $Z(u) = D$. It follows from (2.15) that $Z(t)$ is nonsingular for each $t \in J$ and hence $Z(b)$ is nonsingular by (2.34) and (2.15). (Note that it does not follow directly from (2.34) alone that $Y(b)$ is nonsingular since the rank of a matrix is not a continuous function of its coefficients.) Therefore $Z_1(b), ..., Z_r(b)$ are linearly independent. From the uniqueness part of the existence-uniqueness theorem - Theorem 2.2 - $Y(t) = Z(t)$ for $t \in J$ and hence also for $t = b$ by (2.34). The proof for the endpoint a is similar. This establishes (2.37) and (2.40). To prove the moreover parts of the Theorem consider the fundamental matrix Φ , see Definition 2.5, choose $u \in J$ and determine the solution Y of (2.13) by the initial condition $Y(u) = \Phi(b, u)\, C$ then $Y(b) = C$. Note that $\Phi(b, u)$ exists by (2.34). The proof of (2.38) is similar. □

2.6 Continuous dependence of solutions on the problem.

THEOREM 2.11. Let $u, v \in J = (a, b)$, $-\infty \leq a < b \leq \infty$, $C, D \in M_{n,m}(\mathbb{C})$, $P, Q \in M_n(L(J))$, $F, G \in M_{n,m}(L(J))$. Assume

$$(2.42) \qquad Y' = PY + F \;\; on\ J, \; Y(u) = C; \quad Z' = Q\, Z + G \;\; on\ J, \; Z(u) = D.$$

Then

$$(2.43) \qquad |Y(t) - Z(t)| \leq K\, e^{\left(\int_a^b |Q|\right)}, \;\; a \leq t \leq b,$$

where

$$(2.44) \qquad K = |C - D| + \left| \int_u^v |F| \right| + M \left| \int_u^v |P| \right| + \int_a^b |F - G| + M \int_a^b |P - Q|,$$

and

$$(2.45) \qquad M = \left(|C| + \int_a^b |F| \right) \exp \left(\int_a^b |P| \right).$$

PROOF: For $a < t < b$ this follows from Theorem 2.8 and the Gronwall inequality. The case $t = a$ and $t = b$ then follows from Theorem 2.9. □

THEOREM 2.12. Let $J = (a, b), -\infty \leq a < b \leq \infty$, $u \in J$, $C \in M_{n,m}(\mathbb{C})$, $P \in M_n(L(J))$, and $F \in M_{n,m}(L(J))$. Let $Y = Y(\cdot, u, C, P, F)$ be the solution of (2.3), (2.4) on J. Then Y is a continuous function of all its variables u, C, P, F uniformly on the closure of J ; more precisely, for fixed P, F, u, C; given any $\epsilon > 0$ there is a $\delta > 0$ such that if $v \in J$, $D \in M_{n,m}(\mathbb{C})$, $Q \in M_n(L(J))$, and $G \in M_{n,m}(L(J))$ satisfy

$$|u - v| + |C - D| + \int_a^b |P - Q| + \int_a^b |F - G| < \delta, \tag{2.46}$$

then

$$|Y(t, u, C, P, F) - Y(t, v, D, Q, G)| < \epsilon \ , \ a \leq t \leq b. \tag{2.47}$$

Note that $Y(t, u, C, P, F)$ is jointly continuous in u, C, P, F, uniformly for t in $\bar{J}$.

PROOF: The absolute continuity of the Lebesgue integral and (2.46) imply that the constant K in (2.44) can be made arbitrarily small. The conclusion then follows from Theorem 2.11. □

THEOREM 2.13. Let $J = (a, b)$, $-\infty \leq a < b \leq \infty$, let $P_k \in M_{n,n}(L_{loc}(J))$, $F_k \in M_{n,m}(L_{loc}(J))$, $C_k \in M_{n,m}$, $u_k \in J$, $k \in N_0 = \{0, 1, 2, \dots\}$. Assume

(i) $P_k \to P_0 \quad as \quad k \to \infty$
locally in $L_{loc}(J)$ in the sense that for each compact subinterval K of J we have

$$\int_K |P_k - P_0| \to 0 \quad as \quad k \to \infty;$$

(ii) $F_k \to F_0 \quad as \quad k \to \infty$
locally in $L_{loc}(J)$ in the sense that for each compact subinterval K of J we have

$$\int_K |F_k - F_0| \to 0 \quad as \quad k \to \infty;$$

(iii) $C_k \to C_0 \in C \quad as \quad k \to \infty;$
(iv) $u_k \to u_0 \in J \quad as \quad k \to \infty.$

Then

$$Y(t, u_k, C_k, P_k, F_k) \to Y(t, u_0, C_0, P_0, F_0) \quad as \quad k \to \infty$$

locally uniformly on J, i.e., uniformly in t on each compact subinterval of J.

Moreover, if $P_k \in M_n(L(J))$, $F_k \in M_{n,m}(L(J))$, and (i), (ii) hold in $L(J)$, i.e. with K replaced by J and (iii), (iv) hold, then

$$Y(t, u_k, C_k, P_k, F_k) \to Y(t, u_0, C_0, P_0, F_0) \quad as \quad k \to \infty$$

uniformly on the closure of J.

PROOF: This follows from Theorem 2.12. □

2.7 Differentiable dependence of solutions on the data including the coefficients.

Theorem 2.12 shows that the solution of the initial value problem (2.3), (2.4) with $P \in M_{n,n}(L_{loc}(J))$, $F \in M_{n,m}(L_{loc}(J))$, $C \in M_{n,m}(\mathbb{C})$ depends continuously on all the given data. In this section we show that this dependence is differentiable.

DEFINITION 2.14 (The Frechet derivative on Banach spaces)**.** A map T from a Banach space X into a Banach space Z, $T : X \to Z$, is differentiable at a point $x \in X$ if there exists a bounded linear map $T'(x) \;: X \to Z$ such that

$$|T(x+h) - T(x) - T'(x)\,h| = o\,(|h|),\; as\; h \to 0\; in\; X.$$

That is, for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|T(x+h) - T(x) - T'(x)\,h| \le \varepsilon\;(|h|)\; for\; all\; h \in X\; with\; |h| < \delta.$$

If such a map $T'(x)$ exists, it is unique and is called the (Frechet) derivative of T at x. A map T is differentiable on a set $S \subset X$ if it is differentiable at each point of S. In this case the derivative is a map : $x \to T'(x)$ from S into the Banach space $L(X, Z)$ of all bounded linear operators from X into Z denoted by T'. To say that T' is continuously differentiable on S or T is C^1on S means that the map T' is continuous in the operator topology of the Banach space $L(X, Z)$.

The differentiability of the solution

$$Y \;=\; Y(t, u, C, P, F)$$

with respect to t follows from the definition of solution. The differentiability of Y with respect to u is established in the next lemma.

LEMMA 2.15. Fix t, C, P, F and consider Y as a function of u. Then $Y \in AC_{loc}(J)$.

PROOF: It follows from the representation (2.16) that the fundamental matrix $\Phi(t,u)$ is differentiable with respect to u, since the inverse of a differentiable matrix is differentiable. The differentiability of Y with respect to u then follows from the representation

$$Y(t,u) = \Phi(t,u)\,C + \int_u^t \Phi(t,s)\,F(s)\,ds.$$

This concludes the proof. □

For fixed t, u, Y is a function of C, P, F mapping $M_{n,m}(\mathbb{C}) \times M_n(L(J)) \times M_{n,m}(L(J))$ into $M_{n,m}(L(J))$. By Theorem 2.12, Y is continuous in C, P, F. Is it differentiable in C? in P? in F?

THEOREM 2.16. For fixed $t, u \in J$, $P \in M_n(L(J))$ and $F \in M_{n,m}(L(J))$, the solution $Y = Y(t,u,C,P,F)$ of (2.3), (2.4) is differentiable in C; its derivative is given by

$$Y'(C) = \frac{\partial Y}{\partial C}(t,u,C,P,F) = \Phi(t,u,P). \tag{2.48}$$

Thus we have

$$Y'(C)\,H = \Phi(t,u,P)\,H, \quad H \in M_{n,m}(\mathbb{C}). \tag{2.49}$$

The derivative $Y'(C)$ is constant in C and in F.

PROOF: This follows directly from the variation of parameters formula and the definition of derivative. □

THEOREM 2.17. Let $J = [a,b]$. Fix $t, u \in J$, $C \in M_{n,m}(\mathbb{C})$, $P \in M_n(L(J))$, and $F \in M_{n,m}(L(J))$; let $Y = Y(t,u,C,P,F)$. We have

(2.50)
$$Y'(F)(H) = \frac{\partial Y}{\partial F}(t,u,C,P,F)(H) = \int_u^t \Phi(t,s)H(s)\,ds, \; H \in M_{n,m}(L(J)).$$

Here the right side of equation (2.50) defines a bounded linear operator on the space $M_{n,m}(L(J))$. The derivative $Y'(F)$ is constant in F.

PROOF: From the variation of parameters formula we get

$$Y(t,u,C,P,F+H) - Y(t,u,C,P,F) = \int_u^t \Phi(t,s,P)H(s)\,ds.$$

The conclusion follows from this equation and the definition of derivative. □

Before stating the next Theorem we give two lemmas. These may be of independent interest.

LEMMA 2.18. Let $J=(a,b)$, $-\infty \le a < b \le \infty$, let $P \in M_n(L_{loc}(J))$, let $u \in J$. Then for any $t \in J$ we have

$$\begin{aligned} \Phi(t,u,P) &= I + \int_u^t P + \int_u^t P(r)\int_u^r P(s)\,drds \\ &\quad + \int_u^t P(r)\int_u^r P(s)\int_u^s P(x)\,dxdsdr + \cdots . \end{aligned} \tag{2.51}$$

PROOF: This follows directly from the successive approximations proof of the existence-uniqueness Theorem : Start with the first approximation $\Phi_0 = I$; then $\Phi_1 = I + \int_u^t P$, etc. □

LEMMA 2.19. Let $P, H \in M_n(L_{loc}(J))$. Then for any $t, u \in J$ we have

$$\Phi(t,u,P+H) \;=\; \Phi(t,u,P)\;\Phi(t,u,S), \tag{2.52}$$

where

$$S \;=\; \Phi^{-1}(\cdot,u,P)\;H\;\Phi(\cdot,u,P). \tag{2.53}$$

PROOF: The proof consists in showing that both sides satisfy the same initial value problem and then using the existence-uniqueness Theorem. □

LEMMA 2.20. If P commutes with the integral of H in the sense that

$$P(t)\,(\int_u^s H) = (\int_u^s H)\,P(t), \quad s,t,u \in J, \tag{2.54}$$

then the exponential law holds:

$$\Phi(t,u,P+H) \;=\; \Phi(t,u,P)\;\Phi(t,u,H). \tag{2.55}$$

PROOF: It follows from Lemmas 2.18, 2.19 and hypothesis (2.54) that $\Phi(\cdot, u, P)\, H = H\, \Phi(\cdot, u, P)$ and hence $S = H$ in (2.53). $\square$

THEOREM 2.21. Let $J = [a, b]$. Fix $t, u \in J$, $C \in M_{n,m}(\mathbb{C})$, $F \in M_{n,m}(L(J))$. For $P \in M_n(L(J))$ let $Y = Y(t, u, C, P, F)$ be the unique solution of (2.3), (2.4). Then the map $P \to Y(t, u, C, P, F)$ from the Banach space $M_n(L(J))$ to $M_{n,m}(\mathbb{C})$ is differentiable and its derivative

$$Y'(P) = \frac{\partial Y}{\partial P}(t,u,C,P,F) \tag{2.56}$$

is the bounded linear transformation from the Banach space $M_n(L(J))$ to the Banach space $M_{n,m}(\mathbb{C})$ given by

$$Y'(P)\,H \;=\; \Phi(t,u,P)\left(\int_u^t \Phi^{-1}(r,u,P)H(r)\Phi(r,u,P)\,dr\right)C$$

(2.57)

$$+\int_u^t \Phi(t,r,P)\left(\int_r^t \Phi^{-1}(s,u,P)H(s)\Phi(s,u,P)\,ds\right)F(r)\,dr,\;\; H\in M_n(L(J)).$$

PROOF: Fix t,u,C,F and let $Y(t,P)=Y(t,u,C,P,F)$. From the variation of parameters formula it follows that for $H\in M_{n,n}(L(J))$ and S defined by (2.53) we have

$$\begin{aligned}
&Y(t,P+H)-Y(t,P)\\
&=\Phi(t,P+H)\,C+\int_u^t \Phi(t,s,P+H)\,F(s)\,ds-\Phi(t,P)\,C\\
&\quad-\int_u^t \Phi(t,s,P)\,F(s)\,ds\\
&=\Phi(t,u,P)\,[\Phi(t,u,S)-I]\,C+\int_u^t \Phi(t,s,P)[\Phi(t,r,S)-I]F(r)dr\\
&=\Phi(t,u,P)\,\left[\int_u^t S+\int_u^t S(x)\int_u^x S(y)dydx+\dots\right]C+\int_u^t \Phi(t,r,P)\\
&\quad\left[\int_r^t S+\int_r^t S(x)\int_r^x S(y)dydx+\dots\right]F(r)dr
\end{aligned}$$

Hence

$$\begin{aligned}
&Y(t,u,P+H)-Y(t,u,P)-\Phi(t,u,P)\left(\int_u^t S(r)dr\right)C\\
&\quad-\int_u^t \Phi(t,r,P)\left(\int_r^t S(x)dx\right)F(r)dr\\
&=\Phi(t,u,P)\,\left[\int_u^t S(x)\int_u^x S(y)dydx+\dots\right]C\\
&\quad+\int_u^t \Phi(t,r,P)\,\left[\int_r^t S(x)\int_r^x S(y)dydx+\dots\right]F(r)dr\\
&=E(H).
\end{aligned}$$

Noting that $|S|(b-a)| \le |k\,H|$ for some $k \in \mathbb{R}$, that $|\Phi(t,u,P)|$ and $|\Phi^{-1}(t,u,P)|$ are bounded on J, there exists an $M > 0$ such that

$$\begin{aligned} |E(H)| &\le M|C|\,[|S(b-a)|^2 + |S(b-a)|^3 \ldots] + M|F|\,[|S(b-a)|^2 + |S(b-a)|^3 \ldots] \\ &\le M|C|\,|kH|\,[|kH| + |kH|^2 + \ldots] + M|F|\,|kH|\,[|kH| + |kH|^2 + \ldots] \end{aligned}$$

¿From this it follows that

$$\frac{|E(H)|}{|H|} \to 0 \; as \; |H| \to 0 \; in \; M_n(L(J)).$$

This completes the proof. □

THEOREM 2.22. Let the hypotheses and notations of Theorem 2.21 hold and assume, in addition, that the commutativity hypothesis (2.54) is satisfied. Then

1.

$$H(t)\;\Phi(t,u,P) \;=\; \Phi(t,u,P)\;H(t), \quad t,u \in J. \tag{2.58}$$

2. The exponential law holds, i.e.

$$\Phi(t,u,P+H) \;=\; \Phi(t,u,P)\,\Phi(t,u,H), \quad t,u \in J. \tag{2.59}$$

3. Formula (2.57) reduces to

$$\begin{aligned} &Y'(P)(H) = \Phi(t,u,P)\left(\int_u^t H(s)\,ds\right) C \;+ \\ &\int_u^t \Phi(t,r,P)\left(\int_r^t H(s)\,ds\right) F(r)\,dr, \; t,u \in J. \end{aligned} \tag{2.60}$$

Note however that $Y\,'(P)$ is not the operator defined by the right hand side of (2.60) since H cannot be restricted to satisfy the commutativity hypothesis (2.54) in the definition of the derivative $Y\,'(P)$.

PROOF: This follows from Theorem 2.21 and Lemma 2.20. □

REMARK 1. In the special case when P and H are constant matrices we have

$$Y'(P)(H) \;=\; \exp((t-u)P)\left(\int_u^t \exp((u-r)P)H\exp((r-u)P)\,dr\right) C$$

$$(2.61) \qquad + \int_u^t \exp((t-r)P) \left(\int_r^t \exp((u-s)P) H \exp((s-u)P) ds \right) F(r)\, dr.$$

Note that if P and H are constant and commute, then (2.61) reduces to

$$Y\,'(P)(H) = (t-u) \exp\left((t-u)P\right) HC + \int_u^t (t-r) \exp((t-r)P) H F(r)\, dr.$$

But this reduction does not hold, in general, for constant matrices which do not commute.

COROLLARY 2.23. Consider the exponential map of matrices:

$$E(A) = e^A, \;\; A \in M_n(\mathbb{C}).$$

The Frechet derivative of E is the bounded linear operator from $M_n(\mathbb{C})$ into $M_n(\mathbb{C})$ given by

$$(2.62) \qquad E'(A)\, H = e^A \int_0^1 e^{-r\,A}\, H\, e^{r\,A}\, dr, \;\; H \in M_n(\mathbb{C}).$$

Note that (2.62) reduces to the more familiar formula $E'(A) = E(A)$ for all $A \in M_n(\mathbb{C})$ only in the one dimentional case $n = 1$. When $n > 1$ (2.62) reduces to $E'(A) = E(A)$ only for constant multiples $A = cI_n$, $c \in \mathbb{C}$, of the identity since only multiples of the identity satisfy the commutativity condition with respect to all matrices in $M_n(\mathbb{C})$.

In Corollary 2.23 $M_n(\mathbb{C})$ can be replaced by $M_n(\mathbb{R})$; in fact $M_n(\mathbb{C})$ can be replaced by an arbitrary Banach algebra. See [56].

PROOF: This is the special case of Theorem 2.21 when $a = 0 = u, \; b = 1, \; P(t) = A$ for all $t \in [0, 1], \; F \equiv 0, \; C = I$. □

For fixed t, u, C, F replace P by $P + zW$ and fix P and W. The next result shows that the solution $Y = Y(t, u, C, P + zW, F)$ of (2.3), (2.4) with P replaced by $P + zW$ is an entire function of z. What is

$$Y'(z) = \frac{\partial Y}{\partial z} \; ?$$

This question is answered by

THEOREM 2.24. Let $J = (a,b)$, $-\infty \le a < b \le \infty$, $t,u \in J$, $C \in M_{n,m}$, $P, W \in M_n(L(J))$, $F \in M_{n,m}(L(J))$; let $Y = Y(t,u,C,P+zW,F)$ denote the unique solution of (2.3), (2.4) for each $z \in C$. Then Y is an entire function of z and

$$Y'(z) = \Phi(t,u,P+zW)\left(\int_u^t \Phi^{-1}(r,u,P+zW)\,W(r)\,\Phi(r,u,P+zW)\,dr\right)C$$
$$+\int_u^t \Phi(t,r,P+zW)\left(\int_r^t \Phi^{-1}(s,u,P+zW)\,W(s)\,\Phi(s,u,P+zW)\,ds\right)F(r)dr$$

PROOF:

$$\begin{aligned}&[Y(t,u,C,P+(z+h)W,F) - Y(t,u,C,P+zW,F)]\\&= [\Phi(t,u,P+(z+h)W) - \Phi(t,u,P+zW)]\,C\\&+\int_u^t [\Phi(t,r,P+(z+h)W) - \Phi(t,r,P+zW)]\,F(r)\,dr.\end{aligned} \tag{2.63}$$

Let

$$S(z) = \Phi^{-1}(\cdot,u,P+zW)W(\cdot)\,\Phi(\cdot,u,P+zW).$$

Proceeding similarly to the proof of Theorem 2.21 we get

$$\begin{aligned}&\Phi(t,u,P+(z+h)W) - \Phi(t,u,P+zW)\\=\;&\Phi(t,u,P+(z+h)W)[\Phi(t,u,hS(z)) - I]\\=\;&\Phi(t,u,P+zW)[h\int_u^t S(z) + o(h)]\\=\;&h\Phi(t,u,P+zW)\int_u^t S(z) + o(h)\end{aligned} \tag{2.64}$$

Combining these two we get

$$\begin{aligned}&[Y(t,u,C,P+(z+h)W,F) - Y(t,u,C,P+zW,F)]\\&= h\left[\Phi(t,u,P+zW)\left(\int_u^t S(z)\right)C + \int_u^t \Phi(t,r,P+zW)\left(\int_r^t S(z)\right)F(r)\,dr\right]\\&+o(h).\end{aligned}$$

And the result follows. □

THEOREM 2.25. Let $J = (a,b)$, $-\infty \le a < b \le \infty$, $t,u \in J$, $C \in M_{n,m}(\mathbb{C})$, $P \in M_n(L(J))$, $F \in M_{n,m}(L(J))$; and for each $z \in C$ let $Y = Y(t,u,C,P+zW,F)$ denote the unique solution of (2.3), (2.4) for each $W \in L(J)$. Then Y is a differentiable function of W and

$$Y'(W)\,H = z\,\Phi(t,u,P+zW)\,\left(\int_u^t \Phi^{-1}(r,u,P+zW)H(r)\,\Phi(r,u,P+zW)\;dr\right)C$$
$$+\,z\int_u^t \Phi(t,r,P+zW)\left(\int_r^t \Phi^{-1}(s,u,P+zW)H(s)\,\Phi(s,u,P+zW)\;ds\right)F(r)\,dr,$$

for $H \in L(J)$.

PROOF: The proof is similar to that of Theorem 2.21 and hence omitted. □

2.8 Adjoint systems.

LEMMA 2.26. Let P, Q be any $k \times k$ complex matrix functions on J. Let F, G be $k \times m$ complex matrix functions on J. If $Y' = PY + F$ and $Z' = QZ + G$ and $C \in E^k(C)$, then

$$(Z^*CY)' = Z^*(Q^*C + CP)Y + Z^*CF + G^*CY. \tag{2.65}$$

PROOF: This follows from a straightforward computation and is therefore omitted. □

COROLLARY 2.27. Let the assumptions and notation be as in Lemma 2.26 . If, in addition, C is invertible and $Q = -C^{-1^*}P^*C^*$, then

$$(Z^*CY)' = Z^*CF + G^*CY. \tag{2.66}$$

PROOF: This follows from (2.65). □

The fundamental matrices of adjoint systems are closely related to each other. The next result gives this relationship. It plays an important role in the theory of adjoint and, in particular, self-adjoint boundary value problems.

THEOREM 2.28 (Adjointness Lemma)**.** Let $P \in M_n(L_{loc}(J))$, let $E \in M_n(\mathbb{C})$. Assume

$$E^{-1}E^* = I \text{ or } E^{-1}E^* = -I \tag{2.67}$$

and define

$$P^+ = -E^{-1}\,P^*\,E. \tag{2.68}$$

Then

$$\Phi(t,s,P) = E^{-1}\,\Phi^*(s,t,P^+)\,E,\ \ s,t\,\in J. \tag{2.69}$$

PROOF: Fix $s \in J$ and let

$$Z(t) = E^{-1*}\Phi^*(t,s,P)\,E^*\Phi(t,s,P^+),\ t \in J.$$

Note that $Z(s) = I$ and

$$\begin{aligned}
Z'(t) &= E^{-1*}[P(t)\Phi(t,s,P)]^*\,E^*\,\Phi(t,s,P^+)\\
&\quad +E^{-1*}\Phi^*(t,s,P)\,E^*\,P^+(t)\,\Phi(t,s,P^+)\\
&= E^{-1*}\Phi^*(t,s,P)\,EE^{-1}P^*(t)E\,E^{-1}\,E^*\,\Phi(t,s,P^+)\\
&\quad +E^{-1*}\Phi^*(t,s,P)\,E^*\,P^+(t)\,\Phi(t,s,P^+)\\
&= -E^{-1*}\Phi^*(t,s,P)\,E^*\,P^+(t)\,\Phi(t,s,P^+)\\
&\quad +E^{-1*}\Phi^*(t,s,P)\,E^*\,P^+(t)\,\Phi(t,s,P^+)\\
&= 0,\ t \in J,
\end{aligned}$$

using (2.67) and (2.68). Hence $Z(t) = I$, for $t \in J$. That this is equivalent to (2.69) follows from the representation $\Phi(t,s,P^+) = Y(t)\,Y^{-1}(s),\ s,t \in J$, for any fundamental matrix Y of $Y' = P^+\,Y$. □

2.9 Inverse Initial Value Problems.

NOTATION. Given d n-dimensional vectors $Y_1, Y_2, \ldots, Y_d$ we denote the $n \times d$ matrix whose $i-th$ column is $Y_i,\ i = 1, \ldots, d$, by

$$Y = [Y_1, Y_2, \ldots, Y_d]. \tag{2.70}$$

Above we started with a coefficient matrix P and, possibly, a nonhomogeneous term F and then studied the existence of solutions and their properties. Here we reverse this. Given a number of functions, under what conditions are they solutions of a first order linear system? For the sake of completeness we state the theorem for both the direct and the inverse problems.

THEOREM 2.29. (i) Let $1 \le d \le n,\ P \in M_n(L_{loc}(J))$. Assume that Y_i, $i = 1, \ldots, d$ are vector solutions of

$$Y' \,=\, PY\,. \tag{2.71}$$

If

$$(2.72) \qquad rank\,[Y_1, Y_2, \ldots, Y_d](t) = d$$

for some t in J, then this is true for every t in J.

(ii) Let $Y_i \in M_{n,1}(AC_{loc}(J))$, $i = 1, \ldots, d$, $1 \le d \le n$. Assume that

$$(2.73) \qquad rank[Y_1, \ldots, Y_d](t) \;=\; d \quad t \in J.$$

Then there exists an $n \times n$ matrix $P \in M_n(L_{loc}(J))$ such that Y_i, $i = 1, \ldots, d$, are solutions of (2.71).

Furthermore, if $Y_i \in M_{n,1}(C^1(J))$, $i = 1, \ldots, d$ then there exists a continuous such P.

PROOF: Part (i) is contained in Theorem 2.2 so we only prove part (ii). If $d = n$ take $P = Y' Y^{-1}$. If $d < n$ we construct an $n \times n$ matrix

$$M = [Y_1, Y_2, \ldots, Y_d, Y_{d+1}, \ldots, Y_n]$$

as follows. For each $t_1 \in J$ there is a $d \times d$ nonsingular submatrix of the $n \times d$ matrix $[Y_1, \ldots, Y_d](t_1)$. Let its rows be numbered by $r_1, \ldots, r_d$. To the right of the first row which in not one of these place the first row of the $(n-d) \times (n-d)$ identity matrix; to the right of the second row which is not one of these place the second row of the $(n-d) \times (n-d)$ identity matrix, and so on. Thus each of Y_i for $i > d$ is a constant matrix with all components zero except one which is the number 1. For each $t_1 \in J$ the matrix M so constructed is nonsingular at t_1 and by continuity $\det M(t) \neq 0$ for all t in some neighborhood N_{t_1} of t_1. Take

$$P(t) = M'(t)\, M^{-1}\,(t)\,,\ for\ t \in N_{t_1}.$$

Any compact subinterval of J can be covered by a finite number of such neighborhoods N_{t_1} and hence P can be defined on J. On points which are covered by more than one such neighborhood, P is multiply defined, we just choose one definition, say the one determined by the lowest numbered neighborhood. Clearly $P \in M_n(L_{loc}(J))$ and Y_i $i = 1, \ldots, d$ are solutions. This completes the proof of the first part of (ii).

To prove the furthermore part we note that the constructed matrix P is piecewise continuous by construction. Thus to get a continuous P we remove the multiply defined aspect of the above construction as follows: On a subinterval which is covered by two or more of the neighborhoods N_t discard all definitions of M used above - just on this subinterval - then connect the two remaining pieces together in such a way as to keep M nonsingular on J. Then construct a new P from the new M as above for all $t \in J$. This results in a continuous P and completes the proof.□

2.10 Comments.

Most of Section 2 is based on the paper [59] by Kong and Zettl. Below we comment on each subsection separately.

1. The notation for matrix functions such as $M_n(L(J))$ is taken from [70].
2. The sufficiency of the local integrability conditions of Theorem 2.2 are well known - see [72] or [81]; the necessity given by Theorem 2.4 is due to Everitt and Race - see [25]. Except for the use of the Bielecki norm the first proof of Theorem 2.2 is the standard successive approximations argument although it is dressed in the clothes of the Contraction Mapping Theorem in Banach space here. The advantage of the Bielecki norm is that it yields a global proof; the sup norm would only give a local proof and then one has to patch together the intervals of existence. The second proof is a minor variant of the usual successive approximations argument.

   The constancy of the rank of solutions given by Theorem 2.3 is known - see [42] or [75] but we haven't seen it stated under these general conditions. It is surprising how many authors, including the two just mentioned assume continuity of the coefficients when only Lebesgue integrability is needed. This is of some consequence both theoretically and numerically when coefficients are approximated by piece-wise constants, piece-wise linear functions, etc.
3. The variation of parameters formula given by Theorem 2.6 is standard, but our notation is not. We use a notation which shows the dependence of the fundamental matrix on the coefficient matrix P. This is handy for the differentiation results that follow.
4. A detailed discussion of the Gronwall inequality is given here because it is a very useful tool and we do not want any continuity assumptions on f and g.

   The Gronwall inequality has many extensions: see - [11],
5. Theorem 2.8 is elementary but we have not seen it stated in this generality. The continuous extensions of solutions given by Theorem 2.9 are a special case of much more powerful results e.g. Levinsons asymptotic theorem - [14]. Often the existence of limits of solutions are stated only for infinite endpoints. *We want to emphasize here that the relevant consideration is not whether the endpoint is finite or infinite but whether the coefficient matrix P and the inhomogeneous term F are integrable or not all the way to the endpoint.* Theorem 2.10 may be new in [59].
6. Theorems 2.11, 2.12, and 2.13 illustrate clearly that the natural space in which to study solutions of linear ode's is $L^1_{loc}(J)$ in the singular case and $L^1(J)$ in the regular case.
7. Sections 7 and 9 were motivated to some extend by the elegant treatment of the inverse spectral theory for regular Sturm-Liouville problems by Poeschel and Trubowitz in [76]. Theorem 2.14 is standard - see [81] or [3]. Theorems 2.14 and 2.15 are trivial consequences of the variation of parameters formula. Theorems 2.21, 2.24, 2.25, are not so trivial consequences of the Variation of Parameters Formula; and may be new in [59].
8. Adjoint systems of this type were used by Atkinson [3] . They will be used in the next chapter to provide an elegant proof of a very general Lagrange identity due to Everitt and Neumann, see [24]. Theorem 2.27, the Adjointness Lemma, is due to Zettl, see [82], [83], [87].

9. These kinds of inverse problems are discussed by Hartman [42] and Petrovski [75] but not in this generality.

3 SCALAR INITIAL VALUE PROBLEMS (IVP)

We study initial value problems (IVP) consisting of the equation

$$(3.1) \qquad -(py')' + qy = f \text{ on } J$$

together with initial conditions

$$(3.2) \qquad y(c) = h,\ (py')(c) = k, \quad c \in J, \quad h,\, k \in \mathbb{C}$$

where

$$(3.3) \qquad J = (a, b),\ -\infty \leq a < b \leq \infty,\ p,\ q,\ f : J \to \mathbb{C}.$$

3.1 Existence and uniqueness.

DEFINITION 3.1 (Solution)**.** By a solution of equation (3.1) we mean a function $y : J \to \mathbb{C}$ such that y and py' are absolutely continuous on each compact subinterval of J and the equation is satisfied a.e. on J. Given a solution y of (3.1) we refer to (py') as its quasi-derivative to distinguish it from the classical derivative y'.

Note that the classical derivative of a solution y, in general, exists only almost everywhere but the quasi-derivative (py') is absolutely continuous on all compact subintervals of J and thus exists and is continuous at each point of J.

THEOREM 3.2. Every initial value problem (IVP) (3.1), (3.2), (3.3) has a solution defined on J and this solution is unique if and only if

$$(3.4) \qquad 1/p,\ q,\ f\ \in L_{loc}(J).$$

If all the data p, q, f, h, k is real, then the solution is real on J.

PROOF: Let

$$(3.5) \qquad P = \begin{pmatrix} 0 & 1/p \\ q & 0 \end{pmatrix},\ F = \begin{pmatrix} 0 \\ f \end{pmatrix},\ Y = \begin{pmatrix} y \\ py' \end{pmatrix}$$

Then the equation (3.1) is equivalent to the first order system

$$Y' = PY + F \text{ on } J, \tag{3.6}$$

in the sense that, given any scalar solution y of (3.1) the vector Y defined by (3.5) is a solution of the system (3.6) and conversely, given any vector solution Y of system (3.6) its top component y is a solution of (3.1). Theorem 3.2 follows from this system representation and Theorems 2.2 and 2.4. □

Notation. Given (3.3) and (3.4), the unique solution y of (3.1), (3.2) and its quasi-derivative py' are denoted by

$$y = y(\cdot, c, h, k, 1/p, q, f), \quad py' = (py')(\cdot, c, h, k, 1/p, q, f), \tag{3.7}$$

to highlight their dependence on these quantities. In the theory of boundary value problems the spectral parameter λ and the weight function w play important roles; thus we also study the equation

$$-(py')' + qy = \lambda\, w\, y \text{ on } J,\ \lambda \in \mathbb{C}, \tag{3.8}$$

where

$$p,\ q,\ w:\ J \to \mathbb{C},\ 1/p,\ q,\ w \in L_{loc}(J) \text{ on } J. \tag{3.9}$$

For this case we use the notation

$$y = y(\cdot, c, h, k, 1/p, q, w, \lambda),\ (py') = (py')(\cdot, c, h, k, 1/p, q, w, \lambda). \tag{3.10}$$

Theorem 3.2 and its proof readily extend to the case when q is replaced by $q - \lambda w$.

Below, when we study the dependence of y and py' on one of these quantities with all the others fixed we further abbreviate this notation by simply omitting all the fixed variables. Thus we write $y = y(\cdot, c)$ when we wish to study the unique solution as a function of c, $y = y(\cdot, q)$ to study the dependence of y on q, etc.

3.2 Continuous extensions to the endpoints.

DEFINITION 3.3 (Regular and singular endpoints)**.** Let $J = (a, b)$, $-\infty \leq a < b \leq \infty$, and consider the equation (3.8) with conditions (3.9).

The end-point a is said to be *regular* if

$$1/p,\ q,\ w \in L(a,d) \tag{3.11}$$

for some $d \in J$; otherwise it is called *singular*. Similarly, the end-point b is said to be *regular* if

$$1/p,\ q,\ w \in L(d,b) \tag{3.12}$$

for some $d \in J$; otherwise it is called *singular*.

Note that, given (3.9), if (3.11) or (3.12) hold for some $d \in J$ then they hold for any such d.

REMARK 2. In much of the literature an infinite endpoint is automatically classified as singular in contrast with Definition 3.3. We propose this definition in view of the fact that, given (3.9), (3.12) is necessary and sufficient for all solutions y of (3.8) and their quasi-derivatives py' to have a finite limit at b. See Theorem 3.4 below.

It is not the finite or infinite nature of the endpoint b but condition (3.12) which determines whether or not all solutions of (3.8) and their quasi-derivatives have finite limits at b. This is a natural definition of "regular" behavior at b. Similar remarks apply at the endpoint a.

THEOREM 3.4. The limits

$$y(a) = \lim_{t\to a^+} y(t), \quad (py')(a) = \lim_{t\to a^+} (py')(t) \tag{3.13}$$

both exist and are finite for the solution y of every initial value problem (3.1), (3.2), (3.3), (3.4) if and only if

$$1/p,\ q,\ f \in L(a,d) \tag{3.14}$$

for some $d \in (a,b)$; the limits

$$y(b) = \lim_{t\to b^-} y(t), \quad (py)(b) = \lim_{t\to b^-} (py')(t) \tag{3.15}$$

both exist and are finite for the solution y of every initial value problem (3.1), (3.2), (3.3), (3.4) if and only if

$$1/p,\ q,\ f \in L(d,b) \tag{3.16}$$

for some $d \in (a, b)$.

PROOF: The sufficiency of the conditions (3.16), (3.14) follows from Theorem 2.9. A proof of the necessity can be constructed along the lines of the proof of Theorem 2.4. □

THEOREM 3.5. Let (3.1), (3.2), (3.3), (3.4) hold.

- Assume (3.14) and define $y(a)$ and $(py')(a)$ by (3.13). Then each "initial value problem " consisting of equation (3.1) and the terminal value conditions

$$y(a) = h,\ \ (py')(a) = k,\ \ h \in \mathbb{C} \tag{3.17}$$

  has a unique solution on J; this solution is real if all the data is real.
- Assume (3.16) and define $y(b)$ and $(py')(b)$ by (3.15). Then each "initial value problem " consisting of equation (3.1) and the terminal value conditions

$$y(b) = h,\ (py')(b) = k,\ \ h,\ \ k \in \mathbb{C} \tag{3.18}$$

  has a unique solution on J; this solution is real if all the data is real.

PROOF: This follows from the moreover part of Theorem 2.10. □

3.3 Continuous dependence of solutions on the problem.

THEOREM 3.6. Let (3.1), (3.2), (3.3) and (3.4) hold. Using the notation (3.7) each solution y of (3.1), (3.2) and its quasi-derivative py' is a jointly continuous function of all its variables, uniformly on compact subintervals of $J = (a, b)$. More precisely, given $c_j \in J$, h_j, $k_j \in \mathbb{C}$, $1/p_j$, q_j, $f_j \in L_{loc}(J)$, $j = 1, 2$, and given $\epsilon > 0$ and a compact subinterval $K = [a_1, b_1]$ of J containing $c_1$and c_2, there exists a $\delta > 0$ such that if

(3.19)
$$|c_1 - c_2| + |h_1 - h_2| + |k_1 - k_2| + \int_K (|1/p_1 - 1/p_2| + |q_1 - q_2| + |f_1 - f_2|)\ < \delta,$$

then

$$|y(t, c_1, h_1, k_1, 1/p_1, q_1, f_1)\ -\ |y(t, c_2, h_2, k_2, 1/p_2, q_2, f_2)|\ < \epsilon \tag{3.20}$$

and

$$|(py')(t, c_1, h_1, k_1, 1/p_1, q_1, f_1)\ -\ (py')(t, c_2, h_2, k_2, 1/p_2, q_2, f_2)| < \epsilon \tag{3.21}$$

both for all $t \in K$.

Furthermore, if $1/p, q, w \in L(J)$, and (3.19) holds with $K = J$, then (3.20) and (3.21) hold on J.

PROOF: This is a consequence of Theorem 2.12. □

3.4 Differentiable dependence of solutions on the data.

The differentiability of the solution

$$y(t, c, h, k, 1/p, q, w, \lambda)$$

of (3.8), (3.9) and its quasi-derivative (py') with respect to t follows from the definition of solution; the differentiability of y and (py') with respect to c is a consequence Lemma 2.16. The differentiability of y and of (py') with respect to the other variables is studied in this subsection.

THEOREM 3.7. Let (3.3), (3.8), (3.9) hold. Let u, v be solutions of (3.8) determined by the initial conditions

$$u(c) = 0,\ (pu')(c) = 1;\ v(c) = 1,\ (pv')(c) = 0,\ c \in J.$$

Using the notation (3.10) with the associated convention mentioned in the paragraph below (3.10), we have that each of the following maps from $\mathbb{C}$ to $\mathbb{C}$:

$$\begin{array}{lll} h & \to & y(t, c, h, k, 1/p, q, w, \lambda),\ h \to (py')(t, c, h, k, 1/p, q, w, \lambda), \\ k & \to & y(t, c, h, k, 1/p, q, w, \lambda),\ k \to (py')(t, c, h, k, 1/p, q, w, \lambda) \end{array}$$

is differentiable and the derivatives are given by:

$$y'(h) = v(t)\, h,\ h \in \mathbb{C}$$

$$(py')'(h) = (pv')(t)\, h,\ h \in \mathbb{C}$$

$$y'(k) = u(t)\, k,\ k \in \mathbb{C}$$

$$(py')'(k) = (pu')(t)\, k, k \in \mathbb{C},$$

respectively. Note that here $y'(h)$ denotes the derivative of y with respect to h, and in $(py')'(h)$ the outside prime denotes the derivative of the quasi-derivative (py') with respect to h. Thus the two primes in $(py')'$ have different meanings in this formula - the outside one is for differentiation with respect to h , the inside one for diffentiation with respect to t - but since t is fixed here this should not cause confusion. Similar remarks apply to the formulas for differentiation with respect to k.

Let $K = [a_1, b_1]$ be a compact subinterval of J. Each of the following maps from $\mathbb{C}$ to the Banach space $C(K)$:

$$\begin{array}{rcl} h & \to & y(\cdot, c, h, k, 1/p, q, w, \lambda),\ h \to (py')(\cdot, c, h, k, 1/p, q, w, \lambda), \\ k & \to & y(\cdot, c, h, k, 1/p, q, w, \lambda),\ k \to (py')(\cdot, c, h, k, 1/p, q, w, \lambda) \end{array}$$

is differentiable and its Frechet derivative is given by

$$y'(k)\,(g) = v\,g,\ g \in C(K)$$

$$(py')'(k)\,(g) = (pv')\,g,\ g \in C(K)$$

$$y'(k)(g) = u\,g,\ g \in C(K)$$

$$(py')'(k)(g) = (pu')\,g,\ g \in C(K),$$

respectively.

PROOF: This is a straightforward consequence of the variation of parameters formula and the definition of the Frechet derivative. See the above remarks about notation. □

To compute the derivatives of y and py' with respect to $1/p$, q, w and λ we use the fundamental matrix of the system representation of equation (3.8): Let

$$P = \begin{pmatrix} 0 & 1/p \\ q & 0 \end{pmatrix},\ W = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}$$

and define $\Phi(t, s, P, w, \lambda) = (\phi_{ij})$ to be the fundamental matrix of the system $Y' = (P - \lambda W)Y$ determined by the initial condition $Y(s) = I$ for each $s \in J$ where I is the identity matrix. (See Definition 2.5 for Φ.)

THEOREM 3.8. Let (3.3), (3.8), (3.9) hold, and let K be a compact subinterval of J. Fix t, $c \in K$, h, k, $\lambda \in \mathbb{C}, 1/p$, $w \in L(K)$; then the maps $q \to y(t, q)$, $w \to y(t, w)$, $1/p \to y(t, 1/p)$ from $L(K)$ to $\mathbb{C}$ as well as the map $\lambda \to y(t, \lambda)$ from $\mathbb{C} \to \mathbb{C}$ are differentiable and their derivatives are given by

$$y'(t, q)(r) = -\int_c^t \phi_{1,2}(t, s)\, y(t, s)\, r(s)\, ds,\ r \in L(K),$$

$$(py')(t, q)(r) = -\int_c^t \phi_{2,2}(t, s)\, y(t, s)\, r(s)\, ds,\ r \in L(K),$$

$$y'(t,1/p)(r) = \int_c^t \phi_{1,2}(t,s)\,q(s)\left(\int_c^s (py')(x)\,r(x)\,dx\right)ds$$
$$+\int_c^t (py')(x)\,r(x)\,dx,\ r \in L(K),$$

$$(py')(t,1/p)(r) = \int_c^t \phi_{2,2}(t,s)\,q(s)\left(\int_c^s (py')(x)\,r(x)\,dx\right)ds,\ r \in L(K),$$

$$y'(t,w)(r) = \lambda \int_c^t \phi_{1,2}(t,s,w)\,y(s,w)\,r(s)\,ds,\ r \in L(K);$$

$$(py')(t,w)(r) = \lambda \int_c^t \phi_{2,2}(t,s,w)\,y(s,w)\,r(s)\,ds,\ r \in L(K);$$

$$y'(t,\lambda) = \int_c^t \phi_{1,2}(t,s,\lambda)\,w(s)\ y(s,\lambda)\,ds,\ \lambda \in \mathbb{C}.$$

PROOF: We prove some of these, the proofs of the others are similar. Let

$$-(py')' + qy = 0,\ y(c) = h,\ (py')(c) = k;$$
$$-(pz')' + (q+r)z = 0,\ z(c) = h,\ (pz')(c) = k.$$

Let $x = z - y$. Then

$$-(px')' + qx = -r\,z,\ x(c) = 0,\ (px')(c) = 0.$$

From the variation of parameters formula it follows that

$$x(t) = \int_c^t \phi_{1,2}(t,s)(-r(s))\,z(s)ds.$$

Letting $z = y + (z - y)$ we get

$$z(t) - y(t) + \int_c^t \phi_{1,2}(t,s)(r(s)\,y(s)ds = -\int_c^t \phi_{1,2}(t,s)[\,z(s) - y(s)]\,r(s)\,ds$$
$$= o(r)\ as\ r \to 0\ in\ L(J).$$

The last equality follows from the fact that $\phi_{1,2}$ is bounded on $K \times K$ and $z \to y$ uniformly on K by the furthermore part of Theorem 3.6. Similarly we get

$$(pz' - py')(t) = (px')(t) = \int_c^t \phi_{2,2}(t,s)(-r(s)\,z(s)ds$$

and from this, proceeding as above, we obtain the formulas for $y(t,q)(r)$ and for $(py')(t,q)(r)$.

To derive the formulas for the derivatives with respect to $1/p$ we proceed as follows. Let

$$\frac{1}{p_r} = \frac{1}{p} + r, \ r \in L(J)$$

and let $y = y(t, 1/p), \ z = z(t, 1/p_r)$. Set

$$x(t) = \int_c^t \frac{1}{p}(py' - p_r z').$$

Then

$$-(px')' + qx = f, \ x(c) = 0, \ (px')(c) = 0, \ f(t) = -q(t) \int_c^t (p_r z')\, r.$$

From the variation of parameters formula we get

$$x(t) = -\int_c^t \phi_{1,2}(t,s)\, q(s) \left(\int_c^s (p_r z')(u)\, r(u) du \right) ds,$$

$$(px')(t) = -\int_c^t \phi_{2,2}(t,s)\, q(s) \left(\int_c^s (p_r z')(u)\, r(u) du \right) ds.$$

and note that

$$\begin{aligned} z(t) - y(t) &= \int_c^t [\frac{1}{p_r}(p_r z') - \frac{1}{p}(py')] \\ &= \int_c^t [\frac{1}{p}(p_r z') - \frac{1}{p}(py')] + \int_c^t (p_r z')\, r \\ &= -x(t) + \int_c^t (p_r z')\, r \\ &= \int_c^t \phi_{1,2}(t,s)\, q(s) \left(\int_c^s (p_r z')(u)\, r(u) du \right) ds + \int_c^t (p_r z')\, r. \end{aligned}$$

Setting

$$p_r z' = py' + [p_r z' - py']$$

we obtain

$$\begin{aligned} & z(t) - y(t) - \int_c^t \phi_{1,2}(t,s)\, q(s) \left(\int_c^s (py')(u)\, r(u) du \right) ds + \int_c^t (py')\, r \\ = \quad & \int_c^t \phi_{1,2}(t,s)\, q(s) \left(\int_c^s [(p_r z') - py'](u)\, r(u) du \right) ds + \int_c^t [(p_r z') - py']\, r \\ = \quad & o(r) \; as \; r \to 0 \; in \; L(J). \end{aligned}$$

The last equality follows from the boundedness of $\phi_{1,2}$ on $K \times K$, from $q \in L(K)$, and from the fact that, by Theorem 3.6, $(p_r z') \to py'$ uniformly on K as $r \to 0$ in $L(J)$. □

There is an interesting and subtle point involved in the proof of Theorem 3.9: $\frac{1}{p} + r$ may be identically zero on a subinterval of J. Note that the solutions y depend on $\frac{1}{p}$, not on p. Therefore $\frac{1}{p}$ may be identically zero on a subinterval of J or even on all of J. This is allowed by the existence-uniqueness Theorem 2.2 and the subsequent theorems. However, the equation (3.8) has to be interpreted properly in this case as Atkinson [3] has pointed out. In fact Atkinson uses the notation

$$-(\frac{1}{p}\, y')' + qy = \lambda\, w\, y$$

for equation (3.8) but this notation has not been widely accepted.

For regular equations each solution y and its quasi-derivative py' are not only entire functions of λ but have order at most 1/2.

THEOREM 3.9. Let (3.3), (3.9) hold. Assume that

$$1/p,\; q,\; w \in L(J). \tag{3.22}$$

Then every nontrivial solution y of (3.8) and its quasi-derivative py' are entire functions of λ of order at most 1/2. More precisely, there exist positive constants M, B, δ such that

$$|y(t,\lambda)| \le B\, e^{M\sqrt{|\lambda|}},\; a \le t \le b,\; |\lambda| \ge \delta$$

$$|(py')(t,\lambda)| \le B\, e^{M\sqrt{|\lambda|}},\; a \le t \le b,\; |\lambda| \ge \delta$$

PROOF: Let $v = py'$ then $v' = (q - \lambda w)\, y$. Fix λ and let prime "′" denote differentiation with respect to t. Then

$$\begin{aligned} [|\lambda|\, |y|^2 + |v|^2]' \quad &= \quad [|\lambda|\, \bar{y}\, y + \bar{v}\, v]' \\ &= \quad |\lambda|\, (\frac{1}{\bar{p}} y\, \bar{v} + \frac{1}{p}\, v\, \bar{y}) + \bar{v}\, (q - \lambda w)\, y + v\, (\bar{q} - \bar{\lambda}\, \bar{w})\, \bar{y}. \end{aligned}$$

¿From this and the elementary inequality

$$2\,|a\,b| \leq \frac{|\lambda|\,|a|^2 + |b|^2}{\sqrt{|\lambda|}},\ |\lambda| \neq 0$$

we get

$$[|\lambda|\,|y|^2 + |v|^2]' \leq \frac{|\lambda|\,|y|^2 + |v|^2}{\sqrt{|\lambda|}} \left(|\lambda|\,\frac{1}{|p|} + |q| + |\lambda|\,|w| \right)$$

and hence

$$[\log(|\lambda|\,|y|^2 + |v|^2)]' \leq \sqrt{|\lambda|}\,\frac{1}{|p|} + \frac{1}{\sqrt{|\lambda|}}\,|q| + \sqrt{|\lambda|}\,|w|.$$

An integration yields

$$|\lambda|\,|y(t,\lambda)|^2 + |v(t,\lambda)|^2 \leq C\,e^{\sqrt{|\lambda|}\int_a^t (\frac{1}{|p|}+|w|) + \frac{1}{\sqrt{|\lambda\backslash}}\int_a^t |q|}$$

$$\leq B\,e^{M\sqrt{|\lambda|}},\ 0 < M = \int_a^b (\frac{1}{|p|} + |w|) < \infty,\ e^{\left(\frac{1}{\sqrt{|\lambda|}}\int_a^b |q|\right)} < B < \infty.$$

This completes the proof. □

3.5 Endpoint classifications: R, LC, LP, O, NO, LCNO, LCO.

Definition of regular (R), limit-circle (LC), limit-point (LP), oscillatory (O), and nonoscillatory (NO) end-points.

Consider the equation

$$-(py')' + qy = \lambda\,w\,y,\ \lambda \in \mathbb{C},\ on\ J, \tag{3.23}$$

with

$$J = (a,b),\ -\infty \leq a < b \leq \infty,\ p,q,w : J \to \mathbb{C},\ 1/p,q,w \in L_{loc}(J). \tag{3.24}$$

The (finite or infinite) endpoint a is regular if, in addition to (3.24),

$$1/p, q, w \in L(a,d)$$

holds for some (and hence any) $d \in J$; is limit-circle if all solutions of the equation (3.23) are in $L^2_w(a,d) = \{f : (a,d) \to \mathbb{C}, \int_a^d |f|^2 w < \infty\}$ for some (and hence any) $d \in (a,b)$; is LP if it is not LC; is O if there is a nontrivial solution with an infinite number of zeros in any right neighborhood of a ; is NO if it is not O; is LCO if it is both LC and O; and is LCNO if it is both LC and NO. Similar definitions are made at b. An endpoint is called singular if it is not regular.

It is well known [81] that the LC, LP, classifications are independent of $\lambda \in \mathbb{C}$ and that the LCO and LCNO classifications are independent of $\lambda \in R$. At an LP endpoint the O classification, in general, depends on λ.

PROPOSITION 3.10. Let (3.23), (3.24) hold. In addition assume that

$$p, q, w : J \to \mathbb{R},\ p > 0,\ a.e.,\ \lambda \in \mathbb{R}.$$

Then the zeros of every nontrivial solution y of (3.23) are isolated in the interior of J and also at regular endpoints of J i.e. if a nontrivial solution y has a zero at a regular endpoint of J then there is an appropriate one sided neighborhood of this endpoint in which y has no other zero. Thus only the singular endpoints of J can be accumulation points of zeros of y.

PROOF: Let y be a nontrivial solution of (3.23). First we show that if y has consecutive zeros at $c, d \in (a,b),\ c < d$, then $(py')(h) = 0$ for some $h \in (c,d)$. We have

$$0 = y(d) - y(c) = \int_c^d y' = \int_c^d \frac{1}{p}(py') = (py')(h) \int_c^d \frac{1}{p}$$

by the Mean Value Theorem for the Lebesgue integral. (Recall that (py') is continuous on J.) Hence either $(py')(h) = 0$ or $\int_c^d \frac{1}{p} = 0$, but the latter would imply that $\frac{1}{p} = 0$ a.e. in (c,d) in contradiction to the hypothesis that $p > 0$ a.e. in (a,b).

Now to prove the main Proposition suppose there exists a sequence $\{t_n \in (a,b) : n \in N_0\}$ such that $t_n \to t_0$ and $y(t_n) = 0,\ n \in N_0$. Then $y(t_0) = 0$ and from the first part of the proof we get a sequence $\{s_n : n \in N\}$ with $s_n \to t_0$ such that $(py')(s_n) = 0$. Since (py') is continuous in (a,b) it follows that $(py')(t_0) = 0$. But $y(t_0) = 0$ and $(py')(t_0) = 0$ implies that y is identically zero on J by the uniqueness of initial value problems. □

We end this subsection with an example to show that when p changes sign the behavior of the classical and quasi-derivatives of a solution can be quite different.

EXAMPLE 3.11. Let

$$p(t) = \frac{1}{cos(log(t))},\ 0 < t \leq 1.$$

Then $1/p \in L(0,1)$ and the equation

$$-(py')' = 0 \; on \; (0,1)$$

has

$$y = \int cos(log(t))\, dt, \;\; v(t) = 1$$

as solutions. Note that the point 0 is an accumulation point of zeros of y' since $y'(t_k) = 0$ where

$$t_k = e^{-k\pi/2} \to 0, \;\; as \; k \to \infty,$$

but $(py')(t) = 1$ for $t \in [0,1]$. The Wronskian

$$W(y,v)(t) = \begin{vmatrix} y & v \\ py' & pv' \end{vmatrix} (t) = -1, \;\; 0 \leq t \leq 1,$$

but the classical Wronskian

$$\begin{vmatrix} y & v \\ y' & v' \end{vmatrix} (t) = -cos(log(t)), \;\; 0 < t \leq 1$$

is nonconstant with an accumulation point of zeros at 0.

3.6 The maximal domain and Lagrange form.

Let (3.9) hold and assume $w > 0$ a.e. on J; let

$$M\,y = [-(py')' + qy]$$

The maximal domain $\Delta = \Delta(M, w, J)$ is defined by

$$\Delta = \{y : J \to \mathbb{C} : y, \;\; py' \in AC_{loc}(J), \;\; y, \; w^{-1}\,M\,y \in L^2_w(J)\}$$

The Lagrange sesquilinear form is given by

$$[y,z] = yp\bar{z}' - \bar{z}py', \;\; y, \; z \in \Delta.$$

3.7 Regularizing functions.

Here we construct a pair of functions u, v which we call "regularizing" functions since they can be used to "regularize" singular equations with LCNO endpoints.

THEOREM 3.12. Let (3.3), (3.8), and (3.9) hold; and suppose that p, q, w are real-valued and $p > 0,\ w > 0$ a.e. on J. Assume each endpoint is either regular or LCNO. Let Δ and M be defined as in subsection 6. Then there exist functions $u, v \in \Delta$ satisfying the following conditions:

1. They are real valued.
2. For some real $\lambda = \lambda_a$, u is a principal solution at a and v is a nonprincipal solution at a.
3. For some real $\lambda = \lambda_b$, u is a principal solution at b and v is a nonprincipal solution at b.
4. These functions u, v need not be solutions through the interior of (a, b), and, in case $\lambda_a = \lambda_b$, they need not be the same solution near a and near b.
5. $[u, v](a) = \lim_{t \to a^+} [u, v](t) = 1$,
6. $[u, v](b) = \lim_{t \to b^-} [u, v](t) = 1$,
7. $v > 0$ on $J = (a, b)$.

PROOF: See Subsection 5.2 of Section 5 below for the definition of principal and non-principal solution; and see Lemma 7 in Niessen and Zettl [74] for a proof. □

We call such functions u and v "regularizing functions" of the equation (3.8), on (a, b). The reason for this terminology will become clear in subsection 9 where we show that with the help of such functions u, v, particularly v, one can construct a regular equation which is "equivalent" in a natural sense to the singular equation (3.8). We call equation (3.8) singular if at least one endpoint of the underlying interval J is singular; the equation (3.8) is said to be regular if both endpoints are regular.

3.8 Limit-Circle "Initial value problems".

THEOREM 3.13. Let (3.3), (3.8), (3.9) hold. Assume p, q, w are real valued, $w > 0$ a.e. and the left endpoint a is R *or* LC. Suppose u, v are real valued linearly independent solutions on some interval $(a, d]$ for some fixed real λ_0. Given any $\lambda \in \mathbb{R}$ and any $h,\ k \in \mathbb{R}$ the singular initial value problem consisting of the equation

$$-(py')' + qy = \lambda\, w\, y \ on\ J$$

and the singular "initial condition"

$$[y, u](a) = h,\ [y, v](a) = k$$

has a unique real solution y on J. Similarly at b.

PROOF: Since u, v are linearly independent solutions near a we have $[u, v](a) \neq 0$ and we can assume that $[u, v](a) = 1$. Let

$$U = \begin{pmatrix} u & v \\ pu' & pv' \end{pmatrix}, \; Z = U^{-1}Y, \; Y' = (P - \lambda W)Y, \; U' = (P - \lambda_0 W)U,$$

and, using the notation from (3.5) for P, Y, let with $W = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}$. Note that U is a fundamental matrix solution for a fixed λ_0 but Y is a vector solution for an arbitrary $\lambda \in \mathbb{R}$. A direct computation reveals that

$$Z' = (\lambda_0 - \lambda)(U^{-1}WU)Z = (\lambda_0 - \lambda)\, G\, Z \; on \; (a, d]$$

where

$$G = U^{-1}WU = \begin{pmatrix} -uvw & -v^2 w \\ u^2 w & uvw \end{pmatrix} \in L(a, d).$$

Note that $G \in L(a, d)$ follows from the Cauchy-Schwarz inequality coupled with the assumption that a is in the LC case. Hence by Theorem 2.10 all initial value problems

$$Z' = (\lambda_0 - \lambda)\, G\, Z \; on \; (a, d], \; Z(a) = C,$$

have a unique solution. From $Y = UZ$, $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ we get

$$u\, z_1 + v\, z_2 = y, \; (pu')\, z_1 + (pv')\, z_2 = py' \; both \; on \; (a, d].$$

¿From Cramer's rule we get

$$z_1(t) = \begin{vmatrix} y(t) & v(t) \\ (py')(t) & (pv')(t) \end{vmatrix} = [y, v](t), \; a < t < d$$

$$z_2(t) = \begin{vmatrix} u(t) & y(t) \\ (pu')(t) & (py')(t) \end{vmatrix} = -[y, u](t), \; a < t < d.$$

By letting $t \to a$ we get $z_1(a) = k$, $z_2(a) = -h$. Since this holds for arbitrary h, k the proof is complete. □

REMARK 3. Note that the assumption $p > 0$ is not needed in Theorem 3.13, nor have we assumed that a is LCNO but only that a is LC. The transformation $Z = U^{-1}Y$ transforms the singular scalar equation (3.8) into a first order *regular* system.

3.9 Factorization of solutions near an LCNO endpoint.

A singular equation with an LCNO endpoint can be "regularized" using the function v from a regularizing pair u, v of functions as defined in subsection 8.

THEOREM 3.14. Consider the equation (3.8) with J given by (3.3) and with

$$p, q, w : J \to \mathbb{R},\ 1/p, q, w \in L_{loc}(J),\ p > 0,\ w > 0\ a.e.\ \lambda \in \mathbb{C}.$$

Assume that the left endpoint a is LCNO and let u, v be a pair of regularizing functions at a i.e. on (a, d) for some $d \in (a, b)$ as defined in section 7. Define

$$P = v^2 p, W = v^2\, w, Q = w\, v\, M\, v \text{ on } J,$$

and consider the equation

$$-(Pz')' + Qz = \lambda\, W\, z \text{ on } J. \tag{3.25}$$

Then we have

$$1/P,\ Q,\ W \in L(a, d),\ a < d < b,$$

i.e. the equation (3.25) is regular at a and

1. If y is a solution of (3.8) on (a, d), then $z = y/v$ is a solution of (3.25) on (a, d). Conversely, if z is a solution of (3.25) on (a, d) then $y = vz$ is a solution of (3.8) on (a, d).
2. The limits

$$z(a) = \lim_{t \to a^+} z(t);\ (Pz')(a) = \lim_{t \to a^+} (Pz')(t)$$

exist and are finite. Thus the solution z and its quasi-derivative (Pz') can be continuously extended to the (finite or infinite) endpoint a.

3. Note that v is independent of $\lambda \in \mathbb{R}$ but does depend on (M, w) i.e. on $1/p, q, w$ and on the endpoint a i.e. on some neighborhood (a, d) for $d \in (a, b)$.
4. The one-to-one mappings $y(t, \lambda) = v(t)\ z(t, \lambda)$ and $(py')(t, \lambda) = v(t)\,(Pz')(t, \lambda)$ can be given more explicitly, using the notation (3.10), by

$$y(t, c, h, k, 1/p, q, w, \lambda) = v(t) z(t, c, h/v(c), kv(c) - h(pv')(c), 1/P, Q, W, \lambda)$$

$$\begin{aligned} & (py')(t, c, h, k, 1/p, q, w, \lambda) \\ = \ & (pv')(t)(Pz')(t, c, h/v(c), kv(c) - h(pv')(c), 1/P, Q, W, \lambda) \end{aligned}$$

for $c \in J$, and $h, k \in \mathbb{C}$.

PROOF: See Niessen and Zettl [74]. Although the explicit formulas for the 1-1 map $y \to vz$ are not given by these authors it can easily be obtained from there. □

REMARK 4. Note that, in general, y and v do not exist at a. Thus we have

$$z(a) = \frac{y}{v}(a);\ (Pz')(a) = (vpy' - ypv')(a) = [v, y](a)$$

but neither the numerator y nor the denominator v , nor the individual terms in $(Pz')(a)$, can be evaluated separately at a. Of course there is an entirely analogous theorem and remark for the endpoint at b. If each endpoint is either R or LCNO then Theorem 3.14 holds on the entire interval J.

THEOREM 3.15. Let the notation of Theorem 3.14 hold. If each endpoint of the interval J is either R or LCNO, then there exist a pair of regularizing functions u, v defined on all of J such that the conclusions of Theorem 3.14 hold on the whole interval J.

PROOF: See Niessen and Zettl [74]. □

The next example illustrates Theorem 3.15.

EXAMPLE 3.16. For the classical Legendre equation

$$-((1-t^2)\, y')' = \lambda\ y \text{ on } (-1, 1),$$

we have

$$u(t) = 1,\ -1 < t < 1,$$

$$v(t) = \begin{cases} -(1/2)\ln((1-t)/(1+t)) & 1/2 \le t < 1 \\ (1/2)\ln((1-t)/(1+t)) & -1 < t \le -1/2 \end{cases}.$$

Note that $v > 0$ near $+1$ and near -1. Every solution y can be factored as follows:

$$y(t, \lambda) = v(t)\ z(t, \lambda),\ -1 < t < 1,\ \lambda \in R$$

where z is continuous on the closed interval $[-1, 1]$. This shows, in particular, that the asymptotic behavior of solutions of the Legendre equation satisfying a fixed initial condition is independent of $\lambda \in \mathbb{C}$.

3.10 Comments.

These are made for each Subsection separately.

1. In Theorem 3.2 the sufficiency of condition (3.4) is well known, the necessity is due to Everitt and Race [25].
2. The results of Theorems 3.4, 3.5 and 3.6 are surely not new, but we don't know of a reference where they can be found in this generality.
3. Again, other than [59], we don't know of a reference where the continuous dependence of solutions of initial value problems on $1/p, q, w$ *in the* L^1 *norm* is established. The continuous dependence of solutions on initial conditions i.e. on c, h, k is discussed by Hille [43].
4. The differentiable dependence of solutions and their quasi-derivatives on each parameter as well as the formulas for the derivatives seems to be new. The proof of the differentiable dependence of y and of py' on $1/p$ is due to Qingkai Kong and published here for the first time with his permission.

   Is y jointly differentiable in all its variables: $c, h, k, 1/p, q, w$? Ditto for py'. What are the derivatives ?

   Theorem 3.9 is adapted from Atkinson [3]. It follows from the asymptotic form of the eigenvalues of regular self-adjoint SLP that for p, q, w real-valued $p \geq 0,\ w > 0$ that the non-trivial solutions - as functions of λ - are of order exactly 1/2.

   Under what more general conditions on p, q, w are the non-trivial solutions of exact order 1/2 ?

   Under what general conditions (excluding trivial cases such as $p = q = w = 0$) are the solutions of the SL equation of order zero as functions of λ ? Are there general classes of equations for which the solutions have order r, for $0 < r < 1/2$ as functions of λ ?
5. The definitions of R, LC, LP, O, NO, LCNO, LCO are standard except, as pointed out in Remark 2, we classify an infinite endpoint as regular if $1/p, q, w$ are integrable in a neighborhood of this point. This contrasts with the usual practice. For a definition of oscillation of difference equations see [58].

   The standard proof of the invariance of the $L^2_w(J)$ solutions with respect to λ is based on the variation of parameter formula [14].

   The invariance of the LCO case with respect to real λ follows from the spectral theory of ordinary differential operators, see [81]. In general for the symmetric case i.e. p, q, w real and $1/p, q, w \in L_{loc}(a, d), a < d < b,\ p \geq 0,\ w > 0$ there exists a σ_0, $-\infty \leq \sigma_0 \leq \infty$, such that the equation (3.8) is O at a for $\lambda > \sigma_0$ and is NO for $\lambda < \sigma_0$. Examples show that for $\lambda = \sigma_0$ the equation can be O or NO. This "oscillation number" σ_0 is also the starting point of the essential (continuous) spectrum of every self-adjoint realization of the equation. The case $\sigma_0 = -\infty$ is interpreted as meaning that the essential spectrum is not bounded below; the case $\sigma_0 = \infty$ means that the essential spectrum is empty. The latter holds for all SLP for which each endpoint is either R or LC since in this case the spectrum is discrete. The essential but not discrete spectrum is the same for all self-adjoint realizations of the equation (3.8). For proofs of these statements as well as further information the reader is referred to [81].

6. The definition of the maximal domain and of the Lagrange sesquilinear form is standard. These play an important role in the theory of boundary value problems.
7. This section is based on Niessen and Zettl [74].
8. The "system regularization" of LC endpoints based on the fundamental matrix U is not new. It has been used by Fulton and by Fulton and Krall. It has other applications besides the one given here to singular IVP:
   (a) It can be used to prove the LC invariance with repect to λ : ¿From

   $$y(t,\lambda) = u(t,\lambda_0)\, z_1(t,\lambda) + v(t,\lambda_0)\, z_2(t,\lambda)$$

   and $z_j(t,\lambda) \to z_j(a,\lambda)\ as\ t \to a,\ j = 1,2$ it follows that z_j are bounded ("nearly constant") in a neighborhood of a. Hence $y(\cdot,\lambda)$ is in $L^2_w(a,d)$,$a < d < b$, since both u, and v are; this for every $\lambda \in \mathbb{C}$. This proof is much simpler than the standard one based on variation of parameters and seems to be new.
   (b) In connection with boundary value problems the transformation $Z = U^{-1}Y$ has been used by Fulton and by Fulton and Krall.
   Another application of the transformation $Z = U^{-1}Y$ is to prove the invariance of the O classification with respect to all real λ at an LC endpoint:
   Assume a is LC and O for some real λ_0 and $p > 0$. For any real λ we have

   $$y(t,\lambda) = u(t,\lambda_0)\, z_1(t,\lambda) + v(t,\lambda_0)\, z_2(t,\lambda)$$

   case 1. $z_j(a) > 0,\ j = 1,2$. It is easy to see that between any three zeros of u there must be a zero of y : Sketch the graph of u, v keeping in mind the Sturm Separation Theorem. Now it is easy to identify two disjoint intervals on one of which y is positive and on the other negative. Hence y must have a zero between these intervals. The other cases are all established similarly. This "oscillation theory proof " is much simpler than the spectral theory proof and seems to be new.
9. Theorem 3.14 is taken from Niessen and Zettl [74] and Theorem 3.15 is also adapted from this paper although phrased in terms of factoring rather than regularizing.

   Example 3.15 illustrates Theorem 3.14 by showing that the classical Legendre equation, which is singular at both endpoints, is "equivalent" to a regular equation on the same interval. The leading coefficient P and the weight function W of the regular equation are not bounded on $(-1,1)$ but, nevertheless, satisfy the regularity conditions $1/P, W \in L(-1,1)$. The Legendre equation and its regularization are equivalent in the sense that the transformation

   $$y(t,\lambda) = v(t)\, z(t,\lambda)$$

($\lambda_0 = 0$ in this case) maps solutions y of the Legendre equation into solutions z of the regular equation in a 1-1 onto manner. All solutions z of the regular equation are continuous on the closed interval $[-1, 1]$. So the singular behavior is contained in the transformation function v. Since v is independent of λ this shows that the singular behavior of the Legendre equation is independent of $\lambda \in \mathbb{C}$. The invariance of the LC classification with respect to $\lambda \in \mathbb{C}$ and the invariance of the LCNO classifications with respect to $\lambda \in \mathbb{R}$ are merely specific instances of this general invariance property.

Of course, these remarks apply to all other equations where each endpoint is either regular or LCNO.

4 REGULAR TWO POINT BOUNDARY VALUE PROBLEMS

4.1 Introduction.

In this section we study regular Sturm-Liouville problems (SLP) with self-adjoint and non-self-adjoint two point boundary conditions (BC). There are two basic methods available for such a study: operator theory and complex function theory. Both are employed here, singly and in combination.

4.2 Characterization of the eigenvalues.

A regular two point SLP consists of the equation

$$-(py')' + qy = \lambda\, w\, y \ on \ (a,b) = J, \ -\infty \leq a < b \leq \infty, \tag{4.1}$$

where

$$1/p, \ q, \ w : J \to \mathbb{C}, \ 1/p, \ q, \ w \in L(J), \ \lambda \in \mathbb{C}, \tag{4.2}$$

together with boundary conditions

$$AY(a) + BY(b) = 0, \ Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \ A, B \in M_2(C). \tag{4.3}$$

By Theorem 3.4, $Y(a)$, $Y(b)$ exist and are finite so that (4.3) is well defined. Let

$$P = \begin{pmatrix} 0 & 1/p \\ q & 0 \end{pmatrix}, \ W = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}. \tag{4.4}$$

Then the scalar equation (4.1) is equivalent with the first order system

$$(4.5)\qquad Y' = (P - \lambda W)Y = \begin{pmatrix} 0 & 1/p \\ q - \lambda w & 0 \end{pmatrix} Y,\ Y = \begin{pmatrix} y \\ py' \end{pmatrix}.$$

Let $\Phi(\cdot, u, P, w, \lambda)$ be the matrix solution of the initial value problem

$$(4.6)\qquad \Phi' = (P - \lambda W)\,\Phi,\ \Phi(u) = I,\ u \in J,\ \lambda \in \mathbb{C},$$

and define the characteristic function Δ, see Lemma1 below, by

$$(4.7)\qquad \Delta(\lambda) = \det[A + B\,\Phi(b, a, P, w, \lambda)],\ \lambda \in \mathbb{C}.$$

Although the notation Δ was used in section 3 to denote the maximal domain there should be no confusion with its use here as the characteristic function.

LEMMA 4.1. Let (4.1) to (4.7) hold. Then the characteristic function Δ is defined and continuous at a and b for fixed P, w, λ and is an entire function of λ for fixed a, b, P, w.

PROOF: It follows from Theorems 2.9 and 2.10 that, for fixed P, w, λ, $\Delta(a, b)$ exists and is continuous at a and b. The entire dependence on λ follows from the second proof using successive approximations of Theorem 2.2, the existence-uniqueness theorem. Each successive approximation is a polynomial in λ. Since these converge uniformly on each compact subset K of the complex plane to the solution, this solution is analytic on K. Thus the solution is entire in λ since this holds for each such K. □

LEMMA 4.2. Let (4.1) to (4.7) hold. A complex number λ is an eigenvalue of the BVP (4.1), (4.2), (4.3) if and only if $\Delta(\lambda) = 0$. Furthermore the multiplicity of the eigenvalue λ is equal to the number of linearly independent vector solutions $C = Y(a)$ of the linear algebra system

$$(4.8)\qquad [A + B\,\Phi(b, a, \lambda)]\,C = 0.$$

PROOF: Suppose $\Delta(\lambda) = 0$. Then (4.8) has a nontrivial vector solution for C. Let $Y(a) = C$ and solve the IVP

$$Y' = (P - \lambda W)Y,\ Y(a) = C,\ \textit{on } J.$$

Then

$$Y(b) = \Phi(b, a, \lambda)\,Y(a) \textit{ and } [A + B\,\Phi(b, a, \lambda)]\,Y(a) = 0.$$

From this it follows that the top component of Y, say, y is an eigenvector of the BVP (4.1), (4.2), (4.3); that means λ is an eigenvalue of this BVP. Conversely, if λ is an eigenvalue and y an eigenvector of λ, then $Y = \begin{pmatrix} y \\ py' \end{pmatrix}$ satisfies $Y(b) = \Phi(b, a, \lambda)\, Y(a)$ and consequently $[A + B\,\Phi(b, a, \lambda)]\, Y(a) = 0$. Since $Y(a) = 0$ would imply that y is the trivial solution in contradiction to it being an eigenvector, we have that $\det[A{+}B\,\Phi(b, a, \lambda)] = 0$. If (4.8) has two linearly independent solutions for C, say C_1, C_2, then solve the IVP with the initial conditions $Y(a) = C_1, Y(a) = C_2$ to obtain solutions Y_1, Y_2. Then Y_1, Y_2 are linearly independent vector solutions of (4.5) and their top components y_1, y_2 are linearly independent solutions of (4.1). Conversely, if $y_{1,}\, y_2$ are linearly dependent solutions of (4.1) we can reverse the steps above to obtain two linearly independent solutions of the algebraic system (4.8).

LEMMA 4.3. For the BVP (4.1), (4.2), (4.3) exactly one of the following four cases holds:

1. There are no eigenvalues in $\mathbb{C}$.
2. Every complex number is an eigenvalue.
3. There are a nonzero finite number of eigenvalues in $\mathbb{C}$.
4. There are an infinite but countable number of eigenvalues in $\mathbb{C}$ and these have no finite accumulation point in $\mathbb{C}$.

PROOF: This follows directly from Lemmas 4.1 and 4.2 and the fact that the zeros of an entire function are isolated and have no accumulation point in the finite plane $\mathbb{C}$. □

It is convenient to separate the boundary conditions (BC) (4.3) into two mutually exclusive classes: separated and coupled. Note that, since the BC are homogeneous, multiplication by a nonzero constant or a nonsingular matrix leads to equivalent boundary conditions.

LEMMA 4.4 (Separated BC)**.** Let (4.1) to (4.7) hold. Fix P, W, J and assume

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 \\ B_1 & B_2 \end{pmatrix}. \tag{4.9}$$

Then

$$\Delta(\lambda) = A_2\, B_1 \phi_{11}(b, a, \lambda) + A_2 B_2 \phi_{21}(b, a, \lambda) - A_1 B_1 \phi_{12}(b, a, \lambda) - A_1 B_2 \phi_{22}(b, a, \lambda)$$

for $\lambda \in \mathbb{C}$.

PROOF: This follows directly from the definition of Δ. □

LEMMA 4.5 (Coupled self-adjoint BC)**.** Let (4.1) to (4.7) hold. Fix P, W, J and assume that

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ A = e^{i\alpha} K, \ -\pi \leq \alpha \leq \pi, \ K \in SL_2(\mathbb{R}), \tag{4.10}$$

i.e. K is a real 2×2 matrix with determinant 1. Let $K = (k_{ij})$ and define

(4.11)
$$D(\lambda, K) = k_{11}\, \phi_{11}(b, a, \lambda) + k_{12}\, \phi_{21}(b, a, \lambda) + k_{21}\, \phi_{12}(b, a, \lambda) + k_{22}\, \phi_{22}(b, a, \lambda)$$

for $\lambda \in \mathbb{C}$. Note that $D(\lambda, K)$ does not depend on α. Then

1. the complex number λ is an eigenvalue of BVP (4.1), (4.2), (4.3), (4.10) if and only if

$$D(\lambda, K) = -2\, \cos\alpha, \ -\pi \leq \alpha \leq \pi; \tag{4.12}$$

2. for fixed K a complex number λ is an eigenvalue for α if and only if it is an eigenvalue for $-\alpha$ and if u is an eigenfunction of α then its conjugate $\bar{u}$ is an eigenfunction of $-\alpha$.

PROOF: By Theorem 2.3 $\det \Phi(b, a, \lambda) = 1$. This and $\det K = 1$ together with a straightforward but tedious computation yields that

$$\Delta(\lambda) = 1 + e^{2i\alpha} + e^{i\alpha} D(\lambda, K).$$

Thus $\Delta(\lambda) = 0$ is equivalent to

$$D(\lambda, K) = -e^{i\alpha} - e^{i\alpha} = -2\cos\alpha.$$

REMARK 5. Although the matrices (4.10) determine self-adjoint boundary conditions (they are the canonical form of all coupled self-adjoint BC) , no conditions other than (4.2) are assumed on p, q, w in Lemma 4.5. In particular no symmetry (formal self-adjointness) or definiteness assumption is made on equation (4.1). Thus the characterization of the eigenvalues given by (4.12) applies not only to so called left-definite, right-definite and indefinite SLP but the coefficients p, q and the weight function w can be complex valued.

Eberhard and Freiling [20], Mennicken and Möller [67], have established the existence of infinitely many eigenvalues for the symmetric equation (4.1) but with non-self-adjoint BC. We have

THEOREM 4.6. Assume that

$$\begin{array}{rcl} p, q, w & : & J = [a, b] \to R;\ -\infty < a < b < \infty,\ \ p > 0,\ w > 0, \\ p & = & w \in AC(J),\ p'/p, q/w \in L^r(J),\ 1 < r \leq \infty. \end{array}$$

Then

1. the BVP (4.1), (4.9) with $(A_1, A_2) \neq (0, 0) \neq (B_1, B_2)$ has an infinite number of eigenvalues;
2. the BVP (4.1) together with the coupled BC

$$Y(b) = A\,Y(a)$$

has an infinite number of eigenvalue for any $A \in M_2(\mathbb{C})$ satisfying either $a_{12} \neq 0$ or $a_{12} = 0$ and $a_{11} + a_{22} \neq 0$.

PROOF: See [67], [20]. □

These authors also establish an expansion theorem for these cases.

4.3 The Fourier equation.

We now pause to consider the simplest SLP for at least two reasons: (i) to illustrate the results of the previous section and (ii) to indicate some of the coming attractions of section 5. It is remarkable how many properties of SLP for the simplest SL equation

$$-y'' = \lambda\, y$$

hold for the general case. This is a third reason for discussing this equation here.

Consider the equation

$$-y'' = \lambda\, y \text{ on } (a, b),\ -\infty \leq a < b \leq \infty,\ \lambda \in \mathbb{C}. \tag{4.13}$$

We include the case when one or both endpoints are infinite here, even though this is a singular problem then ($p = 1 = w$ are not in $L(J)$ if J is unbounded) and singular problems are not discussed, in general, until the next section, to highlight the interplay between regular and singular problems.

We content that to fully understand regular SLP requires a perspective which includes the singular case.

Each infinite endpoint is in the LP case since the constant 1 is a non L^2 solution for $\lambda = 0$. Thus there is one and only one self-adjoint realization, say S, of the equation (4.13) in the space $L^2(-\infty, \infty)$. The spectrum of S, $\sigma(S)$, contains no eigenvalues and thus coincides with the essential (continuous) spectrum $\sigma_e(S)$; we have

$$\sigma(S) = \sigma_e(S) = [0, \infty).$$

In this case

$$P = \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix}, \; W = \begin{pmatrix} 0, & 0 \\ 1, & 0 \end{pmatrix}.$$

Since these are fixed for this example we will omit them in the notation for Φ. This fundamental matrix $\Phi = \Phi(t, u, P, W, \lambda) = \Phi(t, u, \lambda)$ is determined as the unique solution of the initial value problem

$$\Phi' = (P - \lambda W)\,\Phi, \; \Phi(u) = I, \; t, u \in \mathbb{R}, \; \lambda \in \mathbb{C}.$$

To compute Φ we choose an analytic branch of the square root function $\sqrt{z}$ as follows :

$$\mu = \sqrt{\lambda} = s + it, \; s > 0, \; t > 0, \; for \; \lambda \neq 0,$$

and obtain

$$\begin{aligned} \Phi(t, u, \lambda) \quad &= \quad \begin{pmatrix} \cosh(i\mu(t-u)), & \frac{1}{i\mu}\sinh(i\mu(t-u)) \\ i\mu\sinh(i\mu(t-u)), & \cosh(i\mu(t-u)) \end{pmatrix}, \\ t, u \quad &\in \quad \mathbb{R}, \; \lambda \in \mathbb{C}, \; \lambda \neq 0, \; \mu = \sqrt{\lambda}, \end{aligned} \tag{4.14}$$

and

$$\Phi(t, u, 0) = \begin{pmatrix} 1, & t-u \\ 0, & 1 \end{pmatrix}, \; t, u \in \mathbb{R}. \tag{4.15}$$

Note that for fixed $t, u \in \mathbb{R}$ the fundamental matrix $\Phi(t, u, \lambda)$ is analytic at λ for each $\lambda \in \mathbb{C}$ including $\lambda = 0$. (This can be confirmed from the series expansions of the hyperbolic sinh and cosh functions.)

For the convenience of the reader we now recall some definitions and properties of hyperbolic functions which will be used below.

1. $2\sinh z = e^z - e^{-z}\; z \in \mathbb{C}, 2\cosh z = e^z + e{-}^z, \; z \in \mathbb{C}, \; \tanh z = \sinh z / \cosh z,$
2. $\sinh z = -i \sin iz, \; \cosh z = \cos iz, \; \tanh z = -i \tan iz$
3. $\sinh(z + 2k\pi i) = \sinh z, \; \cosh(z + 2k\pi i) = \cosh z$
4. $(\sinh z)' = \cosh z, \; (\cosh z)' = \sinh z$

5. $\sin z = 0$ if and only if $z = k\pi,\ k \in Z$; $\cos z = 0$ if and only if $z = (2k + 1)\pi/2,\ k \in Z$
6. The general solutions of the equations $\sin x = z,\ \cos x = z,\ \tan x = z$ are given by, respectively,
7. $x = (-1)^k \arcsin z + k\pi,\ k \in Z,$
8. $x = \pm \arccos z + 2k\pi,\ k \in Z,$
9. $x = \arctan z + k\pi,\ k \in Z,$
10. For $-1 \le t \le 1$, $\arcsin t,\ \arccos t$ are real and $-\pi/2 \le \arcsin t \le \pi/2,\ 0 \le \arccos t \le \pi$.

Now let $-\infty < a < b < \infty$ and consider the two point boundary condition

$$AY(a) + BY(b) = 0, \quad A, B \in M_2(\mathbb{C}). \tag{4.16}$$

We now discuss coupled and separated boundary conditions separately.

- Coupled BC. Let

$$A = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix},\ B = \begin{pmatrix} c, & 0 \\ 0, & d \end{pmatrix}. \tag{4.17}$$

From (4.7), (4.14), (4.15), (4.17) we get

$$\Delta(\lambda) = 1 + cd + (c + d)\cosh(i\mu(b - a)),\ \mu = \sqrt{\lambda} \neq 0. \tag{4.18}$$

In all of the examples below the case $\lambda = 0$ needs to be checked separately since $\lambda = 0$ plays a special role in these formulas.

We now consider a number of special cases of (4.17).

1. $c = -d$. Then $\Delta(\lambda) = 1 - c^2$, a constant independent of λ. If this constant is zero then every complex number is an eigenvalue; if this constant is not zero, then no complex number is an eigenvalue. In particular we have
   (a) For $c = 1,\ d = -1$ every complex number is an eigenvalue.
   (b) For $c = -1,\ d = 1$ every complex number is an eigenvalue.
   (c) For $c \in \mathbb{C}$, $c \neq 1,\ c \neq -1$, no complex number is an eigenvalue.
2. $c \neq -d$. The characteristic equation for the eigenvalues is :

$$\cosh((i\mu(b - a)) = \cos(-\mu(b - a)) = -\frac{1 + cd}{c + d} = r.$$

The roots of this equation are given by

$$\begin{aligned} -\mu(b - a) &= \pm \arccos r + 2k\pi,\ k \in Z, \\ \arccos z &= \int_z^1 \frac{dt}{(1 - t^2)^{1/2}} = \pi/2 - \arcsin z, \\ \arcsin z &= \int_0^z \frac{dt}{(1 - t^2)^{1/2}}, \end{aligned}$$

and both integrals must be taken along a path which does not cross the real axis.
When r is real and $-1 \leq r \leq 1$ then the roots for $\mu(b-a)$ are real and we get

$$\mu(b-a) = \mp \arccos r - 2k\pi,\ k \in Z$$

From this and $\mu = s > 0$ we get

$$\mu(b-a) = \arccos r + 2k\pi,\ k \in N_0.$$

We now consider some special cases of this case :
(a) $c = i = d$. Here $r = 0$ and

$$\mu(b-a) = \pi/2 + 2k\pi,\ k \in N_0$$

and the eigenvalues are

$$\lambda_n = \frac{(\pi/2 + 2n\pi)^2}{(b-a)^2},\ n \in N_0.$$

(b) $c = -1 = d$. This is the self-adjoint periodic (P) case with $r = 1$. First note that $\lambda = 0$ is an eigenvalue for this case. Our search for the other eigenvalues leads to $\mu(b-a) = 2k\pi,\ k \in N$ and consequently

$$\lambda_n^P = \frac{(2n\pi)^2}{(b-a)^2},\ n \in N_0.$$

(c) $c = 1 = d$. This is the self-adjoint semi-periodic (S) case and gives $r = -1$. We note that $\lambda = 0$ is not an eigenvalue in this case. Proceeding as above we get

$$\mu(b-a) = \pi + 2k\pi = (2k+1)\pi,\ k \in N_0.$$

Hence the eigenvalues in this case are given by

$$\lambda_n^S = \frac{((2n+1)\pi)^2}{(b-a)^2},\ n \in N_0.$$

(d) $c = 1/d,\ c \in \mathbb{R}$, $c \neq 0$. This gives $r = -(1+1)/(c+1/c)$. Since $c + \frac{1}{c} > 2$ if $c > 0$, and $c \neq 1$; $c + \frac{1}{c} < -2$ if $c < 0$ and $c \neq -1$, we have $-1 < r < 1$. Let

$$t_0(c) = \arccos(r),\ 0 < t_0(c) < \pi.$$

Then the roots are given by

$$(b-a)\mu = t_0(c) + 2k\pi,\ k \in N_0$$

and so the eigenvalues are

$$\lambda_n(c) = \frac{(t_0(c) + 2n\pi)^2}{(b-a)^2},\ n \in N_0$$

(e) $c = e^{i\alpha} = d,\ 0 < \alpha < \pi$. Here

$$r = -\frac{1+e^{2i\alpha}}{2e^{i\alpha}} = -\cos\alpha,\ t_0(\alpha) = \arccos(-\cos\alpha) = \pi - \alpha \in (0,\pi).$$

So

$$(b-a)\mu = \pi - \alpha + 2k\pi,\ k \in N_0,$$

and therefore

$$\lambda_n(\alpha) = \frac{(\pi - \alpha + 2n\pi)^2}{(b-a)^2},\ n \in N_0.$$

- Separated BC. Here we take

$$A = \begin{pmatrix} A_1, & A_2 \\ 0, & 0 \end{pmatrix},\ B = \begin{pmatrix} 0, & 0 \\ B_1, & B_2 \end{pmatrix},\ A_j, B_j \in \mathbb{C},\ j = 1,2.$$

A direct calculation gives

$$\begin{aligned} \Delta(\lambda) &= A_1B_1\Phi_{12}(b,a,\lambda) + A_1B_2\Phi_{22}(b,a,\lambda) - A_2B_1\Phi_{11}(b,a,\lambda) \\ &\quad -A_2B_2\Phi_{21}(b,a,\lambda) \\ &= (A_1B_2 - A_2B_1)\cosh((i\mu(b-a)) + (\frac{1}{\mu}A_1B_1 - i\mu A_2B_2)\sinh((i\mu(b-a)) \end{aligned}$$

Note that $\Delta(\lambda)$ is periodic with fundamental period $2k\pi i$, so if there is one eigenvalue then there is a countable infinity of them. For each eigenvalue there is only one linearly independent eigenvector. We now consider some special cases. For each case we list the characteristic equation, its roots for μ and the corresponding eigenvalues. As in the earlier cases $\lambda = 0$ has to be checked independently.

1. $A_1B_2 - A_2B_1 = 0$. Note that this includes both the Dirichlet and Neumann boundary conditions. Here we have

$$\Delta(\lambda) = (\frac{1}{\mu}A_1B_1 - i\mu A_2B_2)\sinh((i\mu(b-a)).$$

To find all eigenvalues we proceed as follows: (i) we find all eigenvalues produced by the roots of the sinh factor, (ii) check to see if any of these roots yield a non-zero root of the first factor, and (iii) check separately to see if $\lambda = 0$ is an eigenvalue. Clearly the first factor can produce at most one eigenvalue and that only in exceptional cases.

$$\begin{aligned}\sinh((i\mu(b-a)) &= -i\ \sin((-\mu(b-a)) = 0\\ -\mu(b-a) &= (-1)^k \arcsin(0) + k\pi = k\pi,\ k \in Z\end{aligned}$$

Since $\mu = s > 0$ we get the following eigenvalues from the periodic factor:

$$\lambda_n = \frac{(n\pi)^2}{(b-a)^2},\ n \in N.$$

2. $A_1 = 1 = B_1,\ A_2 = 0 = B_2$. For these Dirichlet BC $\lambda = 0$ is not an eigenvalue and also the first factor does not produce an eigenvalue; hence all the eigenvalues are given by

$$\lambda_n = \frac{(n\pi)^2}{(b-a)^2},\ n \in N.$$

3. $A_1 = 0 = B_1,\ A_2 = 1 = B_2$. These are the Neuman (N) BC.

$$-i\mu\sinh((i\mu(b-a)) = \mu\sin(-\mu(b-a)) = 0$$

$$\mu(b-a) = k\pi,\ k \in Z.$$

Since $\lambda = 0$ is also an eigenvalue in this case and the first factor does not produce an eigenvalue we get that

$$\lambda_n^N = \frac{(n\pi)^2}{(b-a)^2},\ n \in N_0.$$

4. $A_1 = 1 = B_2,\ A_2 = 0 = B_1.$

$$\cosh((i\mu(b-a)) = \cos(-\mu(b-a)) = 0$$

$-\mu(b-a) = \pm\arccos(0) + 2k\pi,\ k \in Z,$

$$\lambda_n^{DN} = \frac{(\pi/2 + 2n\pi)^2}{(b-a)^2},\ n \in N_0.$$

5. $A_1 = 0 = B_2,\ A_2 = 1 = B_1.$

$$-\cosh((i\mu(b-a)) = -\cos((-\mu)(b-a)) = 0.$$

Here we have the same roots and hence the same eigenvalues as in the previous case

$$\lambda_n^{ND} = \frac{(\pi/2 + 2n\pi)^2}{(b-a)^2},\ n \in N_0.$$

6. $A_1B_2 - A_2B_1 \neq 0.$

$$\coth(i\mu(b-a)) = \frac{(\frac{1}{\mu}A_1B_1 - i\mu A_2B_2)}{A_1B_2 - A_2B_1}.$$

The roots of this equation are not so easy to find explicitly since the unknown μ appears on both sides. However, numerical approximations can be obtained from a root finder code.

Some observations

1. Let $\{a_k : k \in N\}$ be a decreasing sequence to $-\infty$; $\{b_k : k \in N\}$an increasing sequence to $+\infty$ and let

$$E = \{\lambda_n^D(a_k, b_k) : n \in N_0,\ k \in N.$$

   Then E is dense in $[0, \infty)$. Thus every point of the essential spectrum of the self-adjoint realization S of the Fourier equation on $(-\infty, \infty)$ is the limit of a sequence of eigenvalues of regular problems on the intervals $(a_k, b_k),\ k \in N$. This illustrates a general result, to be discussed in section 5, about the approximation of the spectrum of singular SLP by eigenvalues of a sequence of regular SLP.
2. For any $n \in N_0$ the Dirichlet eigenvalue λ_n^D is greater than or equal to λ_n for any other self-adjoint boundary condition.
3. $\lambda_n^D(a, b) \to 0$ as $(b-a) \to \infty$ for each $n \in N_0$.
4. $\lambda_n^D(a, b) \to \infty$ as $(b-a) \to 0$ for each $n \in N_0$.

5. $\lambda_0^D(a, b, q) \to \infty$ as $(b-a) \to 0$ for $q(t) = c$, a constant. But
6. $\lambda_0^N(a, b, q) \to c$ as $(b-a) \to 0$ for $q(t) = c$, a constant. Thus, as the length of the interval shrinks to zero the difference between the Dirichlet and Neuman eigenvalues goes to infinity. This is a general phenomenon of SLP found recently by Kong and Zettl, see [55], [57].

4.4 The space of regular Boundary Value Problems.

We want to show that if two SLP are "close" to each other then their eigenvalues and eigenfunctions are also "close" to each other. To study the "closeness" of two BVP we introduce a "boundary value problem space" with a distance function. Let

$$J = (a', b'),\ -\infty \leq a' < b' \leq \infty,$$

$$\Omega = \{\omega = (a, b, A, B, 1/p, q, w)\} \tag{4.19}$$

such that

$$-\infty \leq a' < a < b < b' \leq \infty,\ A, B \in M_2(\mathbb{C}),\ p, q, w : J \to \mathbb{C},\ 1/p, q, w \in L_{loc}(J).$$

By an eigenvalue of $\omega \in \Omega$ we mean an eigenvalue of the BVP determined by ω. Note that we have changed the notation for the endpoints of the interval J from a to a' and b to b'; this is so that we can use a and b to denote endpoints of varying subintervals of J.

We want to show that the eigenvalues and eigenfunctions depend continuously on the problem, i.e. a small change of the problem results in a small change of each eigenvalue and each eigenfunction. This means we have to compare the spectrum of different problems which may be defined on different intervals. Each $\omega \in \Omega$ determines a unique SL problem: a, b the interval, A, B the boundary condition, and the restrictions of p, q, w on $[a, b]$ the equation. Observe that the values of p, q, w outside the interval $[a, b]$, i.e. in $(a', b') \setminus [a, b]$, do not affect the spectrum of the problem determined by ω. To account for this and to facilitate comparisons between eigenvalues of problems defined on different intervals we let

$$\tilde{\Omega} = \{\tilde{\omega} = (a, b, A, B, \widetilde{1/p}, \tilde{q}, \tilde{w})\} \tag{4.20}$$

where

$$\tilde{q} = \begin{cases} q & on\ [a, b] \\ 0 & otherwise \end{cases} \tag{4.21}$$

and $\widetilde{1/p}$, $\tilde{w}$ are defined similarly. Now we introduce the Banach space

$$(4.22) \qquad X = R \times R \times M_{2,2}(C) \times M_{2,2}(C) \times L^1(a',b') \times L^1(a',b') \times L^1(a',b')$$

with its "natural" norm

$$(4.23) \qquad ||\omega|| = ||\tilde{\omega}|| = |a| + |b| + ||A|| + ||B|| + \int_{a'}^{b'} \left(|\widetilde{1/p}| + |\tilde{q}| + |\tilde{w}| \right)$$

where $||A||$ is any fixed matrix norm.

This space X is a "natural" setting for the study of regular SL problems. Note that, since $1/p, q, w$ are only assumed to be in $L_{loc}(a',b')$, Ω is not a subset of X but $\tilde{\Omega}$ is since $\widetilde{1/p}, \tilde{q}, \tilde{w}$ are in $L^1(a',b')$. Now we identify Ω with $\tilde{\Omega}$ as a subset of X. Then Ω inherits the norm from X, and the convergence in Ω is determined by this norm. It is easy to see that every point in Ω is an accumulation point of Ω with respect to this norm in X.

4.5 Continuity of Eigenvalues and Eigenfunctions.

Are the eigenvalues of a regular SLP continuous functions of the problem? The answer is YES and NO.

NO, because for a fixed index n, the n-th eigenvalue λ_n of a self-adjoint SLP is, in general, not a continuous function of the problem i.e. of ω.

YES, because every isolated eigenvalue can be embedded in a "continuous eigenvalue branch".

In this section these statements will be made precise. Also the continuous dependence of the eigenfunctions on the problem is studied here.

Consider the SLP consisting of the equation

$$(4.24) \qquad -(py')' + qy = \lambda\, wy, \ on\ (a,b), \ -\infty < a < b < \infty,$$

together with separated boundary conditions

$$(4.25) \qquad A_1 y(a) + A_2 (py')(a) = 0, \ (A_1, A_2) \neq (0,0), \ A_1, A_2 \in \mathbb{R},$$

$$(4.26) \qquad B_1 y(b) + B_2 (py')(b) = 0, \ (B_1, B_2) \neq (0,0), \ B_1, B_2 \in \mathbb{R},$$

and coefficients satisfying

$$(4.27) \qquad p, q, w : (a,b) \to \mathbb{R}, \ 1/p, q, w \in L(a,b), \ p > 0, \ w > 0, \ a.e.\ on\ (a,b).$$

It is well known that this problem is self-adjoint and has an infinite but countable number of eigenvalues $\{\lambda_n : n \in N_0\}$, these are all real, simple, bounded below and can be indexed to satisfy

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \dots ; \; and \; \lambda_n \to \infty \; as \; n \to \infty.$$

THEOREM 4.7. Let (4.24) to (4.27) hold. Fix a, b, p, q, w.

- Fix B_1, B_2 and let $A_1 = 1$. Consider $\lambda_n = \lambda_n(A_2)$ as a function of $A_2 \in \mathbb{R}$. Then for each $n \in N_0$, $\lambda_n(A_2)$ is continuous at A_2 for $A_2 > 0$ and $A_2 < 0$ but has a jump discontinuity at $A_2 = 0$. More precisely we have:
  1. $\lambda_n(A_2) \to \lambda_n(0) \; as \; A_2 \to 0^-, \; n \in N_0$.
  2. $\lambda_0(A_2) \to -\infty \; as \; A_2 \to 0^+$.
  3. $\lambda_{n+1}(A_2) \to \lambda_n(0) \; as \; A_2 \to 0^+$.
- Fix A_1, A_2 and let $B_1 = 1$. Consider $\lambda_n = \lambda_n(B_2)$ as a function of $B_2 \in \mathbb{R}$. Then for each $n \in N_0$, $\lambda_n(B_2)$ is continuous at B_2 for $B_2 > 0$ and $B_2 < 0$ but has a jump discontinuity at $B_2 = 0$. More precisely we have:
  1. $\lambda_n(B_2) \to \lambda_n(0) \; as \; B_2 \to 0^+, \; n \in N_0$.
  2. $\lambda_0(B_2) \to -\infty \; as \; B_2 \to 0^-$.
  3. $\lambda_{n+1}(B_2) \to \lambda_n(0) \; as \; B_2 \to 0^-$.

PROOF. See Everitt, Möller and Zettl [23]. □

REMARK 6. Note that $\lambda_0(A_2)$ has an infinite jump discontinuity at $A_2 = 0$, but for all $n \geq 1$, $\lambda_n(A_2)$ has a finite jump discontinuity at $A_2 = 0, \; \lambda_n(A_2)$ is left but not right continuous at 0. Similarly, $\lambda_0(B_2)$ has an infinite jump discontinuity at $B_2 = 0$, but for all $n \geq 1$, $\lambda_n(B_2)$ has a finite jump discontinuity at $B_2 = 0; \; \lambda_n(B_2)$ is right but not left continuous at 0. In all cases $\lambda_n(0)$ is embedded in a continuous branch of eigenvalues as A_2 or B_2 passes through zero but this branch is not given by a fixed index n; the index "jumps" from n to $n+1$ as A_2 or B_2 pass through zero from the appropriate direction.

REMARK 7. This forced "index jumping" in order to stay on a continuous branch of eigenvalues plays an important role in some of the algorithms and their numerical implementations used in the code SLEIGN2 [9] for the numerical approximation of the spectrum of regular and singular SLP.

REMARK 8. Kong and Zettl [57] have shown that each continuous eigenvalue branch is in fact differentiable everywhere including the point A_0 (or B_0) where the index jumps. This also follows from Möller and Zettl [69].

EXAMPLE 4.8. Consider the BVP with Dirichlet BC and the equation

$$-(p_\varepsilon \, y')' = \lambda \, y \; on \; (0, 1),$$

where $\varepsilon \in [0,1]$ and

$$p_\varepsilon(t) = \begin{cases} -1, & if \quad 0 \le t \le \varepsilon, \\ 1, & \quad\quad \varepsilon < t \le 1. \end{cases}$$

Then for $\varepsilon = 0$ the spectrum is bounded below but for each $\varepsilon > 0$ the spectrum is unbounded below. Note that $1/p_\varepsilon \to 1/p_0$ in $L(0,1)$.

LEMMA 4.9 (Continuity of the roots as functions of parameters)**.** Let A be an open set in $\mathbb{C}$, F a metric space, f a continuous complex valued function on $A \times F$ such that for each $\alpha \in F$, the map $z \to f(z,\alpha)$ is an analytic function on A . Let B be an open subset of A whose closure $\bar{B}$ in $\mathbb{C}$ is compact and contained in A, and let $\alpha_0 \in F$ be such that no zero of $f(z,\alpha_0)$ is on the boundary of B. Then there exists a neighborhood W of α_0 in F such that :

1. For any $\alpha \in W$, $f(z,\alpha)$ has no zero on the boundary of B;
2. for any $\alpha \in W$, the sum of the orders of the zeros of $f(z,\alpha)$ contained in B is independent of α.

PROOF. See page 248 in Dieudonné [16] . □

THEOREM 4.10. Let $\omega_0 = (a_0, b_0, A_0, B_0, 1/p_0, q_0, w_0) \in \Omega$. Let $\lambda(\omega_0)$ be an isolated eigenvalue of the SL problem (4.2), (4.2), (4.3). Then, given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\omega = \omega(a, b, A, B, 1/p, q, w) \in \Omega$ satisfies

$$||\omega - \omega_0|| = |a - a_0| + |b - b_0| + ||A - A_0|| + ||B - B_0|| +$$

$$+ \int_{a'}^{b'} \left(|\frac{1}{\tilde{p}} - \frac{1}{\tilde{p_0}}| + |\tilde{q} - \tilde{q}_0| + |\tilde{w} - \tilde{w}_0| \right) < \delta \tag{4.28}$$

then ω has an eigenvalue $\lambda(\omega)$ satisfying

$$|\lambda(\omega) - \lambda(\omega_0)| \; < \; \epsilon. \tag{4.29}$$

Furthermore, if $\lambda(\omega_0)$ is simple then there is exactly one $\lambda(\omega)$ satisfying (4.29); if $\lambda(\omega_0)$ is a double eigenvalue, then either (4.29) holds for a double eigenvalue of ω or for exactly two simple ones.

PROOF. See Kong and Zettl [57] Theorem 3.1. □

REMARK 9. Besides establishing the continuity of the eigenvalues within continuous eigenvalue branches for self-adjoint and non-self-adjoint SLP, Theorem 4.10 is also an existence theorem for eigenvalues. Since any self-adjoint SLP ω_0 is known to have a countably infinite number of isolated eigenvalues, we can conclude from

Theorem 4.10 that every SLP ω which is sufficiently close to ω_0, *whether it is self-adjoint or not,* must have n eigenvalues close to n eigenvalues of $\lambda(\omega_0)$, for any positive integer n. Note that it does not follow directly from Theorem 4.10 that ω must have an *infinite* number of eigenvalues close to those of ω_0.

REMARK 10. Below, each eigenvalue will be assumed to be embedded in a continuous eigenvalue branch in the sense of Theorem 4.10.

By a normalized eigenfunction u of an SL problem we mean an eigenfunction u that satisfies

$$\int_a^b |u|^2 \, w \; = \; 1. \tag{4.30}$$

Next we state a result for normalized eigenfunctions. Note that these are not uniquely determined. In the case of a simple eigenvalue they are unique up to sign, but for a double eigenvalue there are pairs of linearly independent normalized eigenfunctions.

Below when considering SLP on two subintervals of J we extend the solutions and their quasi-derivatives continuously to the whole of the open interval J. This is done to facilitate comparisons between solutions and their quasi-derivatives on different intervals.

THEOREM 4.11. Let the notation and hypotheses of Theorem 4.10 hold.

(i) Assume the eigenvalue $\lambda(\omega_0)$ is simple for some $\omega_0 \in \Omega$ and let $u(\cdot,\, \omega_0)$ denote a normalized eigenfunction of $\lambda(\omega_0)$. Then there is a neighborhood M of ω_0 in Ω such that $\lambda(\omega)$ is simple for every $\omega \in M$ and there exist normalized eigenfunctions $u(\cdot, \omega)$ of $\lambda(\omega)$ for $\omega \in M$ such that

$$u(\cdot,\, \omega) \to u(\cdot,\, \omega_0), \quad (pu')(\cdot,\, \omega) \to (pu')(\cdot,\, \omega_0), \quad \textit{as } \omega \to \omega_0 \textit{ in } \Omega, \tag{4.31}$$

both uniformly on any compact subinterval K of (a', b').

(ii) Assume that $\lambda(\omega)$ is a double eigenvalue for all ω in some neighborhood M of ω_0 in Ω. Let $u(\cdot, \omega_0)$ be any normalized eigenfunction of $\lambda(\omega_0)$. Then there exist normalized eigenfunctions $u = (\cdot, \omega)$ of $\lambda(\omega)$ such that

$$u(\cdot,\, \omega) \to u(\cdot,\, \omega_0), \quad (pu')(\cdot,\, \lambda) \to (pu')(\cdot,\, \omega_0), \quad \textit{as } \omega \to \omega_0 \textit{ in } \Omega, \tag{4.32}$$

both uniformly on any compact subinterval K of (a', b'). Note that in this case, given two linearly independent normalized eigenfunctions u_j of $\lambda(\omega_0)$ there exist a pair u_j of linearly independent normalized eigenfunctions of $\lambda(\omega)$ such that $u_j(\cdot, \omega) \to u_j(\cdot, \omega_0)$ as $\omega \to \omega_0$ *in* Ω for $j = 1, 2$.

PROOF. See [57], Theorem 3.2. □

4.6 Self-adjoint problems with $w > 0$.

The space of self-adjoint SLP is denoted by Ω_{s-a} i.e.

$$\Omega_{s-a} = \{\omega = (a, b, A, B, 1/p, q, w)\}$$

where we assume that

$$\begin{aligned} -\infty &\leq a' < a < b < b' \leq \infty,\ A,\ B \in M_2(C), \\ A\,E\,A^* &= B\,E\,B^*,\ rank\,(A|B) = 2,\ E = \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}, \\ p,\ q,\ w &: J \to R,\ 1/p,\ q,\ w \in L_{loc}(J),\ w > 0,\ J = (a', b'). \end{aligned}$$

1. **Note that no sign condition is placed on p.**
   For our purposes here it is convenient to divide the self-adjoint boundary conditions into three mutually exclusive subclasses and to use the following canonical representations of these subclasses:
2. **Separated self-adjoint BC.** These are

$$A_1 y(a) + A_2 (py')(a) = 0,\ A_1, A_2 \in R,\ (A_1, A_2) \neq (0, 0) \tag{4.33}$$

$$B_1 y(b) + B_2 (py')(b) = 0,\ B_1, B_2 \in R,\ (B_1, B_2) \neq (0, 0) \tag{4.34}$$

   These separated conditions can be parameterized as follows:

$$\cos\alpha\, y(a)\ -\ \sin\alpha\, (py')(a)\ =\ 0,\quad 0 \leq \alpha < \pi; \tag{4.35}$$

$$\cos\beta\, y(b)\ -\ \sin\beta\, (py')(b)\ =\ 0,\quad 0 < \beta \leq \pi. \tag{4.36}$$

   Note the different normalization in (4.36) for β than that used for α in (4.35). This is for convenience in stating some of the results below.
3. **All *real* coupled self-adjoint BC.** These can be formulated as follows:

$$\begin{pmatrix} y(b) \\ (py')(b) \end{pmatrix} = K \begin{pmatrix} y(a) \\ (py')(a) \end{pmatrix} \tag{4.37}$$

   where $K \in SL_2(\mathbb{R})$, i.e. K satisfies

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}, \ k_{ij} \in R, \ \det K = 1. \tag{4.38}$$

4. **All *complex* coupled self-adjoint BC.** These can be formulated as follows:

$$\begin{pmatrix} y(b) \\ (py')(b) \end{pmatrix} = \exp(i\,\alpha)\, K \begin{pmatrix} y(a) \\ (py')(a) \end{pmatrix} \tag{4.39}$$

where K satisfies (4.6) and $-\pi < \alpha < 0$, or $0 < \alpha < \pi$.

For the canonical form (4.3), (4.4) of the special case of separated self-adjoint BC we use the notation

$$\Omega_s = \{\omega = (a, b, \alpha, \beta, 1/p, q, w)\}; \tag{4.40}$$

for the general self-adjoint coupled case we let

$$\Omega_{cc} = \{\omega = (a, b, \alpha, K, 1/p, q, w), \quad -\pi < \alpha < 0, \text{ or } \ 0 < \alpha < \pi\}\,. \tag{4.41}$$

When $\alpha = 0$ we shorten (4.41) to

$$\Omega_{rc} = \{\omega = (a, b, K, 1/p, q, w)\}\,. \tag{4.42}$$

Most of the following results are well-known. See [81] for some proofs with only integrable coefficients; see [68] for the case when p changes sign, and see [8], [18] for the case of complex couple BC.

THEOREM 4.12. Let $\omega \in \Omega_{s-a}$; then ω is in exactly one of the subclasses: Ω_s, Ω_{cc}, Ω_{rc}.

(a) Assume that

$$p \geq 0 \text{ a.e. on } [a, b].$$

Then for $\omega \in \Omega_s$ the BVP ω has only real and simple eigenvalues; there are an infinite but countable number of them; they are bounded below and can be ordered to satisfy

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \cdots\,; \quad \lambda_n \to +\infty \text{ as } n \to \infty \tag{4.43}$$

If u_n is an eigenfunction of λ_n, then u_n is unique up to constant multiples and u_n has exactly n zeros in the open interval (a, b), $n \in N_0 = \{0, 1, 2, \dots\}$.

Notation. Let

$$\lambda_n = \lambda_n(a, b, \alpha, \beta, 1/p, q, w); \quad u_n = u_n(\cdot, a, b, \alpha, \beta, 1/p, q, w), \quad n \in N_0, \tag{4.44}$$

to highlight the dependence on these quantities.

For $\omega \in \Omega_{rc}$ the BVP ω has only real eigenvalues; each of these may be simple or double; there are an infinite but countable number of them and they can be ordered to satisfy

$$-\infty < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots; \quad \lambda_n \to +\infty \ , \ as \ \ n \to \infty. \tag{4.45}$$

Notation. Let

$$\lambda_n = \lambda_n(a, b, K, 1/p, q, w); \quad u_n = u_n(\cdot, a, b, K, 1/p, q, w), \quad n \in N_0. \tag{4.46}$$

Note that there is some arbitrariness in the indexing of the eigenfunctions corresponding to a double eigenvalue.

For $\omega \in \Omega_{cc}$ the BVP ω has only real and simple eigenvalues; there are an infinite but countable number of them and they can be ordered to satisfy

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \dots \lambda_n < ... \tag{4.47}$$

Notation. Denote these eigenvalues by

$$\lambda_n = \lambda_n(a, b, \alpha, K, 1/p, q, w); \quad u_n = u_n(\cdot, a, b, \alpha, K, 1/p, q, w), \quad n \in N_0. \tag{4.48}$$

For any $\omega \in \Omega_{s-a}$ we have the following asymptotic formula:

$$\frac{\lambda_n}{n^2} \to c = \pi^2 \left(\int_a^b \sqrt{\frac{w}{p}} \right)^{-2}, \ as\ n \to \infty. \tag{4.49}$$

If we fix all variables except α and shorten the notation to $\lambda_n = \lambda_n(\alpha)$, then we have $\lambda_n(-\alpha) = \lambda_n(\alpha)$,and the complex conjugate of an eigenfunction of $\lambda_n(\alpha)$ is an eigenfunction of $\lambda_n(-\alpha)$.

(b) **Assume that p changes sign in the interval [a,b]**, i.e. p is positive on a subset of $[a, b]$ of positive Lebesgue measure and p is negative on a subset of the interval $[a, b]$ of positive Lebesgue measure. Then

Each BVP $\omega \in \Omega_s$ has only real and simple eigenvalues; there are an infinite but countable number of them; they are unbounded below and above and can be ordered to satisfy

$$\begin{aligned} \dots &< \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < \dots ; \\ \lambda_n &\to +\infty \; as \; n \to \infty; \;\; \lambda_n \to -\infty \; as \; n \to -\infty. \end{aligned} \tag{4.50}$$

Each BVP $\omega \in \Omega_{rc}$ has only real eigenvalues; each of these may be simple or double; there are an infinite but countable number of them; they are unbounded below and above and can be ordered to satisfy

$$\begin{aligned} \dots &\leq \lambda_{-2} \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots ; \\ \lambda_n &\to +\infty \;\; n \to \infty, \;\; \lambda_n \to -\infty \;\; n \to -\infty. \end{aligned} \tag{4.51}$$

The notations for eigenvalues λ_n and eigenfunctions u_n, $n \in Z$, for part (b) are the same as those introduced in part (a) for $n \in N_0$.

PROOF. The fact that the eigenvalues are unbounded below when p changes sign was estalished by M. Möller. This holds even if there is no subinterval on which p is negative. The fact that the eigenvalues for any $\omega \in \Omega_{cc}$ are all simple follows from Weidmann [81], although it doesn't appear to be stated explicitly there for the general case. See also [8]. The other results are standard. □

Notation. In the following we denote by λ_n and u_n the n-th eigenvalue and the n-th eigenfunction of a SL problem where $n \in N_0$ if $p \geq 0 \, a.e.$ on (a,b) and $n \in Z$ if p changes sign on (a,b), respectively. When p changes sign the eigenvalues are indexed by $\mathbb{Z}$ and we follow the sleign2 convention and denote by λ_0 the first non-negative eigenvalue. This determines the indexing scheme uniquely.

THEOREM 4.13. Let $\omega = (a,b,A,B,1/p,q,w) \in \Omega_{s-a}$. Fix a,b,p,q,w and let $A = e^{i\alpha}K, \; K = (k_{ij}), \; B = -I$, where $K \in SL_2(\mathbb{R})$ i.e. we have the BC

$$Y(b) = e^{i\alpha} K \, Y(a), \; -\pi \leq \alpha \leq \pi.$$

Also assume that $p > 0$ a.e. on J and denote the eigenvalues for this boundary condition by $\lambda_n(\alpha, K)$, abbreviated to $\lambda_n(K)$ when $\alpha = 0$, for $n \in N_0$.

Suppose that either $k_{12} < 0$ *or* $k_{12} = 0$ *and* $k_{11} + k_{22} > 0$. Then

1. $\lambda_0(K)$ is simple;
2. $\lambda_0(K) < \lambda_0(-K)$ and
3. the following inequalities hold for $-\pi < \alpha < 0$ and $0 < \alpha < \pi$:

$$\begin{aligned} -\infty &< \lambda_0(K) < \lambda_0(\alpha, K) < \lambda_0(-K) \leq \lambda_1(-K) < \lambda_1(\alpha, K) < \lambda_1(K) \\ &\leq \lambda_2(K) < \lambda_2(\alpha, K) < \lambda_2(-K) \leq \lambda_3(-K) < \dots \end{aligned}$$

Furthermore, for $0 < \alpha < \beta < \pi$ we have

$$\begin{aligned}\lambda_0(\alpha, K) &< \lambda_0(\beta, K) < \lambda_1(\beta, K) < \lambda_1(\alpha, K) < \lambda_2(\alpha, K) < \lambda_2(\beta, K) \\ &< \lambda_3(\beta, K) < \lambda_3(\alpha, K) < \dots\end{aligned}$$

Suppose that either $k_{12} > 0$ *or* $k_{12} = 0$ *and* $k_{11} + k_{22} < 0$. Then

1. $\lambda_0(-K)$ is simple;
2. $\lambda_0(-K) < \lambda_0(K)$ and
3. the following inequalities hold for $-\pi < \alpha < 0$ and $0 < \alpha < \pi$:

$$\begin{aligned}-\infty &< \lambda_0(-K) < \lambda_0(\alpha, K) < \lambda_0(K) \le \lambda_1(K) < \lambda_1(\alpha, K) < \lambda_1(-K) \\ &\le \lambda_2(-K) < \lambda_2(\alpha, K) < \lambda_2(K) \le \lambda_3(K) < \dots\end{aligned}$$

Furthermore, for $0 < \alpha < \beta < \pi$ we have

$$\begin{aligned}\lambda_0(\beta, K) &< \lambda_0(\alpha, K) < \lambda_1(\alpha, K) < \lambda_1(\beta, K) < \lambda_2(\beta, K) < \lambda_2(\alpha, K) \\ &< \lambda_3(\alpha, K) < \lambda_3(\beta, K) < \dots\end{aligned}$$

PROOF. This general result is due to Eastham in a private communication [18], special cases were established in [8], [81], [19]. □

THEOREM 4.14. Let $\omega = (a, b, 0, \pi, 1/p, q, w) \in \Omega_s$ and fix a, p, q, w. Assume that

$$p \ge 0 \ a.e. \ and \ q^2/w \in L_{loc}(a', b').$$

Then for any $n \in N_0$, $\lambda_n^D(b)$ is strictly decreasing in (a, b') and

$$\lambda_n^D(b) \to \infty \ as \ b \to a^+.$$

PROOF. See [55]. □

THEOREM 4.15. Let $\omega = (a, b, \alpha, \pi/2, 1/p, q, w) \in \Omega_s$ (i.e. we have an arbitrary separated self-adjoint BC at a but a Neuman condition at b.) Assume that

$$Q = q/w \in AC_{loc}[a, b'), \ p(b) \ge \delta > 0 \ for \ b \in (a, b').$$

Then

1. $\lambda_0(b) \to Q(a) = q(a)/w(a)$ *as* $b \to a^+$.
2. $\lambda_n(b) \to \infty$ *as* $b \to a^+$ *for* $n = 1, 2, 3, \ldots$
3. If Q is decreasing in (a, b') then $\lambda_n(b)$ is decreasing in (a, b') and $\lambda_n(b) \geq Q(b)$ for each $n \in N_0$.
4. If Q is increasing in (a, b') and $Q(b) \to \infty$ as $b \to b'$ the $\lambda_0(b)$ is increasing in (a, b') and $\lambda_0(b) \leq Q(b)$; for $n \in N$, $\lambda_n(b)$ has a unique extremum in (a, b') and this extremum is a strict minimum.
5. If Q has a unique extremum in (a, b') and this extremum is a strict minimum and $Q(b) \to \infty$ as $b \to b'$ then for any $n \in N_0$, $\lambda_n(b)$ has a unique extremum in (a, b') and this extremum is a strict minimum.

PROOF. See Theorem 4.4 in [55]. □

4.7 Differentiability Properties of Eigenvalues.

In this section we study the differentiable dependence of the simple eigenvalues on the problem. It turns out that the continuous eigenvalue branches studied above in Subsection 4.5 are differentiable.

THEOREM 4.16. Let $\omega = (a, b, A, B, 1/p, q, w) \in \Omega_{s-a}$.

1. Fix all the components of ω except the left endpoint a and consider $\lambda = \lambda(a)$ as a function of a. Assume that for some a, $\lambda(a)$ is a simple eigenvalue of $\omega(a)$. Then there is a neighborhood U of a and a neighborhood V of $\lambda(a)$ such that for every c in U the BVP $\omega(c)$ has exactly one eigenvalue in V and it is simple. The map $\lambda : U \to V$ is differentiable almost everywhere in U and we have

$$\lambda'(a) = \frac{1}{p(a)}|pu'|^2(a) - |u|^2(a)[q(a) - \lambda(a)w(a)] \quad a.e. \text{ in } U. \tag{4.52}$$

   Furthermore, if p, q, w are continuous at a and $p(a) \neq 0$, then (4.52) holds at the point a.
2. Fix all the components of ω except b. Assume that for some b, $\lambda(b)$ is a simple eigenvalue of $\omega(b)$. Then there is a neighborhood U of b and a neighborhood V of $\lambda(b)$ such that for every c in U the BVP $\omega(c)$ has exactly one eigenvalue in V and it is simple. The map $\lambda : U \to V$ is differentiable almost everywhere in U and we have

$$\lambda'(b) = -\frac{1}{p(b)}|pu'|^2(b) + |u|^2(b)\,[q(b) - \lambda(b)w(b)]. \tag{4.53}$$

   Furthermore, if p, q, w are continuous at b and $p(b) \neq 0$, then (4.53) holds at the point b.

REMARK 11. In his well known monograph on variational methods for eigenvalue problems Hans Weinberger states, without proof or reference to a proof, that the Dirichlet eigenvalues are decreasing functions of the length of the interval but that this is not true for the Neuman eigenvalues. Theorem 4.16 sheds a great deal of light on this.

Assume that $p > 0$. For Dirichlet BC $u(a) = u(a, a, 0, \pi) = 0$ and $u(b, b, 0, \pi) = 0$ hence the second term in (4.52) and in (4.53) is zero; thus it is clear from these formulas that the Dirichlet eigenvalues are increasing functions of the left endpoint and decreasing functions of the right endpoint. It is also clear from (4.52), (4.53) that this is not true, in general, *for any other boundary conditions.* However if q/w is bounded above, say by C, then *for any boundary conditions* all eigenvalues greater than C are increasing functions of the left endpoint a and decreasing functions of the right endpoint b. Since for any fixed regular SLP the eigenvalues $\lambda_n \to \infty$ asymptotically as n^2 it is clear that if q/w is bounded above only the lower eigenvalues may fail to be monotonic functions of the length of the interval.

THEOREM 4.17. Let $\omega = \omega(a, b, A, B, 1/p, q, w) \in \Omega_{s-a}$ and fix a, b, p, q, w. (Recall that A, B are replaced by α, β for BC (4.35), (4.36); by α, K for (4.39) and by K for (4.37)). Fix all components of ω except α and let $\lambda = \lambda(\alpha)$ and $u = u(\cdot, \alpha)$. Then λ is differentiable and

$$\lambda'(\alpha) = -u^2(a) - (pu')^2(a), \quad 0 \le \alpha < \pi. \tag{4.54}$$

1. Fix all components of ω except β and let $\lambda = \lambda(\beta)$ and $u = u(\cdot, \beta)$. Then λ is differentiable and

$$\lambda'(\beta) = u^2(b) + (pu')^2(b), \quad 0 < \beta \le \pi. \tag{4.55}$$

2. Fix all components of ω except $\alpha\,\min(4.39)$ and let $\lambda = \lambda(\alpha)$ and $u = u(\cdot, \alpha)$. Then λ is differentiable at α for any α satisfying $-\pi < \alpha < 0$ or $0 < \alpha < \pi$ and

$$\lambda'(\alpha) = -2\,\mathrm{Im}[u(b)\ (p\overline{u}')(b)], \tag{4.56}$$

   where $\mathrm{Im}[z]$ denotes the imaginary part of z.
3. Fix all components of ω except K. Assume that $\lambda = \lambda(K)$ is a simple eigenvalue and $u = u(\cdot, K)$ a normalized eigenfunction of K. Then there exists a neighborhood U of K and a neighborhood V of $\lambda(K)$ such that for every G in U the (self-adjoint or non-self-adjoint since G is not assumed to be in $SL_2(\mathbb{R})$) BVP $\omega(G)$ has exactly one eigenvalue in V and it is simple. The map $\lambda : U \to V$ is differentiable at K and its Frechet derivative is given by:

$$\lambda'(K)\,H \;=\; [p\overline{u}'(b), -\overline{u}(b)]\,HK^{-1}\begin{pmatrix} u(b) \\ (pu')(b) \end{pmatrix}, \quad H \in M_{2,2}(\mathbb{C}). \tag{4.57}$$

PROOF. See [57] for parts (1) and (2) and see [69] for part (3). □

THEOREM 4.18. Let $\omega = (a, b, A, B, 1/p, q, w) \in \Omega$. Fix a, b, A, B and assume that A, B satisfy the self-adjointness conditions:

$$A\,E\,A^* = B\,E\,B^*, \;\; rank\,(A|B) = 2, \; E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Suppose that $\lambda(\omega)$ is an isolated simple eigenvalue of ω. Then there is a simple closed curve Γ in $\mathbb{C}$ with $\lambda(\omega)$ in its interior and a neighborhood U of ω in Ω such that for any ρ in U the BVP identified with ρ has exactly one simple eigenvalue inside Γ. This map $\lambda : U \to \mathbb{C}$ is differentiable with respect to

1. q with p, w fixed and

$$\lambda'(q)\,h = \int_a^b |u(\cdot, q)|^2\,h, \;\; h \in L(a, b), \tag{4.58}$$

   where $u = u(\cdot, q)$ is a normalized eigenfunction of $\lambda(q)$;
2. $1/p$ with q, w fixed and

$$\lambda'(1/p)\,h = -\int_a^b |(pu')(\cdot, 1/p)|^2\,h, \;\; h \in L(a, b), \tag{4.59}$$

   where $u = u(\cdot, 1/p)$ is a normalized eigenfunction of $\lambda(1/p)$;
3. w with $1/p, q$ fixed and

$$\lambda'(w)\,h = -\lambda(w)\int_a^b |u(\cdot, w)|^2\,h, \;\; (h \in L(a, b)), \tag{4.60}$$

   where $u = u(\cdot, w)$ is a normalized eigenfunction of $\lambda(w)$.

PROOF. See [69]. □

THEOREM 4.19. Let $\omega = (a, b, A, B, 1/p, q, w) \in \Omega_{s-a}$ and fix a, b, A, B. Assume that $\lambda(\omega)$ is a simple eigenvalue of ω. (Recall that each eigenvalue is assumed to lie on a continuous eigenvalue branch; see Theorem 4.10 and Remark 10.)

1. Fix p, w. Suppose $Q \in L(a, b)$ and $Q \geq q$ on $[a, b]$. Assume that $\lambda(s(t))$ is on the same continuous eigenvalue branch as $\lambda(q)$ for all $t \in [0, 1]$, where

$$s(t) = q + t(Q - q).$$

   Then $\lambda(Q) \geq \lambda(q)$. If $Q > q$ on a subset of $[a, b]$ having positive Lebesgue measure, the $\lambda(Q) > \lambda(q)$.

2. Fix q, w. Suppose $P \in L(a,b)$ and $P \leq p$ on $[a,b]$. Assume that $\lambda(s(t))$ is on the same continuous eigenvalue branch as $\lambda(1/p)$ for $t \in [0,1]$, where

$$s(t) = 1/p + t(1/P - 1/p).$$

   Then $\lambda(1/P) \geq \lambda(1/p)$. If $1/P < 1/p$ on a subset of $[a,b]$ having positive Lebesgue measure, the $\lambda(1/P) < \lambda(1/p)$.
3. Fix p, q. Suppose $W \in L(a,b)$ and $W \geq w > 0$ on $[a,b]$. Assume that $\lambda(s(t))$ is on the same continuous eigenvalue branch as $\lambda(w)$ for all $t \in [0,1]$, where

$$s(t) = w + t(W - w).$$

   Then $\lambda(W) \geq \lambda(w)$ if $\lambda(W) < 0$ and $\lambda(w) < 0$; but $\lambda(W) \leq \lambda(w)$ if $\lambda(W) > 0$ and $\lambda(w) > 0$. Furthermore, if strict inequality holds in the hypothesis on a set of positive Lebesgue measure, then strict inequality holds in the conclusion.

PROOF. We give the proof for (1), the proofs of (2) and (3) are similar. Define

$$f(t) = \lambda(s(t)),\ t \in [0,1].$$

From the chain rule and (4.58) we have

$$f'(t) = \lambda'((s(t))\, s'(t) = \int_a^b |u^2(r, s(t))|\, (Q(r) - q(r))\, dr \geq 0,\ t \in [0,1].$$

Hence $f(1) = \lambda(Q) \geq \lambda(q) = f(0)$. The strict inequality part of the theorem also follows from this. □

THEOREM 4.20. Fix $a, b \in J$, $a < b$. Let $\omega = (A, B, 1/p, q, w) \in \Omega = (M_2(C))^2 \times (L(a,b))^3$ Assume that $\lambda(\omega)$ is an isolated simple eigenvalue of ω. Then there is a simple closed curve Γ in C with $\lambda(\omega)$ in its interior and a neighborhood U of ω in Ω such that for each $\varphi \in U$ the BVP identified with φ has exactly one simple eigenvalue $\lambda(\varphi)$ inside Γ. This map $\lambda : U \to C$ is differentiable at ω and its derivative is given by:

$$\lambda'(\omega)\, \rho = \int_a^b \{-py'\bar{p}\bar{z}'(1/r) + [g - \lambda(\rho)v]y\bar{z}\} + d^*[CY(a) + DY(b)], \tag{4.61}$$

where $\rho = (C, D, 1/r, g, v) \in \Omega$, y, z are biorthogonal solutions of the given and its adjoint boundary value problems at $\lambda(\omega)$, *i.e.*

$$-(py')' + qy = \lambda(\omega)\, w\, y,\ AY(a) + BY(b) = 0,\ Y = \begin{pmatrix} y \\ py' \end{pmatrix} \tag{4.62}$$

(4.63) $$-(\bar{p}z')' + \bar{q}z = \bar{\lambda}(\omega)\bar{w}z,\ Z = \begin{pmatrix} z \\ \bar{p}z' \end{pmatrix},\ \int_a^b y\,\bar{z}\,w = 1,$$

and where $d \in \mathbb{C}^2$ is such that

(4.64) $$Z(a) = EA^* d,\ Z(b) = -EB^* d,\ E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

PROOF. See [69]. □

REMARK 12. Note that for Theorem 4.20 no self-adjointness hypothesis is needed: the coefficients p, q and the weight function w may be complex-valued; the boundary conditions need not be self-adjoint. The existence of infinitely many eigenvalues for non-self-adjoint SLP is well known for so called Birkhoff regular and Stone regular SLP, see [67], [20].

4.8 Comments.

1. These are made separately for each subsection.
2. Characterizing the eigenvalues as the zeros of an entire function is a standard technique but we don't know of a refernce where it is done for the general Δ function given by (4.7). Lemmas (4.1) to (4.3) are standard. The Fourier equation in subsection 3 provides examples for cases (1), (2) and (4) but not (3) of Lemma 4.3. Atkinson indicates in his book [3] that there are examples of case (3) of lemma (4.3) with a leading coefficient p such that $1/p$ is identically zero on subintervals. Are there such examples with $p > 0$?

   Lemma 4.5 was established by Bailey, Everitt and Zettl [8] for self-adjoint, regular and singular SLP. It was only during the writing of these notes that the author realized that no symmetry (formal self-adjointness) or definiteness assumption is needed on the equation.

   Theorem 4.6 is only a special case of results of Mennicken and Möller and of Eberhard and Freiling. More results of this kind and detailed proofs can be found in the forthcoming book by the former two authors [67]
3. Coddington and Levinson [14] have examples showing that the Fourier equation with non-self-adjoint two point boundary conditions can have either no eigenvalues or every complex number may be an eigenvalue. The writer found it rather interesting that dozens of properties of the hyperbolic functions come into play just to compute eigenvalues of regular problems for the Fourier equation. This was a humbling experience. Thank goodness for the book by Abramowitz and Stegun[1] .
4. For regular problems on an interval J it seems that the $L^1(J)$ norm for $1/p, q, w$ is the "natural" norm to use.
5. The "index jumping" phenomenon in order to preserve continuity given by Theorem 4.7 is due to Everitt, Möller and Zettl [23]. Another version of it, which involves changing the endpoint and the boundary conditions " in harmony with each other " was discovered by Bailey, Everitt and Zettl and

is used in their numerical code SLEIGN2 to compute eigenvalues of singular problems with oscillatory endpoints. See the forthcoming article by Everitt, Möller and Zettl.

The continuity result of Theorem 4.10 is basically a consequence of a theorem in complex variables which extends the result on the continuity of the roots of polynomial equations as functions of the coefficients, to analytic functions. Given an entire function whose coefficients from its series expansion depend continuously on a parameter which lives in a metric space, then its zeros are continuous functions of this parameter. The previous subsection on the space of BVP served to provide such a metric space, actually a Banach space for our application. The continuity result of Theorem 4.10 allows us to work on "continuous eigenvalue branches" which, as Theorem 4.7 shows, are not always specified by a fixed eigenvalue index.

Theorem 4.11 appears to be new in [57], see also [54].

6. M. Möller [68] showed that the eigenvalues are not bounded below when p changes sign even if there is no subinterval on which p is negative. The other results summarized in Theorem 4.12 are standard.

   Theorem 4.13 is due to Eastham [18] in a private communication and is presented here for the first time with his permission; special cases were established in [8], in Weidmann's Lecture Notes [81] for the special case $K = \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix}$ with c a non zero real number. For a very different proof for the case $c = 1$ see the book of Eastham [19].

   The asymptotic formula (4.49) has many extensions, see Atkinson and Mingarelli [5], Harris [39], [52].
7. The differentiability results of Subsection 7 are due to Kong and Zettl [55], [57]. The differentiability results with respect to the endpoints were inspired by earlier such results of Dauge and Helffer [15]. For extensions of all these results to higher order ode's see Kong, Wu and Zettl [54]. For a far reaching extension to operator theory with applications to matrix theory, ordinary differential equations, partial differential equations, etc. see the article of Möller and Zettl [69].

5 SINGULAR BOUNDARY VALUE PROBLEMS

5.1 Introduction.

In this section we discuss singular self-adjoint SLP. This is a field so vast that we can only hope to give a brief introduction here. Following a review of the some of the basic theory we will focus on two topics: (i) The behavior of eigenvalues of regular SLP near a singular boundary and (ii) The approximation of the discrete as well as continuous (essential) spectrum of a given singular problem with spectra of regular problems. On these two topics we aim to bring the reader to the frontier. To illustrate some of the basic behavior of the spectrum of SLP we discuss briefly the 29 examples from the sleign2 package in Subsection 5.8.

5.2 Principal and non-principal solutions.

Consider the symmetric equation

$$(5.1)\quad \begin{aligned} My &\equiv -(py')' + qy = \lambda\, w\, y \text{ on } J = (a,b),\ -\infty \le a < b \le \infty, \\ \lambda &\in \mathbb{R},\ p,\, q,\, w : J \to R,\ 1/p,\, q,\, w \in L_{loc}(J),\ p \ge 0,\ w > 0 \text{ a.e. on } J. \end{aligned}$$

Note that for this section we assume that $p \ge 0$ and $w > 0$ unless explicitly stated otherwise.

Recall that, according to Proposition 3.10 in section 3 no nontrivial solution of (5.1) can have an accumulation point of zeros in the interior of J. The zeros, if any, of any nontrivial solution of (5.1) inside the interval J are isolated. Thus only an endpoint of J can be an accumulation point of zeros of a nontrivial solution of (5.1) and that can happen only at a singular endpoint.

DEFINITION 5.1 (Principal Solution)**.** Let u, v be real solutions of (5.1). Then

- u is called a *principal solution* at a if
  1. $u(t) \neq 0$ for $t \in (a,d]$ and some $d \in J$,
  2. every solution y of (5.1) which is not a multiple of u satisfies

$$(5.2)\qquad u(t) = o(y(t)) \text{ as } t \to a.$$

- v is called a *non-principal solution* at a if
  1. $v(t) \neq 0$ for $t \in (a,d]$ and some $d \in J$,
  2. v is not a principal solution at a.

Principal and non-principal solutions at b are defined similarly.

To simplify things we state definitions and assertions only for the left endpoint a. Similar definitions and assertions for the right endpoint b always hold and are freely used.

LEMMA 5.2. If (5.1) has a principal solution u at a, then every non-zero real multiple of u is also a principle solution and no other solution is a principal solution at a.

PROOF. This follows directly from the definition. □

REMARK 13. By Lemma (5.2) the principal solution u at an endpoint, if it exists, is unique up to real constant multiplicative factors. Non-principal solutions are never unique, since if v is non-principal and u is principal, both at the same endpoint, then $v+cu$ is also a non-principal solution for any $c \in \mathbb{R}$. Simple examples show that the same solution may be principal at one endpoint and non-principal at the other. Clearly principal and non-principle solutions do not exist at an oscillatory endpoint.

REMARK 14. If the equation in (5.1) is regular at a then for any solution y, y and py' can be continuously extended to a and principal solutions u exist and satisfy the initial conditions : $u(a) = 0$, $(pu')(a) \neq 0$. Any non-principal solution v at a satisfies : $v(a) \neq 0$.

THEOREM 5.3. The equation (5.1) is non-oscillatory at a if and only if there exists a principal solution at a.

PROOF. See p. 547 in Niessen and Zettl [74]. □

The next result gives a characterization of principal and non-principal solutions. This will be used below in “regularizing ” singular LCNO endpoints.

THEOREM 5.4. Assume that (5.1) is non-oscillatory at a for some $\lambda \in \mathbb{R}$. Let u, v be real solutions of (5.1) satisfying $u(t) \neq 0$, $v(t) \neq 0$ for $t \in (a, d]$ and some $d \in J$. Then

1. u is a principal solution at a if and only if

$$\int_a^d \frac{1}{p\,u^2} = \infty; \tag{5.3}$$

2. v is a non-principal solution at a if and only if

$$\int_a^d \frac{1}{p\,v^2} < \infty; \tag{5.4}$$

3. if u is a principal solution and v is a non-principal solution at a , then there exists a $c \in \mathbb{R}$, $c \neq 0$, such that

$$u(t) = v(t) \int_a^t \frac{c}{p\,v^2},\ a < t \leq d, \tag{5.5}$$

4. and

$$|u(t)\,v(x)| < |u(x)\,v(t)|,\ for\ a < t < x \leq d. \tag{5.6}$$

PROOF. See p. 548 in [74]. □

5.3 Singular Boundary Conditions.

Boundary conditions of the form (4.3) do not make sense when one endpoint, say a, is singular since $Y(a)$ does not exist, in general. What takes their place ? This depends on the endpoint classification e.g. LP or LC. Before going into the details we need to set the stage first. Recall the definitions of the maximal domain Δ and the sesquilinear form $[\cdot,\cdot]$ from Subsection 3.6 of Section 3.

DEFINITION 5.5 (Maximal and Minimal Operators)**.** Let (5.1) hold and let Δ and M be defined as in
Subsection 3.6. Define

$$T_1 f = Mf, \ for\ f \in \Delta.$$

$$T_0' f = Mf, \ f \in \Delta, \ f \textit{ has compact support in } J.$$

Then T_1 is called the maximal operator of (M,w) (or of the equation (5.1)) on J, and the minimal operator T_0 of (M,w) is defined as the closure of T_0' .

LEMMA 5.6. The maximal and minimal domains are dense in the Hilbert space

$$H = L^2(J,w) = \{f : J \to \mathbb{C}, \int_J |f|^2 w < \infty\},$$

T_0 is a closed symmetric operator and $T_0^* = T$, $T^* = T_0$. Hence any self-adjoint extension of T_0 is also a self-adjoint restriction of T and conversely.

PROOF. See [72], [81]. □

Any self-adjoint extension S of the minimal operator T_0 satisfies

$$T_0 \subset S = S^* \subset T_1, \tag{5.7}$$

and can be determined by two point boundary conditions. These, however, are vacuous at an LP endpoint. To describe these conditions it is convenient to take cases depending on the LP/LC classification of the endpoints. Here LC/LP will mean that the left endpoint a is LC and the right endpoint b is LP, etc.

An operator S satisfying (5.7) is called a self-adjoint extension of T_0 on J, or a self-adjoint restriction of T_1 on J, or simply a self-adjoint realization of the equation (5.1) on J, or a self-adjoint realization of (M,w) on J.

THEOREM 5.7. Let (5.1) hold.

- Assume each endpoint is LP. Then the minimal operator T_0 is itself self-adjoint and has no proper self-adjoint extensions since $T_0 = S = S^* = T_1$. Thus there are no boundary conditions needed nor allowed in this case.
- Assume a is either R or LC and b is LP. Then there is no boundary condition needed or allowed at b and all the self-adjoint BC at a can be characterized as follows: Let $u, v \in \Delta$ be real-valued such that $[u,v](a) \neq 0$, and let

$$A_1[u,y](a) + A_2[v,y](a) = 0, \ A_1, A_2 \in \mathbb{R}, \ (A_1,A_2) \neq (0,0). \tag{5.8}$$

In other words, if S is the restriction of the maximal operator to functions $y \in \Delta$ satisfying (5.8) then S is a self-adjoint extension of T_0 (and a self-adjoint restriction of T_1), and, conversely, given any self-adjoint extension of T_0 its domain consists of all $y \in \Delta$ satisfying (5.8) for some such A_1, A_2.

- Assume a is LP and b is either R or LC. Then there is no boundary condition needed or allowed at a and all the self-adjoint BC at b can be characterized as follows: Let $u, v \in \Delta$ be real valued and satisfy $[u,v](b) \neq 0$ and let

$$B_1[u,y](b) + B_2[v,y](b) = 0,\ B_1, B_2 \in \mathbb{R},\ (B_1, B_2) \neq (0,0). \tag{5.9}$$

In other words, if S is the restriction of the maximal operator to functions $y \in \Delta$ satisfying (5.9) then S is a self-adjoint extension of T_0 (and a self-adjoint restriction of T_1), and, conversely, given any self-adjoint extension of T_0 its domain consists of all $y \in \Delta$ satisfying (5.9) for some B_1, B_2.

- Assume the left endpoint a is R or LC and the right endpoint b is also in the R or LC case. Then there are boundary conditions required at both endpoints in order to determine a self-adjoint extension of T_0. These may be separated (i.e. separate conditions at each endpoint) or coupled. Choose real valued $u, v \in \Delta$ satisfying $[u,v](a) \neq 0$ and $[u,v](b) \neq 0$. Then all self-adjoint extensions of T_0 are determined by BC which have the following form:

$$\begin{aligned} AY(a) + BY(b) &= 0,\ Y = \begin{pmatrix} [y,u] \\ [y,v] \end{pmatrix},\ A, B \in M_2(\mathbb{C}), \\ rank(A : B) &= 2,\ AEA^* = BEB^*,\ E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{5.10}$$

In other words, every BC (5.10) determines a self-adjoint extension of T_0 and every such extension is determined by (5.10) for some matrices A, B satisfying these conditions.

The existence of $Y(a)$, as a finite limit, follows from Green's formula and the hypothesis that each endpoint is either R or LC.

PROOF. This is well known; see [6], [61], [8], [81]. □

REMARK 15. Functions u, v needed in (5.10) can often, but not always, be obtained by choosing linearly independent real solutions of (5.1) for some particular real value of λ, e.g. $\lambda = 0$.

Below, an operator S satisfying (5.7) will be called a self-adjoint realization of the equation (5.1) or of (M, w) where the expression M is defined in Subsection 3.6.

Now that we know the self-adjoint realizations S of M we next discuss their spectrum $\sigma(S)$. Let $\sigma_d(S)$ denote the discrete spectrum i.e. the set of isolated eigenvalues, if any, of S. Set $\sigma_e = \sigma - \sigma_d$, then σ_e is called the essential spectrum; in some of the literature this is also referred to as the continuous part of the spectrum. Either one, but not both, of σ_d and σ_e may be empty.

Next we summarize some basic properties of the spectrum.

PROPOSITION 5.8. Let S be a self-adjoint realization of (M, w). Then

1. the spectrum of S is a closed subset of the reals $\mathbb{R}$ which is not bounded above; it may or may not be bounded below;
2. $\sigma_e(S) = \sigma_e(T_0)$, and so the essential spectrum of all self-adjoint extensions is the same; the essential spectrum is also invariant under an $L^1(a, b)$ perturbation of q, (See Hinton and Shaw [45]);
3. the operator T_0 is bounded below if and only if $\sigma(S)$ is bounded below for each self-adjoint realization S; (But not uniformly for all S);
4. Each one of the sets $\sigma(S), \sigma_e(S)$ is a closed subset of $\mathbb{R}$. Either one of these two sets, but not both, may be empty.

REMARK 16. We see from this Proposition that the essensial spectrum does not depend on the boundary conditions; thus $\sigma_e = \sigma_e(J, p, q, w)$. The eigenvalues do depend on the boundary condition. To the left of the continuous spectrum there may be no eigenvalues, a finite number of them, or an infinite number of them. In the case of an infinite number of eigenvalues to the left of $\sigma_0 = \inf \sigma_e$ these can have no accumulation point other than, possibly, σ_0 and $-\infty$. There also may be eigenvalues embedded in the continuous spectrum as well as in gaps of the continuous spectrum. The essential spectrum may have no gaps, a finite number of them, or an infinite number of them. A remarkable result of Hartmann [41] states that any closed set of real numbers which is not bounded above is the spectrum of a Sturm-Liouville operator ! See also Halvorsen [37]. In particular the spectrum of an SLP can be a Cantor-like set which is not bounded above.

For specific examples of SLP which illustrate some of these features (but not the Cantor-like behavior of the spectrum) the reader is referred to the examples discussed in Subsection 5.8 which come with the code SLEIGN2; these, along with the code, can be downloaded from the WWW using netscape or mosaic or lynx by specifying the URL :

ftp://ftp.math.niu.edu./pub/papers/Zettl/Sleign2

or by accessing the web page

http://www.math.niu.edu./~zettl/SL2/

5.4 The Friedrichs extension.

If there is one proper self-adjoint extension of the minimal operator then there are an infinite number of such extensions. Friedrichs singled out one of these which he called "ausgezeichnete" and which has come to be known as "the Friedrichs extension". He singled one out by giving a construction which constructed such an extension while preserving the lower bound. (However having the same lower bound does not characterize the Friedrichs extension since there are, in general, other self-adjoint extensions which also have the same lower bound.) To get the Friedrichs extension you must use the Friedrichs construction or some equivalent version of it. This construction works for any symmetric, densely defined, bounded below, operator in a Hilbert space. When it is applied to T_0 it makes no explicit use of boundary conditions. Thus the question arises : What boundary conditions determine the Friedrichs extension of the minimal operator T_0 ?

DEFINITION 5.9 (The Friedrichs Extension)**.** Suppose S is a densely defined symmetric (but not necessarily closed) operator in a Hilbert space H which is bounded below. Let $D(S_F)$ denote the set of all $y \in D(S^*)$ for which there exists a sequence $\{y_n : n \in N\}$ such that

1. $y_n \to y$ *in* H *as* $n \to \infty$,
2. $(S(y_n - y_m),\ y_n - y_m) \to 0$ *as* $n, m \to \infty$.

Define the operator S_F by

$$S_F\, y = S^* y,\ for\ y \in D(S_F).$$

Then S_F is called the Friedrichs extension of S. According to the well known result of Friedrichs [28], S_F is a self-adjoint operator with the same lower bound as S.

Friedrichs himself [29] addressed the question of which boundary condition determines the Friedrichs extension for regular SLP ? For regular SLP the answer in general is : the Dirichlet condition, see [73], [50], [10], [78]. We take up the singular case next.

THEOREM 5.10. If the minimal operator T_0 is bounded below with lower bound c then the equation (5.1) is NO at a and at b for any $\lambda < c$. Conversely, if the equation (5.1) is NO at a for some real λ_a and at b for some real λ_b, then the minimal operator T_0 is bounded below.

PROOF. See Niessen and Zettl [74] Corollary 2.1 and Theorem 4.2. □

THEOREM 5.11. If (M, w) is in the LC case at a and if $My = \lambda wy$ is NO at a for some real λ, then $My = \lambda w\, y$ is NO at a for every real λ.

PROOF. See Theorem 4.1 in Niessen and Zettl [74]. □

THEOREM 5.12. Let (5.1) hold. Let the maximal domain Δ, the Lagrange form $[\cdot,\cdot]$ and the expression M be defined as in Subsection 3.6. Assume that each endpoint is either regular or LCNO. Then the minimal operator is bounded below and thus has a Friedrichs extension S_F. Let u_a be a principal solution at a for some real λ_a and let u_b be a principal solution at b for some real λ_b. Then the domain $D(S_F)$ of S_F is given by

$$D(S_F) = \{y \in \Delta : [y, u_a](a) = 0 = [y, u_b](b)\}. \tag{5.11}$$

Note that (5.11) is independent of the principal solution chosen for λ_a and is independent of $\lambda_a \in \mathbb{R}$; similarly at b.

PROOF. See Niessen and Zettl [74], Theorems 4.2 and 4.3. □

REMARK 17. At a regular endpoint, say a, (5.11) reduces to $y(a) = 0$. Thus the BC (5.11) can be viewed as the singular analogues of the regular Dirichlet boundary conditions. It is also shown in [74] that the conditions (5.11) are equivalent to

$$\lim_{t \to a^+} \frac{y(t)}{v_a(t)} = 0 = \lim_{t \to b^-} \frac{y(t)}{v_b(t)},$$

where v_a, v_b are arbitrary non-principal solutions at a and b , respectively, for arbitrary real λ_a, λ_b.

5.5 R or LCNO/ R or LCNO.

We now study the case when each endpoint is either regular or LCNO. The properties of the eigenvalues and eigenfunctions in this case are similar to the regular case. In fact Niessen and Zettl [74] have shown that, given any SLP with endpoints which are either regular or LCNO there exists a regular SLP which has exactly the same spectrum as this singular problem and furthermore the eigenfunctions of the given singular problem $\{y_n : n \in N_0\}$ are related to the eigenfunctions $\{z_n : n \in N_0\}$ of the corresponding regular problem by the equation

$$y_n(t) = v(t)\, z_n(t), \; t \in (a, b), \; n \in N_0, \tag{5.12}$$

for some function v in the maximal domain of the singular problem which satisfies $v(t) > 0$ for $t \in (a, b)$. Since each z_n is a solution to a regular problem on (a, b) z and its quasi-derivative can be continuously extended to the endpoints by Theorem 3.4. Hence the singular behavior at each endpoint is contained in v. In particular, this shows that at each endpoint the singular (e.g. asymptotic) behavior of all eigenfunctions (in fact of all solutions for all real λ) is the same.

THEOREM 5.13. Let (5.1) hold and assume that each endpoint is either R or LCNO. Let S be a self-adjoint realization of (5.1).

1. Then the spectrum of S is discrete and bounded below. It consists of a countably infinite sequence $\{\lambda_n : n \in N_0\}$ of real eigenvalues tending to $+\infty$ which can be ordered to satisfy

$$-\infty < \lambda_0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots \to \infty. \tag{5.13}$$

   Here the eigenvalues are counted according to their multiplicity. Each eigenvalue can have multiplicity one, in which case it is called simple, or two, in which case it is called double. Therefore, in (5.13), equality cannot hold for more than two consecutive terms.

2. Let y_n be a real eigenfunction of λ_n. Then y_n has at least $n-1$ and at most $n+1$ zeros in (a,b). If $[y_n, u_a](a) = 0$, where u_a is a principal solution at a, then y_n has at most n zeros in (a,b).
3. Let $N(\lambda)$ denote the number of eigenvalues of S in the interval $(-\infty, \lambda]$. Then

$$\frac{N(\lambda)}{\sqrt{\lambda}} \to \frac{1}{\pi}\int_a^b \sqrt{\frac{w(t)}{p(t)}}dt < \infty, \; as \; \lambda \to \infty, \tag{5.14}$$

4. and

$$\frac{\lambda_n}{n^2\pi^2} \to \frac{1}{\left(\int_a^b \sqrt{\frac{w(t)}{p(t)}}dt\right)^2}, \; as \; n \to \infty. \tag{5.15}$$

The finiteness of the integral in (5.14) and (5.15) is a consequence of the assumption that each endpoint is R or LCNO.

PROOF. See Theorem 5.2 in [74]. □

THEOREM 5.14. Let (5.1) hold. Assume that each endpoint is regular or LCNO. Let u, v be real maximal domain functions such that u, v are principal and non-principal solutions at a for some real λ_a and at b for some real λ_b, respectively, normalized to satisfy : $[u,v](a) = 1 = [u,v](b)$. (Such u, v exist by Lemma 5.2 and Theorem 5.3). Let S with spectrum $\sigma(S)$ be the self-adjoint realization determined by the normalized separated boundary conditions

$$\cos(\alpha)\,[y,u](a) + \sin(\alpha)\;[y,v](a) = 0,\; 0 \le \alpha < \pi, \tag{5.16}$$

$$\cos(\beta)\,[y,u](b) + \sin(\beta)\,[y,v](b) = 0,\; 0 < \beta \le \pi. \tag{5.17}$$

Then

1. $\sigma(S) = \{\lambda_n : n \in N_0\}$ and each eigenvalue is simple.These eigenvalues can be ordered to satisfy

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots \to \infty; \tag{5.18}$$

2. if y_n is an eigenfunction corresponding to λ_n, then y_n has exactly n zeros in the open interval (a,b) for any $n \in N_0$;
3. $\lambda_n(\alpha, \beta)$ is a continuous function of α, β , $n \in N_0$;
4. $\lambda_n(\alpha, \beta)$ is strictly decreasing in α for each fixed β and strictly increasing in β for each fixed α, $n \in N_0$;

5. with the understanding that terms involving $\lambda_{-1}, \lambda_{-2}$ are not present we have the following inequalities for $\alpha \in [0,\pi),\ \beta \in (0,\pi]$, and $n \in N_0$

$$\lambda_{n-2}(0,\pi) < \left\{ \begin{array}{c} \lambda_{n-1}(0,\beta) \\ \lambda_{n-1}(\alpha,\pi) \end{array} \right\} < \lambda_n(\alpha,\beta) \leq \left\{ \begin{array}{c} \lambda_n(0,\beta) \\ \lambda_n(\alpha,\pi) \end{array} \right\} \leq \lambda_n(0,\pi). \tag{5.19}$$

PROOF. The regular case is known, see [81]. The singular case then follows from the regular case and the transformation employed in [74] to "regularize" singular LCNO endpoints. □

THEOREM 5.15 (Canonical Coupled BC)**.** Let (5.1) hold. Assume that each endpoint is regular or LCNO. Then the canonical form of all coupled self-adjoint boundary conditions is

$$Y(b) = e^{i\alpha}\, K\, Y(a) \tag{5.20}$$

where $-\pi \leq \alpha \leq \pi$,

$$\begin{aligned} Y &= \left(\begin{array}{c} [y,\theta] \\ [y,\varphi] \end{array} \right),\ \theta,\varphi \in \Delta,\ \theta,\varphi\ real,\ \ K \in SL_2(\mathbb{R}), \\ [\theta,\varphi](a) &= 1 = [\theta,\varphi](b);\ Y(a) = \lim_{t\to a^+} Y(t),\ \ Y(b) = \lim_{t\to b^-} Y(t). \end{aligned} \tag{5.21}$$

Fix p, q, w, a, b and let $\lambda_n(\alpha, K),\ n \in N_0$ denote the eigenvalues for BC (5.20); when $\alpha = 0$ this notation is abbreviated to $\lambda_n(K)$. Suppose that

$$k_{12} < 0\ or\ k_{12} = 0\ and\ k_{11} + k_{22} > 0,$$

Then

1. $\lambda_0(K)$ is simple;
2. $\lambda_0(K) < \lambda_0(-K)$;
3. the following inequalities hold for -$\pi < \alpha < 0$ and $0 < \alpha < \pi$:

$$\begin{aligned} -\infty &< \lambda_0(K) < \lambda_0(\alpha,K) < \lambda_0(-K) \leq \lambda_1(-K) < \lambda_1(\alpha,K) < \lambda_1(K) \\ &\leq \lambda_2(K) < \lambda_2(\alpha,K) < \lambda_2(-K) \leq \lambda_3(-K) < \dots \end{aligned} \tag{5.22}$$

Furthermore, for $0 < \alpha < \beta < \pi$ we have

$$\begin{aligned} \lambda_0(\alpha,K) &< \lambda_0(\beta,K) < \lambda_1(\beta,K) < \lambda_1(\alpha,K) < \lambda_2(\alpha,K) < \lambda_2(\beta,K) \\ &< \lambda_3(\beta,K) < \lambda_3(\alpha,K) < \dots \end{aligned} \tag{5.23}$$

Note that $K \in SL_2(\mathbb{R})$ implies $-K \in SL_2(\mathbb{R})$ and therefore if the hypothesis :

$$k_{12} < 0 \textit{ or } k_{12} = 0 \textit{ and } k_{11} + k_{22} > 0$$

fails to hold for K then it holds for $-K$.

PROOF. In Niessen and Zettl [74] it is shown for any singular problem with each endpoint either R or LCNO there exists a regular problem which has exactly the same eigenvalues. This result then follows from the regular case, see Theorem 4.13. The regular case was established by Eastham [18] using an argument similar to that in his book [19] for the periodic and semi-periodic cases; and by Bailey, Everitt and Zettl [8] using an argument similar to that used by Weidmann [81]. □

The next result characterizes the eigenvalues of singular SLP consisting of the canonical form (5.20) of the coupled self-adjoint boundary conditions *but for the general equation (5.1) with real or complex valued coefficients.*

THEOREM 5.16. Consider the SLP consisting of the equation

$$-(py')' + qy = \lambda wy \textit{ on } J = (a,b),\ -\infty \le a < b \le \infty, \tag{5.24}$$

where

$$p, q, w : J \to \mathbb{C}, 1/p, q, w \in L_{loc}(J), \tag{5.25}$$

with the boundary conditions (5.20), (5.21) and assume each endpoint is either R or LC. For each $\lambda \in \mathbb{C}$ determine unique solutions $u = u(\cdot,\lambda),\ v = v(\cdot,\lambda)$ by the 'singular initial conditions'

$$[u,\theta](a,\lambda) = 0,\ [u,\varphi](a,\lambda) = 1,\ [v,\theta](a,\lambda) = 1,\ [v,\varphi](a,\lambda) = 0. \tag{5.26}$$

Such solutions u, v exist by Theorem 3.13. Let $K \in SL_2(\mathbb{R})$. Then for any α, $-\pi \le \alpha \le \pi$, a number $\lambda \in \mathbb{C}$ is an eigenvalue of the BVP (5.26), (5.25), (5.20), (5.21), (5.24) if and only if

$$D(K,\lambda) = 2\cos(\alpha), \tag{5.27}$$

where for $\lambda \in \mathbb{C}$

$$\begin{aligned} D(K,\lambda) &= k_{11}[u(\cdot,\lambda),\varphi](b) + k_{22}[v(\cdot,\lambda),\theta](b) \\ &\quad -k_{12}[v(\cdot,\lambda),\varphi](b) - k_{21}[u(\cdot,\lambda),\theta](b). \end{aligned} \tag{5.28}$$

PROOF. See Theorem 3.1 in [8]. Although this Theorem is stated there only for the case when p, q, w are real valued and $w > 0$ the proof given there holds with no significant changes when p, q, w are complex valued. □

COROLLARY 5.17. For any $n \in N_0$ we have $\lambda_n(-\alpha, K) = \lambda_n(\alpha, K)$.

PROOF. This follows directly from (5.27) since $\cos(\alpha) = \cos(-\alpha)$. □

5.6 Behavior of eigenvalues near a singular boundary.

For this subsection we change the notation for the interval J from

$$J = (a, b)$$

to

$$J = (a', b').$$

The reason for this change of notation is that we wish to consider "approximations" of a singular SLP on an interval (a', b') by a sequence of regular SLP on truncated intervals (a_r, b_r) where

$$-\infty \leq a' < a_r < b_r < b' \leq \infty$$

and the sequence $\{a_r : r \in N\}$ converges decreasingly to a and the sequence $\{b_r : r \in N\}$ converges increasingly to b. By S and S_r we denote self-adjoint realizations of (M, w) on the intervals (a', b') and (a_r, b_r), respectively. Thus S and S_r are a self-adjoint operator in the Hilbert spaces $H = L^2((a', b'), w)$ and $H_r = L^2((a_r, b_r), w)$, respectively. Similarly the spectrum and eigenvalues are denoted by $\sigma(S)$, $\sigma(S_r)$, $\lambda_n(S)$, $\lambda_n(S_r)$, $n \in N_0$, $r \in N$. Below, the "inherited" operators S_r^i are defined by "inherited" boundary conditions. These play a special role in the approximation of the singular spectrum.

THEOREM 5.18. Let (5.1) hold. Let S, S_r be arbitrary self-adjoint realizations of (M, w) on the intervals (a', b') and (a_r, b_r), respectively, for $r \in N$.

For all endpoint classifications : If $\sigma(S)$ is not bounded below, then

$$\lambda_n(S_r) \to -\infty, \; as \; r \to \infty, \; for \; each \; n \in N_0. \tag{5.29}$$

PROOF. This is an as yet unpublished result of Everitt and Zettl [26]. □

DEFINITION 5.19 (Inherited BC and Operators)**.**

- Near an LP endpoint the inherited BC is the Dirichlet condition.
- Assume a' is LC and b' is LP. Then there is no singular boundary condition at b' and all singular self-adjoint BC at a' have the form

$$A_1[y,u](a') + A_2\,[y,v](a') = 0,\ A_1, A_2 \in \mathbb{R},\ (A_1, A_2) \neq (0,0), \tag{5.30}$$

  where u, v are real valued maximal domain functions satisfying $[u,v](a') = 1$.

  The inherited BC on (a_r, b_r) are obtained by replacing a' by a_r in (5.30) and by using the Dirichlet conditions $y(b_r) = 0$ at b_r. Note that, although u, v and their quasi-derivatives may not be defined at a' they are well defined at any point $a' < a_r < b_r < b'$.
- Assume b' is LC and a' is LP. Then there is no singular boundary condition at a' and all singular self-adjoint BC at b' have the form

$$B_1[y,u](b') + B_2\,[y,v](b') = 0,\ B_1, B_2 \in \mathbb{R},\ (B_1, B_2) \neq (0,0),$$

  where u, v are real valued maximal domain functions satisfying $[u,v](b') = 1$.

  The inherited BC on (a_r, b_r) are obtained by replacing b' by b_r in these conditions and by using the Dirichlet conditions $y(a_r) = 0$ at a_r. Note that, although u, v and their quasi-derivatives may not be defined at b' they are well defined at any point b_r, $a' < a_r < b_r < b'$.
- Each endpoint is either regular or LC. In this case we have self-adjoint realizations determined by both separated and coupled BC. Let the BC on (a', b') be determined by (5.10) but with our changed notation a' for a and b' for b. To obtain the inherited BC just replace a, b in (5.10) by (a_r, b_r). Note that although the same matrices A, B occur in the BC on (a', b') and on (a_r, b_r) when these inherited boundary conditions are written in the usual form (4.3) for regular BC the coefficient matrices, say $A = A(a_r), B = B(b_r)$ depend on values of u, v and their quasi-derivatives at a_r, b_r. In particular as the endpoint a_r or b_r is changed, e.g. in the code sleign2, the inherited boundary conditions change accordingly.
- The inherited operators and their spectral quantities are identified with the superscript i : S_r^i, $\lambda_n^i(a_r, b_r)$.

DEFINITION 5.20 (Start of the essential spectrum)**.** For any operator S let

$$\sigma_0 = \inf \sigma_e,\ -\infty \leq \sigma_0 \leq \infty, \tag{5.31}$$

where σ_e denotes the essential spectrum. Here $\sigma_0 = -\infty$ is interpreted to mean that the essential spectrum is not bounded below and $\sigma_0 = \infty$ means that the essential spectrum is empty i.e. the spectrum consists entirely of isolated eigenvalues.

THEOREM 5.21. Let S be a self-adjoint realization of (M, w) on (a', b'), let S_r^i be the inherited operator on (a_r, b_r). Assume that $\sigma(S)$ is bounded below and discrete. Let $\sigma(S) = \{\lambda_n(S) : n \in N_0\}, \sigma(S_r) = \{\lambda_n(S_r^i) : n \in N_0\}$. Then

$$\lambda_n(S_r^i) \to \lambda_n(S),\ as\ r \to \infty,\ for\ each\ n \in N_0. \tag{5.32}$$

PROOF. This is an as yet unpublished result of Everitt and Zettl [26]. □

REMARK 18. We comment on the contrast between (5.32) and (5.29). This markedly different behavior of the eigenvalues of regular problems shows the enormous influence that the spectrum of a singular problem has on the regular problems which are "close" to the singular one; in this case by virtue of the fact that the endpoints of the regular problem are close to the endpoints of the singular one. To understand the behavior of the eigenvalues of *regular* problems one needs a perspective which includes the *singular* case. This is even more interesting when viewed in the light of the asymptotic formula (5.15) for the eigenvalues on each fixed interval (a_r, b_r).

THEOREM 5.22. Assume each endpoint is either R or LC and at least one endpoint is O. Let S and S_r be as in Theorem 5.20. Then

$$\sigma(S) = \{\lambda_n : n \in \mathbb{Z}\}\ and\ \lambda_n(S_r^i) \to -\infty,\ as\ r \to \infty,\ n \in \mathbb{N}_0. \tag{5.33}$$

Nevertheless we have: Given any $\lambda_k \in \sigma(S)$ there exists an increasing (index) sequence of positive integers $n(r, k)$, depending on r and on λ_k such that

$$\lambda_{n(r,k)}(S_r^i) \to \lambda_k\ as\ r \to \infty. \tag{5.34}$$

PROOF. This is contained in Theorem 4.1 of [6]; see also Remark 1 (ii) on pages 15-16. □

In this case, i.e. when the eigenvalues are unbounded below as well as above, we follow the SLEIGN2 convention that λ_0 denotes the smallest nonnegative eigenvalue. This makes the indexing scheme unique.

THEOREM 5.23. Let S, S_r^i be as in Theorem 5.21. Assume that

$$-\infty < \sigma_0 < \infty. \tag{5.35}$$

(Hence at least one endpoint is LP.)

1. If S has no eigenvalue less than σ_0, then

(5.36) $$\lambda_n(S_r^i) \to \sigma_0, \;\; r \to \infty, \; n \in \mathbb{N}_0.$$

2. If S has exactly one eigenvalue below σ_0, say $\lambda_0 < \sigma_0$, then

(5.37) $$\lambda_0(S_r^i) \to \lambda_0, \; \textit{and} \; \lambda_n(S_r^i) \to \sigma_0, \; \textit{as} \; r \to \infty, \; n \geq 1.$$

3. If S has exactly two eigenvalues, say λ_0 and λ_1, below σ_0, then

(5.38) $$\lambda_0(S_r^i) \to \lambda_0, \; \lambda_1(S_r^i) \to \lambda_{1,} \; r \to \infty; \; \lambda_n(S_r^i) \to \sigma_0, \; r \to \infty, \; n \geq 2.$$

4. etc
5. If S has an infinite number of eigenvalues $\{\lambda_n : n \in \mathbb{N}_0\}$ to the left of σ_0, then

(5.39) $$\lambda_n(S_r^i) \to \lambda_n, \; r \to \infty, \; n \in \mathbb{N}_0.$$

REMARK 19. Theorem 5.23 can be used to detect the number of eigenvalues to the left of the essential spectrum. Using one of the numerical codes such as sleign2 or the Fulton and Pruess code sledge [30] one can ascertain which of (5.36), (5.37), (5.38) or (5.39) holds.

REMARK 20. The so called Coffee-Evans equation has attracted a good deal of attention in the literature because it has the interesting feature that, despite the asymptotic behavior (5.15), the lower eigenvalues occur in clusters of three which are close together. It is clear from Theorem 5.23 how to construct examples of regular problems with clusters of three million or three billion eigenvalues as close together as you please. This also shows that it easy to construct examples of *regular* SLP which will defeat any numerical code for the computation of eigenvalues. See Zettl [86] for some illustrations of this. (This paper can be downloaded from the author's web page following the instructions for downloading sleign2 given in Subsection 3 above.)

5.7 Approximating the spectrum of a given singular problem with eigenvalues of regular problems.

THEOREM 5.24. Let S be any self-adjoint realization of (M, w). The sequence of inherited operators $\{S_r^i : r \in \mathbb{N}\}$ is spectral included for S i.e. given any $\lambda \in \sigma(S)$ there exists an $n(r, \lambda) \in \mathbb{N}_0$ for each $r \in \mathbb{N}$ such that

$$\lambda_{n(r,\lambda)}(S_r^i) \to \lambda,\ as\ r \to \infty. \tag{5.40}$$

PROOF. This is contained in [6]. □

THEOREM 5.25. Assume that $\sigma(S)$ is bounded below. Then the sequence of inherited operators $\{S_r^i : r \in \mathbb{N}\}$ is spectral exact for S below $\sigma_0(S)$, i.e. it is spectral included and if the convergence (5.40) holds for some $\lambda < \sigma_0$, then $\lambda \in \sigma(S)$.

PROOF. This is contained in [6]. □

By Theorem 5.23 any point of the spectrum of a singular problem can, in principle, be approximated arbitrarily closely by eigenvalues from the inherited sequence of regular problems. In practice this isn't feasible since there are an uncountable number of points in the spectrum of the singular problem and finding an index sequence for each one is a hopeless task.

Nevertheless Theorems (5.23), (5.24), (5.25) together than be used to approximate the spectrum of many singular SLP quite effectively. This is done by approximating, not the individual points of the spectrum, but the spectral bands and gaps. It is remarkable that for so many singular problems the first few spectral bands and gaps - or the absence of gaps - can be detected and approximated from the distribution of a few thousand eigenvalues of the inherited problems which can be computed with sleign2 or sledge. For illustrations of this scheme see Zettl [88]. This paper can be downloaded as part of the SLEIGN2 package of files from the internet; see the instructions in Subsection 3 of this Section.

5.8 Examples.

Here we list a few examples to illustrate some of the concepts and results discussed above. These are taken from the the SLEIGN2 package. We follow the notation established above. We supplement the endpoint classifications given above with the weakly regular (WR) classification used by sleign2 : The endpoint a is WR if

$$1/p,\ q,\ w\ \in L(a,c),\ a < c < b,$$

but at least one of $1/p, q, w$ is not bounded in (a,c) for any $c,\ a < c < b$, or $w(a) = 0$.

1. The Legendre equation on $(-1,1)$.

$$p(t) = 1 - t^2,\ q(t) = 1/4,\ w(t) = 1,$$

$$u(t) = 1,\ v(t) = \frac{1}{2}\log(\frac{1+t}{1-t}),\ -1 < t < 1.$$

The maximal domain functions u, v are solutions for $\lambda = 0$, -1 is LCNO, 1 is LCNO. The BC

$$[y,u](-1) = -(py')(-1) = 0, \ [y,u](1) = -(py)'(1) = 0,$$

determines the Friedrichs extension whose eigenvalues are given by

$$\lambda_n = n(n+1), \ n \in N_0;$$

and the eigenfunctions are the classical Legendre polynomials.

The BC

$$[y,v](-1) = 0, \ [y,v](1) = 0,$$

is a separated non-Friedrichs BC. Analogues of the regular periodic and semi-periodic BC are

$$[y,u](-1) = [y,u](1), \ [y,v](-1) = [y,v](1);$$
$$[y,v](-1) = [y,v](1), \ [y,u](-1) = [y,u](1);$$

respectively.

2. The Liouville form of the Bessel equation on $(0,\infty)$.

$$p = 1, \ q(t) = \frac{\nu^2 - 1/4}{t^2}, \ w = 1.$$

LP at ∞ for all ν.

At 0 :

- LCNO for $-1 < \nu < 1$ but $\nu^2 \neq 1/4$
- R for $\nu^2 = 1/4$
- LP for $\nu^2 > 1/4$.

$$u(t) = t^{\nu+1/2}, \ v(t) = t^{-\nu+1/2}, \ for \ \nu \neq 0, -1/2, \ 1/2,$$
$$u(t) = t, \ v(t) = 1, \ for \ \nu = -1/2,$$
$$u(t) = t, \ v(t) = -1, \ for \ \nu = 1/2,$$
$$u(t) = \sqrt{t}, \ v(t) = \sqrt{t}\ \log(t), \ for \ \nu = 0.$$

For $\nu \geq 0$, u is the principal solution; thus $[y,u](0) = 0$ is the Friedrichs condition at 0; there is no BC at ∞ since this endpoint is LP. But for $-1/2 <$

$\nu < 0$ note that v is the principal solution and hence $[y,v](0) = 0$ is the Friedrichs BC.

3. The Halvorsen equation on $(0,\infty)$.

$$p(t) = 1,\ q(t) = \frac{e^{-2/t}}{t^4},\ w(t) = 1;$$

0 is WR; ∞ is LCNO. In this case the BC vector Y has the form $Y(0) = \begin{pmatrix} y(0) \\ (py')(0) \end{pmatrix}$ at 0 and $Y(\infty) = \begin{pmatrix} [y,u](\infty) \\ [y,v](\infty) \end{pmatrix}$ where $u(t) = 1,\ v(t) = t$.

4. The Boyd equation on $(-\infty, 0)$ and on $(0,\infty)$.

$$p = 1,\ q(t) = -1/t,\ w = 1$$

LP at $-\infty$ and ∞; LCNO at 0^+ and 0^-.

$$u(t) = t,\ v(t) = 1 - t(\log|t|)$$

Solutions can be given in terms of Whittaker functions, see [7]; the given u, v are maximal domain functions which are not solutions for any λ. This equation on the interval $(-1,1)$, *hence with an interior singularity at* 0, arose in a model in connection with the study of eddies in the atmosphere, see Boyd [13].

5. The regularized Boyd equation on $(-\infty, 0)$ and on $(0,\infty)$.

$$p = r^2,\ q = -r^2(\log|t|),\ w = r^2,$$

where

$$r(t) = e^{-(t\log|t|)-t}$$

LP at $-\infty$ and ∞; WR at 0^- and 0^+.

This is a WR form of example 4; the singularity at zero has been "regularized" using quasi-derivatives. There is a one-to-one correspondence between all self-adjoint BC of this example and example 4 considered as a "two interval" problem. Each self-adjoint realization of the Boyd equation on $(-\infty,\infty)$ is unitarily equivalent to a self-adjoint realization of the regularized Boyd equation and conversely. Thus these problems have the same spectrum. Also the eigenfunctions are closely related (but, of course not the same since one is singular and the other regular). For details see [4], [7], [21], [74].

6. The Sears -Titchmarsh equation on $(0,\infty)$.

$$p(t) = t,\ q(t) = -t,\ w(t) = 1/t$$

LP at 0; LCO at ∞.

This equation was studied by Titchmarsh [79] and by the two named authors .

For problems on $[1, \infty)$ the spectrum is discrete but unbounded above and below; for some numerical results see [7].

$$u(t) = \frac{\cos(t) + \sin(t)}{\sqrt{t}}, \; v(t) = \frac{\cos(t) - \sin(t)}{\sqrt{t}}$$

7. The BEZ equation on $(-\infty, 0)$ and on $(0, \infty)$.

$$p(t) = t, \; q(t) = 1/t, \; w(t) = 1;$$

LP at $-\infty$ and ∞; LCO at 0^- and 0^+. See example 5 in [7] for some numerical results.

$$u(t) = \cos(\log |t|), \; v(t) = \sin(\log |t|).$$

8. The Laplace tidal wave equation on $(0, \infty)$.

$$p(t) = 1/t, \; q(t) = \frac{k}{t^2} + \frac{k^2}{t}, \; w = 1, \; k \in \mathbb{R}, \; k \neq 0.$$

LCNO at 0 for all k; LP at ∞ for all k. This is only a very special case of the named equation, see Homer [48] for the general equation and references to the applied literature.

Even for this special case there are no representations of solutions in terms of the well known special functions. Thus to determine boundary conditions one must use maximal domain functions. Such functions are given by

$$u(t) = t^2, \; v(t) = t - 1/k.$$

9. The Latzko equation on $(0, 1)$.

$$p(t) = 1 - t^7, \; q = 0, \; w(t) = t^7,$$

WR at 0 (since $w(0) = 0$); LCNO at 1. The singularity at 1 requires the use of maximal domain functions such as

$$u(t) = 1, \; v(t) = -\log(1 - t)$$

to determine the boundary condition vsctor $Y(1)$. This example has a long and celebrated history; see Fichera [27].

10. A weakly regular equation on $(0, \infty)$.

$$p(t) = \sqrt{t},\ q = 0,\ w(t) = \frac{1}{\sqrt{t}}.$$

LP at ∞ and weakly regular at zero due to both p and w.

11. The Plum equation on $(-\infty, \infty)$.

$$p(t) = 1,\ q(t) = 100\cos^2(t),\ w(t) = 1$$

LP at $-\infty$ and ∞.

This is a form of the Mathieu equation. Since both endpoints are LP there are no boundary conditions needed or allowed; there is a unique self-adjoint realization of this equation which has no eigenvalues. It's spectrum consists of an infinite number of compact intervals, the spectral bands, separated by gaps. The first few spectral band are rather thin, the first has a width on the order of 10^{-4} .

The eigenvalues of this problem on the interval $(-b, b),\ 0 < b < \infty$, with Dirichlet boundary conditions

$$y(-b) = 0 = y(b),$$

tend to "bunch up" in the spectral bands of the whole line problem, particularly the fist band. See the comments in the last paragraph of Subsection 5.7. From Theorem 5.23 we know that

$$\lambda_n(b) \to \sigma_0 = \inf \sigma_e.$$

In particular, for any positive ε and any positive integer n the first n eigenvalues differ from each other by less than ε if b is sufficiently large, e.g. one can choose b large enough so that the first 7 million eigenvalues agree to the first 1000 digits. Given any numerical code for the computation of Sturm-Liouville eigenvalues, it can be defeated simply by choosing a large enough b. All this for such a "simple" regular SLP. The singular problem on $(-\infty, \infty)$ exerts a strong influence on the behavior of the eigenvalues of the regular problems on $(-b, b)$.

12. The Mathieu equation on $(-\infty, \infty)$ and on $(0, \infty)$.

$$p = 1,\ q(t) = \sin(t),\ w = 1.$$

LP at $-\infty$ and at ∞; R at 0.

This classical Mathieu equation has a celebrated history and voluminous literature. There are no eigenvalues for this problem on $(-\infty, \infty)$. On $(0, \infty)$ there may be one eigenvalue depending on the boundary condition at 0. The essential spectrum is the same for the whole line problem and the half line problem and consists of an infinite number of disjoint compact intervals separated by gaps. All the gaps are present. The endpoints of the spectral bands and gaps are periodic and semi-periodic eigenvalues of the problem on the interval $[0, 2\pi]$. These can be computed with sleign2.

The above remarks, with some appropriate modifications, apply to the general socalled Hill's equation. This is the SL equation with periodic coefficients p, q, w all of the with the same period.

Of special intereat is the starting point of the essential spectrum σ_0, which is finite in these cases for $p > 0,\ w > 0$. This point σ_0 is also the "oscillation number" of the equation; this means the equation is NO for $\lambda < \sigma_0$ and O for $\lambda > \sigma_0$. It may be O or NO for $\lambda = \sigma_0$. For the Mathieu equation considered here ($p = 1 = w,\ q(t) = \sin(t)$) σ_0 is not known explicitly but is approximately $\sigma_0 \sim -0.378$. This can be checked by computing the lowest eigenvalue of the periodic problem on $[0, 2\pi]$ i. e. $\sigma_0 = \lambda_0^P(0, 2\pi)$.

13. The hydrogen atom equation on $(0, \infty)$.

$$p(t) = 1,\ q(t) = \frac{k}{t} + \frac{h}{t^2},\ w(t) = 1,\ k, h \in \mathbb{R}.$$

This is the two parameter version of the classical one-dimensional equation for quantum theory modelling of the hydrogen atom; see [49], section 10 where most of the results reported on here can be found. A few of these results can be found in the commentary file xamples.tex of the package of files comprising the SLEIGN2 package. For all h, k there are no positive eigenvalues, ∞ is LP and the essential spectrum is $[0, \infty)$. If $k = 0$ the equation reduces to Bessel, see example #2 with $h = \nu^2 - 1/4$.

LP at ∞ for all h, k.

At 0 :

- R for $h = 0 = k$,
- LCNO for $h = 0$ and all $k \neq 0$
- LCNO for $-1/4 \leq h < 3/4$, but $h \neq 0$ and all k
- LCO for $h < -1/4$ and all k
- LP for $h \geq 3/4$ and all k.

Let

$$\rho = \sqrt{h + 1/4},\ for\ h \geq -1/4.$$

: (a) For $h \geq 3/4$ and $k \geq 0$ there is at most one negative eigenvalue and $\lambda = 0$ may be an eigenvalue; for $h \geq 3/4$ and $k < 0$ there are infinitely many negative eigenvalues given by

$$\lambda_n = \frac{-k^2}{(2n + 2\rho + 1)^2},\ n \in N_0,$$

and $\lambda = 0$ is not an eigenvalue.

: (b) For $h = 0,\ u(t) = t,\ v(t) = 1 + kt \log t$. For some computed eigenvalues see [7] and [49], section 10.

: (c) $-1/4 < h < 3/4,\ 0 < \rho < 1$ but $\rho \neq 1/2$. All the following results hold for the non-Friedrichs boundary condition : $[y, v](0) = 0$, where

$$u(t) = t^{\rho+1/2},\ v(t) = t^{1/2-\rho} + \frac{k}{1-2\rho} t^{3/2-\rho}.$$

(a) $k > 0,\ 0 < \rho < 1/2$ there are no negative eigenvalues

(b) $k > 0,\ 1/2 < \rho < 1$ there is exactly one negative eigenvalue given by

$$\lambda_0 = \frac{-k^2}{(2\rho - 1)^2}$$

(c) if $k < 0,\ 0 < \rho < 1/2$ there are infinitely many negative eigenvalues given by

$$\lambda_n = \frac{-k^2}{(2n - 2\rho + 1)^2},\ n \in N_0$$

(d) if $k < 0,\ 1/2 < \rho < 1$ there are infinitely many negative eigenvalues given by

$$\lambda_n = \frac{-k^2}{(2n - 2\rho + 3)^2},\ n \in N_0$$

The next few results refer to the BC

$$A_1[y, u](0) + A_2[y, v](0) = 0.$$

(e) if $k = 0$ and $A_1 A_2 < 0$ ther is exactly one negative eigenvalue given by :

$$\lambda_0 = -4 \left(\frac{-A_1 \Gamma(1+\rho)}{A_2 \Gamma(1-\rho)} \right)^{1/\rho}$$

(f) Note that for $h = -1/4,\ k \in \mathbb{R}$, the LCNO classification at 0 holds. Here

$$u(t) = \sqrt{t} + kt\sqrt{t},\ v(t) = 2\sqrt{t} + (\sqrt{t} + kt\sqrt{t}) \log(t).$$

For $k = 0$ and $A_1 A_2 < 0$ there is exactly one negative eigenvalue given by

$$\lambda_0 = c\, e^{2A_1/A_2},\ c = 4\, e^{4-2\gamma},\ \gamma = 0.5772156649...,$$

is Eulers constant.

(g) $h < -1/4,\ k \in \mathbb{R}$, the equation is LCO at 0. For $k = 0$ this equation reduces to the Krall equation, see example 20. For $k \neq 0$ explicit formulas for the eigenvalues are not available; some qualitative properties of the spectrum are :
for all k there are infinitely many negative eigenvalues going exponentially to $-\infty$
for $k > 0$ the point 0 is not an accumulation point of eigenvalues
for $k \leq 0$ the eigenvalues also accumulate at 0.

14. The Marletta equation on $(0, \infty)$.

$$p = 1,\ q(t) = \frac{3(t-31)}{4(t+1)(4+t)^2},\ w = 1.$$

This equation is R at 0 and LP at ∞. For some boundary conditions at 0 this equation deceives both codes sleign (not sleign2) and sledge into falsly reporting $\lambda = 0$ as an eigenvalue since there is a "near" eigenfunction there. For details see Marletta's certification report for the code sleign [66].

15. The harmonic oscillator equation on $(-\infty, \infty)$.

$$p = 1,\ q(t) = t^2,\ w = 1$$

LP at $-\infty$ and at ∞. Thus there is a unique self-adjoint extension; it has discrete spectrum given by

$$\lambda_n = 2n + 1,\ n \in N_0.$$

See [79] for a classical treatment.

16. The Jacobi equation on $(-1, 1)$.

$$p(t) = (1-t)^{\alpha+1}(1+t)^{\beta+1},\ q = 0,\ w(t) = (1-t)^{\alpha}(1+t)^{\beta}$$

At -1 for all α :
- LP for $\beta \leq -1$ and for $\beta \geq 1$
- WR for $-1 < \beta < 0$
- LCNO for $0 \leq \beta < 1$

At $+1$ for all β:

- LP for $\alpha \leq -1$ and for $\alpha \geq 1$
- WR for $-1 < \alpha < 0$
- LCNO for $0 \leq \alpha < 1$.

The boundary condition functions u, v can be taken as follows :

- For $t < 0$:
  - : (i) if $-1 < \beta < 0$ then $u(t) = (1+t)^{-\beta},\ v(t) = 1$
  - : (ii) if $\beta = 0$ then $u(t) = 1,\ v(t) = \log \frac{1+t}{1-t}$
  - : (iii) if $0 < \beta < 1$ then $u(t) = 1,\ v(t) = (1+t)^{-\beta}$
- For $t > 0$:
  - : (i) if $-1 < \alpha < 0$ then $u(t) = (1-t)^{-\alpha},\ v(t) = 1$
  - : (ii) if $\alpha = 0$ then $u(t) = 1,\ v(t) = \log \frac{1+t}{1-t}$
  - : (iii) if $0 < \alpha < 1$ then $u(t) = 1,\ v(t) = (1-t)^{-\alpha}$.

To get the classical Jacobi polynomials take $-1 < \alpha,\ -1 < \beta$; then note the following endpoint classifications and required boundary conditions:

At $+1$:
(a) $-1 < \alpha < 0,\ WR,\ (py')(1) = 0$
(b) $0 \leq \alpha < 1,\ LCNO,\ [y, u](1) = 0$
(c) $1 \leq \alpha,\ LP$

At -1:
(a) $-1 < \beta < 0,\ WR,\ (py')(-1) = 0$
(b) $0 \leq \beta < 1,\ LCNO,\ [y, u](-1) = 0$
(c) $1 \leq \beta,\ LP$.

For the classical Jacobi orthogonal polynomials the eigenvalues are given by:

$$\lambda_n = n(n + \alpha + \beta + 1),\ n \in N_0.$$

It is interesting to observe that the required boundary condition for the Jacobi polynomials is the Friedrichs condition in the LCNO case but not in the WR case.

17. The rotation Morse oscillator on $(0, \infty)$.

$$p = 1,\ w = 1,\ q(t) = \frac{2}{t^2} - 2000(2E - E^2),\ E = e^{-1.7(t-1.3)}.$$

LP at 0 and ∞. Hence there is a unique self-adjoint realization; its essential spectrum is $[0, \infty]$ and it has 26 negative eigenvalues. (Ask sleign2 to compute the first 28 eigenvalues and note the appearance of the 26 negative eigenvalues and the starting point of the essential spectrum at 0.)

18. The Dunsch equation on $(-1, 1)$. See Dunford and Schwartz [17] chapter VIII, pp. 1510-1520 for a discussion of this problem.

$$p(t) = 1 - t^2,\ q(t) = \frac{2\alpha^2}{1+t} + \frac{2\beta^2}{1-t},\ w = 1,\ 0 \leq \alpha,\ 0 \leq \beta.$$

At -1:

LP for $\alpha \geq 1/2$ and all β
LCNO for $0 \leq \alpha < 1/2$ and all β
At +1:
LP for $\beta \geq 1/2$ and all α
LCNO for $0 \leq \beta < 1/2$ and all α.
Boundary condition functions can be obtained as follows:

$$At\ -1: u_-(t) = (1+t)^{\alpha},\ v_-(t) = (1+t)^{-\alpha},$$

$$At\ +1: u_+(t) = (1+t)^{\beta},\ v_+(t) = (1+t)^{-\beta}.$$

Note that u, v are maximal domain functions but not solutions. In [17] on p. 1519 it is claimed that the boundary value problem determined by the boundary conditions

$$[y, u_-](-1) = 0 = [y, u_+](1)$$

has eigenvalues given by

$$\lambda_n = (n + \alpha + \beta + 1)\,(n + \alpha + \beta),\ n \in N_0.$$

19. The Donsch equation on $(-1, 1)$. This is a modification of example 18 which illustrates an LCNO/LCO mix. Replace α in 18 by $i\gamma$. This changes the singularity at -1 from LCNO to LCO. For $\gamma > 0$ and $0 < \beta < 1/2$ we have

$$At\ -1: u(t) = \cos(\gamma \log(1+t)),\ v(t) = \sin(\gamma \log(1+t))$$

$$At\ +1: u(t) = (1-t)^{\beta},\ v(t) = (1-t)^{-\beta}.$$

These u, v are maximal domain functions which are not solutions.

20. The Krall equation on $(0, \infty)$. This example is a special case of the Bessel equation, see example 2 above. Its solutions can be obtained in terms of modified Bessel functions. (We have followed the sleign2 package here by adding the constant 1 to q, this is done in sleign2 to facilitate numerical computations.)

$$p = 1,\ q(t) = 1 - \frac{k^2 + 1/4}{t^2},\ w = 1,\ k \in \mathbb{R},\ k \neq 0.$$

LCO at 0 and LP at ∞ for all $k \neq 0$. The essential spectrum is $[1, \infty)$.
For the boundary condition

$$[y, u](0) = 0,\ u(t) = t^{1/2} \cos(k\ \log(t))$$

there are an infinite number of eigenvalues which cluster at $-\infty$ and at 1. (Note that u need only be a maximal domain function on $[0, d)$ for $0 < d < \infty$. To get a maximal domain function on $(0, \infty)$ one can patch u at d appropriately.) The eigenvalues approach $-\infty$ and 1 very rapidly, see [7] or use sleign2 for more details; also see [60] for more information.

21. The Fourier equation. See Subsection 4.3 for a description of the eigenvalues for various boundary conditions, self-adjoint and otherwise.
22. The Laguerre equation on $(0, \infty)$.

$$p(t) = t^{\alpha+1}e^{-t},\ q = 0,\ w(t) = t^{\alpha}e^{-t},\ \alpha \in \mathbb{R}.$$

LP at ∞ for all α.

At 0 :

- LP for $\alpha \leq -1$
- WR for $-1 < \alpha < 0$
- LCNO for $0 \leq \alpha < 1$
- LP for $\alpha > 1$.

This is the classical form of the celebrated equation, which for parameter values $\alpha > 1$ produces the Laguerre polynomials as eigenfunctions; for the appropriate boundary at 0, when needed, the eigenvalues are given by

$$\lambda_n = n,\ n \in N_0.$$

Remarkably, these are independent of α, see Abramovitz and Stegun [1], chapter 22, section 22.6 for more details. See the file xamples.f (this is not a typo) of the sleign2 package for details of the boundary condition functions u, v.

The code sleign2 has only very limited success with this problem; for numerical computations the Laguerre/Liouville equation, which has the same eigenvalues (for the appropriate corresponding boundary conditions) is more convenient, see example 23 to follow.

23. The Laguerre/Liouville equation on $(0, \infty)$.

$$p = 1,\ w = 1,\ q(t) = \frac{\alpha^2 - 1/4}{t^2} - \frac{\alpha + 1}{2} + \frac{t^2}{16},\ \alpha \in \mathbb{R}.$$

LP at ∞ for all α
LCNO for $-1 < \alpha < 1$ but $\alpha^2 \neq 1/4$
R for $\alpha^2 = 1/4$
LP for $\alpha \geq 1$

See the xamples.f file of the sleign2 package for details of appropriate boundary condition functions.

24. Jacobi/Liouville form of the Jacobi equation. See the files xamples.f and xamples.tex of the sleign2 package for details.
25. The Meissner equation on $(-\infty, \infty)$.

$$p = 1,\ q = 0,\ w = \begin{cases} 1 & for \quad t < 0 \\ 9 & for \quad t \geq 0 \end{cases}$$

LP at $-\infty$ and ∞.

This equation is well known in the applied literature in connection with the modelling of crystals in one dimension, see [19], [47].

Periodic boundary conditions on $(-1/2, 1/2)$. We have $\lambda_0 = 0$ and

$$\begin{aligned} \lambda_{4n+1} &= (2n\pi + \alpha)^2,\ \lambda_{4n+2} = (2(n+1)\pi - \alpha)^2, \\ \lambda_{4n+3} &= \lambda_{4n+4} = (2(n+1)\pi)^2,\ \alpha = \cos^{-1}(\frac{-7}{8}),\ n \in N_0. \end{aligned}$$

Note that there are infinitely many simple and infinitely many double periodic eigenvalues on the interval $(-1/2, 1/2)$.

Semi-Periodic eigenvalues on $(-1/2, 1/2)$.

$$\begin{aligned} \lambda_{4n} &= (2n\pi + \beta)^2,\ \lambda_{4n+1} = (2n\pi + \gamma)^2,\ \lambda_{4n+2} = (2(n+1)\pi - \gamma)^2, \\ \lambda_{4n+3} &= (2(n+1)\pi - \beta)^2,\ \beta = \cos^{-1}(\frac{1+\sqrt{33}}{16}),\ \gamma = \cos^{-1}(\frac{1-\sqrt{33}}{16}). \end{aligned}$$

Observe that the semi-periodic eigenvalues are all simple.

26. The Lohner equation on $(-\infty, \infty)$.

$$p = 1,\ w = 1,\ q(t) = 1000t$$

LP at $-\infty$ and ∞. Lohner in [65] computed eigenvalues of this equation on a compact interval with regular boundary conditions using interval arithmetic. He obtained good approximations with rigorously guarenteed upper and lower bounds.

27. The Jörgens equation on $(-\infty, \infty)$. This is a remarkable example from Jörgens, see [49], part II, section 10.

$$p = 1,\ w = 1,\ q(t) = \frac{1}{4}\, e^{2t} - k\, e^{t},\ k \in \mathbb{R}.$$

LP at $-\infty$ and at ∞.

The essential spectrum starts at 0; for $k \leq 1/2$ there are no eigenvalues; for

$$h < k - 1/2 \leq h + 1,\ h = 0, 1, 2, 3, \ldots$$

there are exactly $h + 1$ eigenvalues and these are all below the essential spectrum i.e. they are all negative. They can be given explicitly by

$$\lambda_n = -(k - 1/2 - n)^2,\ n = 0, 1, 2, 3, \ldots, h,\ h \in N_0.$$

28. The Behnke-Goerisch equation on $(-\infty, \infty)$.

$$p = 1,\ w = 1,\ q(t) = k\cos^2(t),\ k \in \mathbb{R}.$$

This is a form of the Mathieu equation previously discussed, see examples 11 and 12 above. These authors computed Neumann eigenvalues on a compact interval using interval arithmetic and obtained good approximations with rigorous upper and lower bounds.

29. The Whittaker equation on $(0, \infty)$.

$$p = 1,\ q(t) = \frac{1}{4} + \frac{k^2 - 1}{4t^2},\ w(t) = \frac{1}{t},\ k\ \in N.$$

LP at 0 and ∞.

This equation is studied in [49], part II, section 10. The spectrum is discrete and can be given explicitly by

$$\lambda_n = n + \frac{k+1}{2},\ n \in N_0.$$

5.9 Comments.

Much of Section 5 is based on three papers : Bailey, Everitt, Weidmann and Zettl [6], Niessen and Zettl [74] and the pre-print of Everitt and Zettl [26].

1. Comments are made separately for each subsection.
2. This treatment of principal and non-principal solutions is based on Niessen and Zettl [74]. According to Hartman [42] these terms were coined by Leighton in [64]. The principal solution is the "small" solution but using a term such as small might lead to confusion since the same solution may be small at one endpoint but not the other.
3. The structure of singular limit circle boundary conditions is well known, see [81], [72], [2], [61] but it is not easy to find a clear and comprehensive treatment of them in the literature. We hope that this paper makes a positive contribution in this area. The conditions (5.10) characterize *all* self-adjoint boundary conditions for the case when each endpoint is either R or LC (either LCO or LCNO). The canonical form (5.16) and (5.17) represents *all separated* self-adjoint BC; (5.20) is the canonical form of *all coupled* self-adjoint BC, both for the case R or LC/ R or LC. Clearly (5.16) is a canonical form of (5.8) and (5.17) represents (5.9). For a different representation of singular self-adjoint BC see the treatment of Dunford and Schwartz [17].

4. The characterization of the Friedrichs extension given here is based on Niessen and Zettl [73]. This in turn is based on Rellich [77]. Also see Kaper, Kwong and Zettl [51], Kalf [50], [78]. For a characterization of the Friedrichs extension of powers of some special Sturm-Liouville operators see Baxley [10]. Besides this paper of Baxley this writer is aware of only one other result which characterizes the Friedrichs extension in terms of boundary conditions for singular problems of order higher than two. This is an unpublished paper of Zettl [88] for a very special class of problems. An illustrative example is, with $n > 1$,

$$(-1)^n y^{(2n)} \pm \frac{c}{t} y = \lambda\, y \text{ } on \text{ } 0 < t < 1,\ c \in \mathbb{R},\ c \neq 0.$$

   For the regular case very general results are known, see Niessen and Zettl [73], Möller and Zettl [71].
5. Möller [68] has shown that, for regular as well as singular SLP, if either p changes sign on the underlying interval then the spectrum is not bounded below. This holds even if there is no subinterval of the underlying interval on which p is negative. The corresponding result for very general higher order problems was established by Möller and Zettl in [70].

   Inequalities (5.19), (5.22), (5.23) can be found in Weidmann [81] for the regular case with

$$K = \begin{pmatrix} c & \\ & \frac{1}{c} \end{pmatrix},\ c \neq 0,\ c \in \mathbb{R}.$$

   For earlier work see Jörgens [49], Rellich [77]. These were extended to the singular case by Niessen and Zettl in [73], then further extended to more general K by Bailey, Everitt and Zettl [8], and finally Eastham established the general case [18].

   In [8] there is also the characterization of the eigenvalues given by (5.27) for the case when the expression M is symmetric and $w > 0$. It was only during the writing of these notes that the author realized that the proof given in [8] still holds, in the regular case, for complex valued p, q, w.

   Fore an extension of the asymptotic formula (5.15) to the case when p or w are allowed to change sign see Atkinson and Mingarelli [5]. Additional terms in this formula can be obtained if the coefficients satisfy some smoothness assumtions, see Harris [40], Hartman [42], and the references therein.
6. This Subsection is based on the unfinished paper of Everitt and Zettl [26] which in turn is based, to a considerable extent, on Bailey, Everitt, Weidmann and Zettl [6].
7. This Subsection is also based on [26].
8. These examples were taken from the sleign2 files: xamples.f, xamples.tex. See Subsection 5.3 for instructions on how to download these files from the internet.

5.10 Comments on some topics not covered.

1. The LP/LC dichotomy. See the monograph by Kauffman, Read and Zettl [53] for a brief introduction to LP and LC criteria. There are many sufficient conditions known for LC and also for LP and even some necessary and sufficient conditions but no necessary and sufficient conditions *which can be checked in each case.* Finding such conditions is still an open problem. One obstacle to finding such conditions is that for the LP case to hold it is enough to give conditions on a sequence of intervals, with almost no requirements on the coefficients outside these intervals, whereas essentially pointwise conditions are required for the LC case. For an entirely different approach see Zettl [85] for a construction of *all* LC expressions.
2. The O/NO alternative. See Kauffman, Read and Zettl [53] for a brief introduction to O/NO criteria, also see the monograph of Havorsen and Mingarelli [38] and its references. Many conditions, both necessary and sufficient, are known but there are no known necessary and suffcient conditions *which can be checked in each case.* As in the LC/LP situation the most general sufficient conditions for O are of "interval" type, see e.g. Kwong and Zettl [62], but such conditions are not appropriate for the NO case to hold. For a completely different approach i.e. a constuction of *all* disconjugate (NO) equations see Zettl [84].
3. Absolutely continuous spectrum. See the contributions to these Proceedings from Last and Jitomirskaya and from Stolz. We have only considered the simplest division of the spectrum into its discrete and essential (continuous) parts. See the seminal paper of Gilbert and Pearson [36] for criteria involving subordinate solutions - an extension of principal solutions - for the absolutely continuous spectrum; see also Hinton and Shaw [46], [44], [45], Gesztesy, Gurarie, Holden, Klaus, Sadun, Simon, and Vogl [33] and the references therein. There is an extensive literature on spectral properties of Sturm-Liouville operators by Simon and his 100 co-authors.
4. There is also a vast literature on so called left definite problems, when the weight function w is allowed to change sign. See the monograph by Mingarelli, and its references. Here the operator theory is studied in the setting of Krein and Pontryagen spaces rather than Hilbert space. For oscillatory properties of the eigenfunctions when the weight function changes sign or is identically zero on subintervals, see Everitt, Kwong and Zettl, [22].
5. Another popular topic we have not discussed is the two, or multi, parameter theory. See Volkmer [80] and the papers by Binding and Sleeman.
6. Multi interval theory. See the contribution of Everitt, Shubin, Stolz and Zettl to these Proceedings and the references therein for an introduction to SLP problems on infinitely many intervals. Also see Geszteszy and Kirsch [34], [35], [32]. If an equation has an interior singularity then, in general, the solutions cannot be continuously moved through this singularity. One can study this problem on two separate intervals each of wich has this singular point on the boundary. Take the direct sum of two self-adjoint operators from the two intervals and you have a two-interval self-adjoint operator which is not particularly interesting because it doesn't "connect" the two intervals together. More interesting operators are obtained by connecting solutions

through the singular point, even if they blow up there, in such a way as to get a new "two interval" self-adjoint operator. Actually this construction is also of interest when the interior point is not singular but regular, obtaining what is known as "point interactions". For an early contribution see Zettl [82].

7. Discreteness criteria. See the contribution of Read to these Proceedings and the references therin. Also see [63].
8. Inverse spectral theory. See the landmark paper of Gelfand and Levitan [31], the elegant exposition of Pöschel and Trubowitz [76] and the references therein.

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