COMPUTING CONTINUOUS SPECTRUM

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Abstract

The continuous, or essential, spectrum of singular Sturm-Liouville (S-L) problems can be approximated numerically just by computing eigenvalues of regular S-L problems with separated boundary conditions - provided one knows which regular problems to use. These have been identified by Bailey, Everitt, Weidmann and Zettl in [BEWZ]. Since there are highly effective codes, such as SLEIGN and SLEDGE, available for the computation of eigenvalues of regular S-L problems with separated boundary conditions this scheme works surprisingly well for a large class of problems.

The purpose of this note is to show that the results in [BEWZ] by Bailey, Everitt, Weidmann and Zettl, together with the software packages SLEIGN or SLEDGE can be used to approximate the essential spectrum of a large class of singular Sturm-Liouville (S-L) problems. Below we demonstrate this in particular for the unperturbed and perturbed Mathieu equation and for what we call the Plum equation. But it is to be emphasized that these examples were chosen solely because of their interest in the literature and because there are independent checks available.

Consider the S-L equation

$$-(py')' + qy = \lambda wy \tag{1}$$

Let $J = (a, b)$, $-\infty \le a < b \le \infty$, be a real interval, let p, q, w map (a, b) into $\mathbb{R}$, the reals, and assume that

$$\frac{1}{p},\ q, w \in L_{loc}(J),\ p > 0\ a.e.,\ w > 0\ a.e. \tag{2}$$

[1]This report is on joint work with P. B. Bailey and W. N. Everitt on a project partially funded by NSF under grant DMS-9106470.

Let $\{a_r;\ r \in N_0\}$, $\{b_r:\ r \in N_0\}$, $N_0 = 0, 1, 2, 3, \ldots$ be (endpoint) sequences such that

$$-\infty \le a < a_r < b_r < b \le \infty,\ a_r \to a,\ b_r \to b. \tag{3}$$

Conditions (2) ensure that equation (1) is regular on each interval $[a_r, b_r]$, $r \in N_0$ but the endpoints a, b may be singular.

If S_r is any self-adjoint realization of (1) on $[a_r, b_r]$ then it is well known [W] that the spectrum $\sigma(S_r)$ of S_r is discrete i.e. consists entirely of isolated eigenvalues and these can be ordered to satisfy

$$-\infty < \lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots,\ \lambda_n \to +\infty \text{ as } n \to \infty, \tag{4}$$

where

$$\sigma(S_r) = \{\lambda_n:\ n \in N_0\},\quad \lambda_n = \lambda_n(S_r). \tag{5}$$

If S is a self-adjoint realization of (1) on (a,b) then since one or both endpoints may be singular the spectrum $\sigma(S)$ of S is, in general, quite complicated. It may consist of a discrete and a continuous part. Let

$$\sigma(S) = \sigma_d(S) \cup \sigma_e(S), \tag{6}$$

where σ_d denotes the set of all isolated eigenvalues and σ_e the rest of the spectrum. Either one of σ_d or σ_e, but not both, may be empty. Let

$$\sigma_0 = inf.\ \sigma_e,\ -\infty \le \sigma_0 \le +\infty. \tag{7}$$

Here $\sigma_0 = +\infty$ means that $\sigma_e = \phi$.

Definition. Let S_r be a self-adjoint realization of (1) on $[a_r, b_r]$, $r \in N_0$; and let S be a self-adjoint realization of (1) on (a, b).

(i) The sequence $\{S_r:\ r \in N_0\}$ is spectral included for S if for any $\lambda \in \sigma(S)$ there exists a sequence of eigenvalues

$$\{\lambda_r \in \sigma(S_r):\ r \in N\} \text{ such that } \lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3, \ldots \text{ converges to } \lambda. \tag{8}$$

(ii) The sequence $\{S_r : r \in N_0\}$ is spectral exact for S if it is spectral included and, in addition, if $\lambda_r \in \sigma(S_r)$, $r \in N_0$ and the sequence $\{\lambda_r : r \in N_0\}$ converges to λ, implies that $\lambda \in \sigma(S)$.

Remark. Note that in this definition each operator S_r "lives" in a different space i.e. $H_r = L^2((a_r, b_r); w)$, no kind of convergence on the operator sequence $\{S_r : r \in N_0\}$ is assumed and, of course, this definition can be made for abstract operators S_r, S not just for S-L operators.

The endpoint a is said to be in the limit-circle case, LC for short, if all solutions of (1) are in $L^2((a, c); w)$ for some c, $a < c < b$. Similarly for the endpoint b. It is well known that the LC classifications is independent of λ. An endpoint is said to be in the limit-point or LP case if it is not LC.

Next we state the result from [BEWZ] upon which our computations are based. This theorem identifies which regular problems, or more accurately, which sequences of regular problems, approximate a given singular one.

Theorem 1. (Bailey, Everitt, Weidmann and Zettl, [BEWZ]). Assume each endpoint a, b is in the LP case. Let $\{a_r : r \in N_0\}$, $\{b_r : r \in N_0\}$ be endpoint sequences satisfying (3).

a) Let S_r be any self-adjoint realization of (1) on $[a_r, b_r]$, $r \in N_0$. (Note that S_r can be determined by any self-adjoint boundary conditions, separated or coupled, specified at a_r and b_r with possibly different boundary conditions for different r's.) Then the sequence $\{S_r : r \in N_0\}$ is spectral included for S but, in general, not spectral exact.

b) Suppose that S is bounded below and each S_r, is determined by Dirichlet boundary conditions:

$$y(a_r) = 0 = y(b_r), \quad r \in N_0. \tag{9}$$

Then

i) The sequence $\{S_r : r \in N_0\}$ is spectral exact for S below $\sigma_0(S)$. (This means that if (8) holds with $\lambda < \sigma_0(S)$ then $\lambda \in \sigma(S)$.)

ii) If S has no eigenvalue below $\sigma_0(S)$, then the sequence

$$\{\lambda_0(S_r);\ r \in N_0\} \text{ converges to } \sigma_0(S), \quad \text{as } r \to +\infty. \tag{10}$$

iii) If S has exactly k eigenvalues below $\sigma_0(S)$, $0 \leq k < \infty$, then

$$\{\lambda_k(S_r);\ r \in N_0\} \to \sigma_0(S), \quad r \to +\infty. \tag{11}$$

(Recall that the eigenvalues are indexed starting with 0 not 1.)

iv) The sequence $\{S_r :\ r \in N_0\}$ cannot be spectral exact for S above $\sigma_0(S)$ for arbitrary endpoint sequences $\{a_r :\ r \in N_0\}$ converging to a and $\{b_r :\ r \in N_0\}$ converging to b.

Proof. This proof is due to W.N.Everitt in a private communication. Parts (a) and (b) (i), (ii), (iii) follow from Theorems 6.1 and 6.2 in [BEWZ]. To prove (b) (iv) note that for any fixed $n \in N_0$, the n-th eigenvalue $\lambda_n(S_r)$ is a continuous function of the endpoints a_r, b_r. Now consider a problem such as the Mathieu equation where the spectrum of S consists of spectral bands separated by gaps. As an eigenvalue "moves" from the second spectral band into the first it must move through the first gap by continuity. Choose n_1 and r_1 such that $\lambda_n(S_{r_1})$ is in the first gap of $\sigma(S)$ and not an eigenvalue of S. Now move the endpoints a_r, b_r further out towards a, b and "watch" the eigenvalues moving through the first gap until one of them is exactly equal to $\lambda_{n_1}(S_{r_1})$. Then record the values of the endpoints a_{r_2}, b_{r_2} for which this happens. Repeating this process we can construct a sequence of endpoints a_{r_j}, b_{r_j} for which there exists a corresponding sequence of eigenvalues converging to $\lambda_{n_1}(S_{r_1})$ which is a point inside the first gap and which is not in $\sigma(S)$.

In [BEWZ] analogues of Theorem 1 for the other endpoint classifications LP/LC, LC/LP and LC/LC are also established. The LC/LC case is illustrated in [BEZ].

The LP/LC and LC/LP cases are similar although technically more complicated to the LP/LP case and hence we illustrate only the LP/LP case. The results in [BEWZ] require that "inherited" boundary conditions on the truncated intervals (a_r, b_r) be used. Near an LP endpoint a Dirichlet condition can be used but near an LC endpoint an "inherited" boundary condition is obtained from a "boundary condition function" which determines the singular boundary conditions of S. For details see [BEWZ].

THE MATHIEU EQUATION

$$-y'' + (\sin\ x)y = \lambda y \quad \text{on} \quad (-\infty, \infty) \ \& \ [0, 2\pi]$$

$$\left\{ \begin{array}{l} y(0) = y(2\pi) \\ y'(0) = y'(2\pi) \end{array} \right\} \quad -\infty < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \to +\infty$$

$$\left\{ \begin{array}{l} y(0) = -y(2\pi) \\ y'(0) = -y'(2\pi) \end{array} \right\} \quad -\infty < \mu_0 \leq \mu_1 \leq \mu_2 \leq \ldots \to \infty$$

SLEIGN2: These eigenvalues were computed by P.B. Bailey using SLEIGN2, a code under development by Bailey, Everitt and Zettl, partially funded by NSF grant: DMS - 9106470.

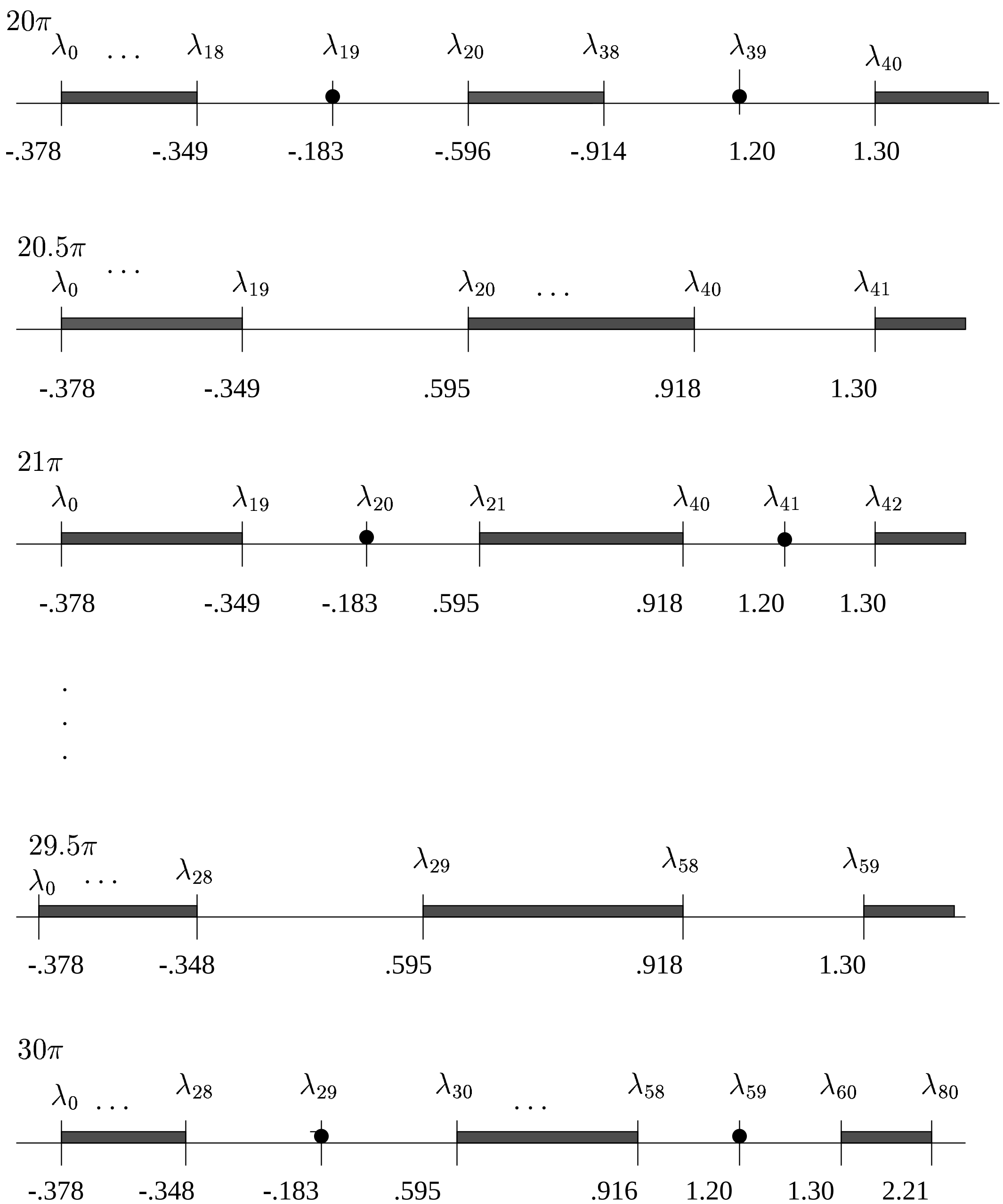
$$\lambda_0 = -0.378492; \quad \lambda_1 = 0.918058; \quad \lambda_2 = 1.29315; \quad \lambda_3 = 4.03192; \quad \lambda_4 = 4.053$$

$$\mu_0 = -0.34767; \quad \mu_1 = 0.594797; \mu_2 = 2.28515; \quad \mu_3 = 2.3425$$

$\sigma(-\infty, \infty)$:

	σ_1		g_1		σ_2		g_2		σ_3		g_3		σ_4		g_4	
λ_0		μ_0		μ_1		λ_1		λ_2		μ_2		μ_3		λ_3		λ_4
~ 1 :		.03		1.		.3		.4		1.		.05		.7		.02

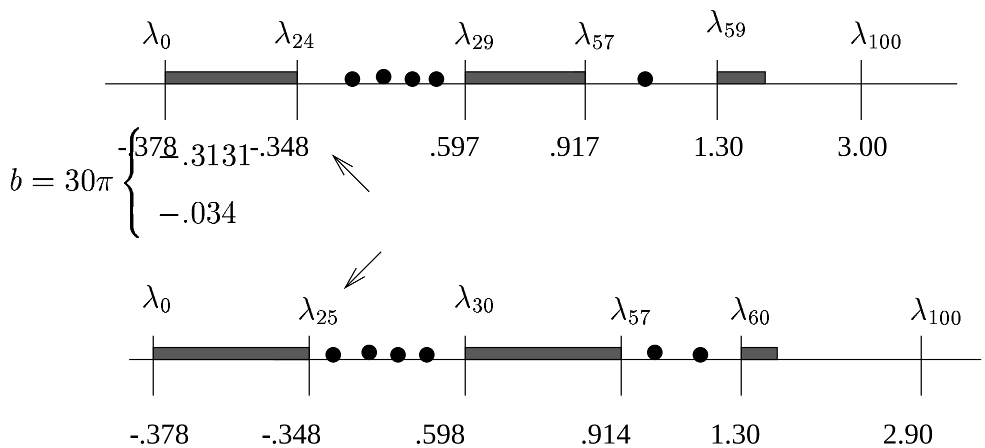
$b_r, \quad a_r = -b_r$



20π
λ_0 . . . λ_{18} λ_{19} λ_{20} λ_{38} λ_{39} λ_{40}
-.378 -.349 -.183 -.596 -.914 1.20 1.30
20.5π
λ_0 . . . λ_{19} λ_{20} . . . λ_{40} λ_{41}
-.378 -.349 .595 .918 1.30
21π
λ_0 λ_{19} λ_{20} λ_{21} λ_{40} λ_{41} λ_{42}
-.378 -.349 -.183 .595 .918 1.20 1.30
29.5π
λ_0 . . . λ_{28} λ_{29} λ_{58} λ_{59}
-.378 -.348 .595 .918 1.30
30π
λ_0 . . . λ_{28} λ_{29} λ_{30} . . . λ_{58} λ_{59} λ_{60} λ_{80}
-.378 -.348 -.183 .595 .916 1.20 1.30 2.21

$$-y'' + q(x)y = \lambda y \quad \text{on} \quad [-b, b], \quad q(x) = \sin(x) + \frac{1}{x^2 + 1}$$

$b = 29.5\pi$



Remark:

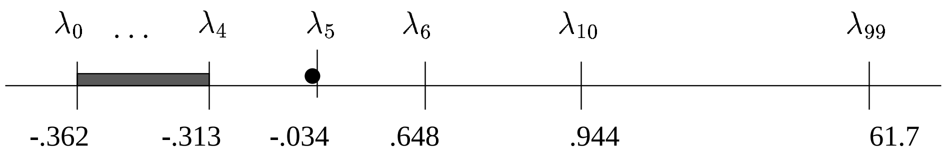
1. The numerical evidence suggests that there are no eigenvalues below the essential spectrum which starts at approximately $\sigma_0 \sim -.378$.

2. There are two eigenvalues, roughly located at -.3131 and at -.034 both in the first gap and both appearing (with the same values to three decimal places) for both BEWZ approximations on the intervals $[-29.5\pi, 29.5\pi]$ and on $[-30\pi, 30\pi]$. These seem to be further illustrations of the "trapping and cascading" phenomenon.

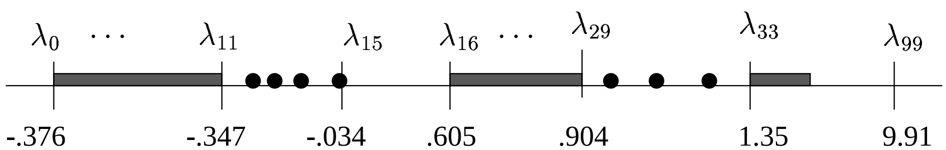
3. On the interval $[-30\pi, 30\pi]$ the first 100 eigenvalues of the Dirichlet problem are all less than 2.90 . Thus the asymptotic behavior has not yet begun to assert its influence.

$$-y'' + q(x)y = \lambda y, \quad [-b, b], \; q(x) = \sin\; x + \frac{1}{1+x^2}$$

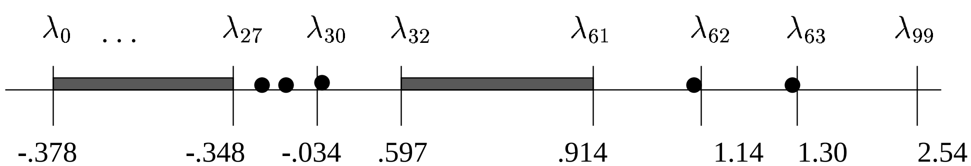
$b = 20$



$b = 50$

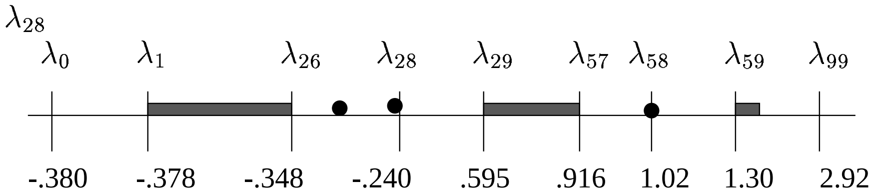


$b = 100$

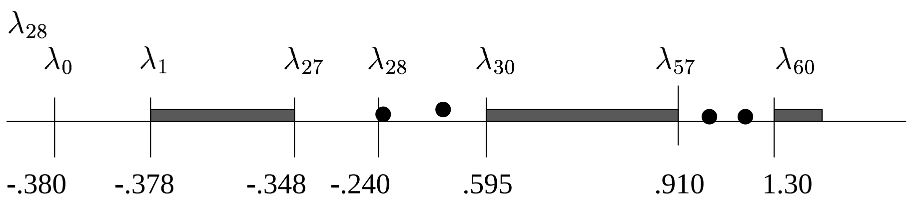


$$-y'' + q(x)y = \lambda y \ \text{ on } \ [-b, b], \quad y(-b) = 0 = y(b), \ q(x) = \sin(x + \frac{1}{1+x^2})$$

$b = 92.6770 \sim 2.95\pi$



$b = 94.2478 \sim 30\pi$



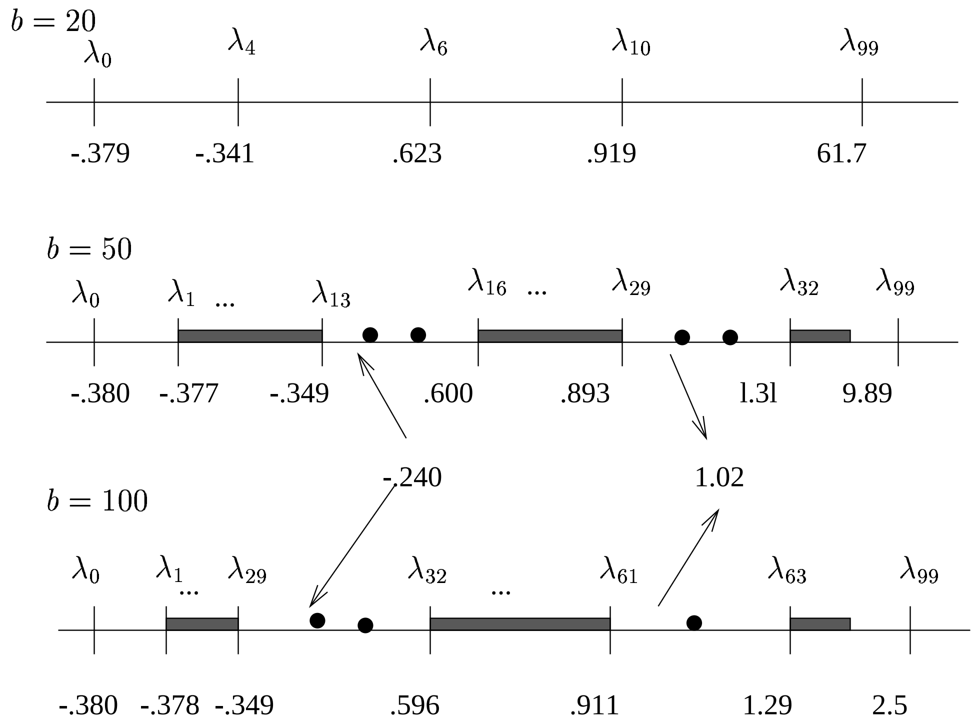
Remark.

1. There is strong numerical evidence for the existence of an eigenvalue near $-.380 < \sigma_0 \sim -.378$.

2. None of the gaps are "clear".

3. For many truncated intervals of the BEWZ approximations the eigenvalue near -.240 persists in the first gap. Is this an indication of the existence of an eigenvalue in the first gap of the singular problem or merely another illustration of "trapping and cascading"?

$$-y'' + q(x)y = \lambda y, \quad [-b,b], \quad q(x) = \sin(x + \tfrac{1}{1+x^2}), \quad y(-b) = 0 = y(b)$$



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Plum Problem

$$-y'' + 100\ \cos^2(x)\ y = \lambda y,\ \text{ on } [0,\ 3.14159265358979323846264 3]:$$

Let the periodic, semi-periodic and Dirichlet eigenvalues be denoted by $\lambda_n^P,\ \lambda_n^S,\ \lambda_n^D$, respectively. These were computed with SLEIGN2. The following inequalities are well known [W].

$$-\infty < \lambda_0^P < \lambda_0^S \le \lambda_0^D \le \lambda_1^S < \lambda_1^P \le \lambda_1^D \le \lambda_2^P < \lambda_2^S \le \lambda_2^D.$$

$$\lambda_0^P = 09.743220453458; \quad \lambda_1^P = 28.685139377765$$

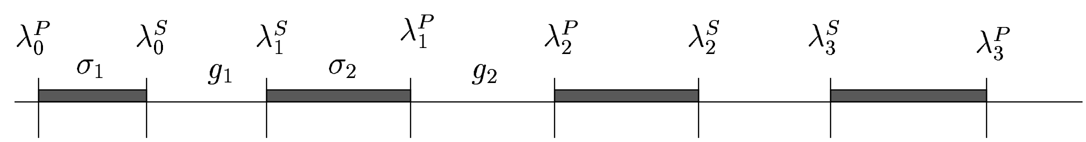
$$\lambda_0^S = 09.743221015341; \quad \lambda_1^S = 28.685100309350$$

$$\lambda_2^P = 46.477835272853; \quad \lambda_3^P = 62.986489952749$$

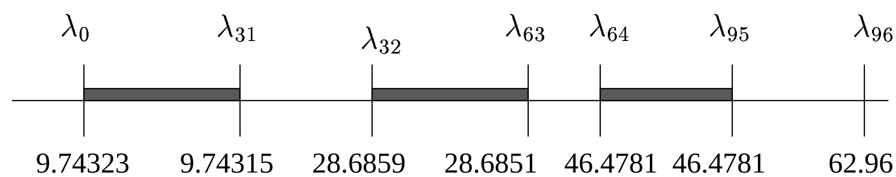
$$\lambda_2^S = 46.479058473390; \quad \lambda_3^S = 62.964079444332$$

$$\lambda_0^D = 9.74322; \qquad \lambda_1^D = 28.6851$$

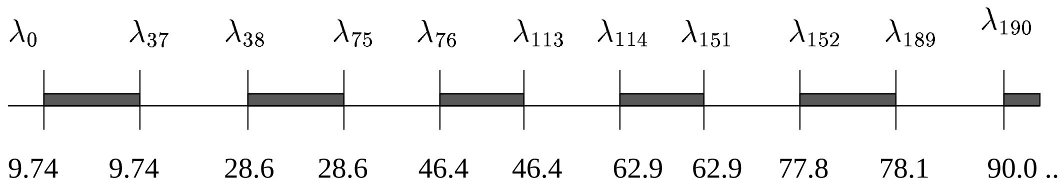
On $(-\infty, \infty)$:



BEWZ appr. on [-50,50] : y(-50) = 0 = y(50).



$y(-60) = 0 = y(60)$



$$\lambda_0^D = 09.7431; \quad \lambda_{38}^D = 28.6850$$
$$\lambda_{37}^D = 09.7432; \quad \lambda_{75}^D = 28.6854$$
$$\lambda_{114}^D = 62.9641; \quad \lambda_{76}^D = 46.4778$$
$$\lambda_{151}^D = 62.9865; \quad \lambda_{113}^D = 46.4791$$
$$\lambda_{152}^D = 77.8056; \quad \lambda_{190}^D = 90.0525$$
$$\lambda_{189}^D = 78.0623;$$

All gaps are clear for both intervals $[-b, b]$, $b = 50$, $b = 60$.

Comments and questions:

1. For the Mathieu equation: Is the sequence $\{S_r : r \in N_0\}$ spectral exact for S when $a_r = -b_r$ and $b_r = (r+1/2)\pi$ and not spectral exact when $a_r = -b_r$ and $b_r = r\pi$?

   Note that for all the cases we checked the first few gaps are clear when $a_r = -b_r$ and b_r is a half-integer multiple of π but not when b_r is an integer multiple of π.

2. Note the numerical detection of an eigenvalue $\lambda_0 \sim -.380$ below the essential spectrum $\sigma_0 \sim -.378$ for the potential $q(x) = \sin(x + \frac{1}{(x^2+1)})$. (Theorem 1 guarantees spectral exactness below σ_0.)

3. For the potential $q(x) = 100 \cos^2 x$ note the extremely thin first few bands and the extraordinary corresponding distribution of the eigenvalues of the BEWZ approximations. All this for the truncated intervals [-50,50] and [-60,60] approximating $(-\infty, \infty)$. It is also interesting that the first few gaps are clear in both cases.

   The first few bands appear to have a width $< 10^{-5}$. (Each band is known to have a positive width [W].)

   These eigenvalue distributions, remarkable as they are, seem even more remarkable in view of the fact that for each problem S_r the eigenvalues are asymptotically like n^2 i.e.

$$\lambda_n(S_r) \sim C_r\, n^2, \quad \text{as } n \to \infty. \tag{12}$$

   Of course $C_r \to 0$ as $r \to \infty$ but still it is interesting to note that the continuous spectrum of S is detected by the low eigenvalues before the asymptotic distribution (12) takes effect.

References

[BEWZ] P. B. Bailey, W. N. Everitt, J. Weidmann and A. Zettl, "Regular approximations of singular Sturm-Liouville problems", Results in Mathematics, v. 23 (1993), 3-22.

[BEZ] P. B. Bailey, W. N. Everitt and A. Zettl, "Computing eigenvalues of singular Sturm-Liouville problems", Results in Mathematics, v. 20 (1991), 391-423.

[W] J. Weidmann, "Spectral theory of ordinary differential operators", Lecture Notes in Mathematics '1258, Springer-Verlag, Heidelberg, 1987.