SLEIGN2
COMMENTARY ON THE INDIVIDUAL EXAMPLES IN XAMPLES.F

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1. Introduction

The examples in this commentary have been chosen to illustrate the capabilities and limitations of the program SLEIGN2. Many of the examples have been chosen from special cases of the well known and well studied "special functions" of mathematical analysis. All possible cases of endpoint classification are represented; all types of self-adjoint boundary conditions are included, *i.e.* regular or singular, and separated or coupled. In the limit-circle case examples are given for which the endpoints may be oscillatory or non-oscillatory.

For a general account of both the analytical and numerical properties of the SLEIGN2 code see the paper by Bailey, Everitt and Zettl [7].

For all 32 examples in this commentary the following data have been entered:

(i) the Sturm-Liouville differential equation and associated interval on the real line $\mathbb{R}$

(ii) the range of any parameters in the differential equation; this serves to remind the reader that numerical values for these parameters have to be entered in some of the examples given in xamples.x, for any such example to run

(iii) the endpoint classification of the differential equation, in the relevant Hilbert function space $L^2((a,b);w)$

(iv) the boundary condition functions u, v required for any LCNO or LCO endpoint

(v) comments on any particular features of the numbered example.

The data in items (i) to (iv) above can also be found in the file xamples.f, but this search requires scrolling through the file as the data items, for any particular example, are located in different sections.

For some of these examples it is possible to give explicit information on the spectrum of associated boundary value problems; this can take the form of providing explicit formulas for eigenvalues against which the program calculated results can be compared.

In all cases of limit-circle endpoints, boundary condition functions u and v have been entered as part of the example data. In the case of limit-circle non-oscillatory endpoints we use the convention that the boundary condition function u determines the principal or Friedrichs boundary condition.

On selecting a numbered example in the file xamples.x, the differential equation is displayed in Fortran, and details of the endpoint classification given. If information on the form of the boundary condition functions u and v is required then the user should scroll separately through the file xamples.f to the appropriate numbered part of the u, v section or refer to the information in this commentary.

Date: 01 March 2001 (File: xamples.tex) $\mathcal{A}_\mathcal{M}\mathcal{S}$-LaTeX; prepared in *Scientific Word*.

Some regular and weakly regular problems can be more successfully run using the limit-circle non-oscillatory (LCNO) algorithm; details are given below for some of the examples.

It should be noted that for limit-circle oscillatory problems it is sometimes difficult to compute numerically more than a few of the eigenvalues. This is due, at least in part, to the rapid growth of the eigenvalues in both the positive and the negative directions; but particularly in the negative direction.

The Laguerre problem, Example 22, has a discrete spectrum and for one particular boundary condition the eigenvalues are known explicitly, leading to the classical Laguerre orthogonal polynomials. In this case numerical values to confirm the details of the spectrum can also be obtained by use of the Liouville transformation; this leads to the Laguerre/Liouville Example 23, for which the program is successful over a wide range of boundary conditions.

The Liouville transformation has also been applied to the Jacobi equation, Example 16, to yield the Jacobi/Liouville Example 24.

The Liouville transformation is sometimes useful in other cases to put a Sturm-Liouville differential equation into a form more suitable for numerical computation; see in particular the Bessel Example 2.

Parameters. The reader is reminded that many of the examples involve the choice of one or more parameters; the range of these parameters is given when the numbered differential equation is displayed in xamples.x; if a choice of parameter is made outside of the stated range the program may abort.

2. Remarks on the individual examples.

(1) **Classical Legendre equation** (see [31, Chapter IV])

$$-\left(\left(1-x^2\right)y'(x)\right)' + \tfrac{1}{4}y(x) = \lambda y(x) \text{ for all } x \in (-1,+1).$$

Endpoint classification in $L^2(-1,+1)$:

Endpoint	Classification
-1	LCNO
$+1$	LCNO

For both endpoints the boundary condition functions u, v are given by (note that u and v are solutions of the Legendre equation for $\lambda = 1/4$)

$$u(x) = 1 \qquad v(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \text{ for all } x \in (-1,+1).$$

(i) The Legendre polynomials are obtained by taking the principal (Friedrichs) boundary condition at both endpoints ± 1 : enter $A1 = 1, A2 = 0,\ B1 = 1, B2 = 0$; *i.e.* take the boundary condition function u at ± 1; eigenvalues: $\lambda_n = (n+1/2)^2$; $n = 0, 1, 2, \cdots$; eigenfunctions: Legendre polynomials $P_n(x)$.

(ii) Enter $A1 = 0,\ A2 = 1,\ B1 = 0,\ B2 = 1$, *i.e.* use the boundary condition function v at ± 1; eigenvalues: μ_n; $n = 0, 1, 2, \cdots$ but no explicit formula is available; eigenfunctions are logarithmically unbounded at ± 1.

(iii) Observe that $\mu_n < \lambda_n < \mu_{n+1}$; $n = 0, 1, 2 \cdots$.

(2) **The Bessel equation** (see [31, Chapter IV])

$$-y''(x) + \left(\nu^2 - 1/4\right) x^{-2} y(x) = \lambda y(x) \text{ for all } x \in (0, +\infty)$$

with the parameter $\nu \in [0, +\infty)$. This is the Liouville form of the classical Bessel equation.

Endpoint classification in $L^2(0, +\infty)$:

Endpoint	Parameter ν	Classification
0	For $\nu = 1/2$	R
0	For all $\nu \in [0, 1)$ but $\nu \neq 1/2$	LCNO
0	For all $\nu \in [1, \infty)$	LP
$+\infty$	For all $\nu \in [0, \infty)$	LP

For endpoint 0 and $\nu \in (0, 1)$ but $\nu \neq 1/2$, the LCNO boundary condition functions u, v are determined by, for all $x \in (0, +\infty)$,

Parameter	u	v
$\nu \in (0, 1)$ but $\nu \neq 1/2$	$x^{\nu+1/2}$	$x^{-\nu+1/2}$
$\nu = 0$	$x^{1/2}$	$x^{1/2} \ln(x)$

(a) Problems on $(0, 1]$ with $y(1) = 0$:
For $0 \leq \nu < 1, \nu \neq \frac{1}{2}$: the Friedrichs case: $A1 = 1, A2 = 0$ yields the classical Fourier-Bessel series; here $\lambda_n = j_{\nu,n}^2$ where $\{j_{\nu,n} : n = 0, 1, 2, \ldots\}$ are the zeros (positive) of the Bessel function $J_\nu(\cdot)$.
For $\nu \geq 1$; LP at 0 so that there is a unique boundary value problem with $\lambda_n = j_{\nu,n}^2$ as before.

(b) Problems on $[1, \infty)$ all have continuous spectrum on $[0, \infty)$:
For Dirichlet and Neumann boundary conditions there are no eigenvalues.
For $A1 = A2 = 1$ at 1 there is one isolated negative eigenvalue.

(c) Problems on $(0, \infty)$ all have continuous spectrum on $[0, \infty)$:
For $\nu \geq 1$ there are no eigenvalues.
For $0 \leq \nu < 1$ the Friedrichs case is given by $A1 = 1, A2 = 0$; there are no eigenvalues.
For $\nu = 0.45$ and $A1 = 10, A2 = -1$ there is one isolated eigenvalue near to the value -175.57.

(3) **The Halvorsen equation**

$$-y''(x) = \lambda x^{-4} \exp(-2/x) y(x) \text{ for all } x \in (0, +\infty)$$

The endpoint classification in the weighted space $L^2((0, +\infty; x^{-4} \exp(-2/x))$:

Endpoint	Classification
0	WR
$+\infty$	LCNO

For the endpoints 0 and $+\infty$ in the WR and LCNO classification the boundary condition functions u, v are determined by

Endpoint	u	v
0	x	1
$+\infty$	1	x

Since this equation is WR at 0 and LCNO at $+\infty$ the spectrum is discrete and bounded below for all boundary conditions. However, this example illustrates that even a R or WR endpoint can cause difficulties for computation. The program fails on R at 0; is successful for WR at 0; is successful for LCNO at 0.

At 0, the principal boundary condition entry is $A1 = 1,\ A2 = 0$; at ∞ with $u(x) = 1,\ v(x) = x$ the principal boundary condition entry is also $A1 = 1,\ A2 = 0$, but note the interchange of the definitions of u and v at these two endpoints.

(4) **The Boyd equation**

$$-y''(x) - x^{-1}y(x) = \lambda y(x) \text{ for all } x \in (-\infty, 0) \cup (0, +\infty).$$

Endpoint classification in $L^2(-\infty, 0) \cup L^2(0, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$0-$	LCNO
$0+$	LCNO
$+\infty$	LP

For both endpoints $0-$ and $0+$

$$u(x) = x \qquad v(x) = x\ln(|x|) \text{ for all } x \in (-\infty, 0) \cup (0, +\infty).$$

This equation arises in a model studying eddies in the atmosphere; see [10]. There is no explicit formula for the eigenvalues of any particular boundary condition; eigenfunctions can be given in terms of Whittaker functions; see [5, Example 3].

(5) **The regularized Boyd equation**

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \text{ for all } x \in (-\infty, 0) \cup (0, +\infty)$$

where

$$p(x) = r(x)^2 \quad q(x) = -r(x)^2 \left(\ln(|x|)\right)^2 \quad w(x) = r(x)^2$$

with

$$r(x) = \exp\left(-(x\ln(|x|) - x)\right) \text{ for all } x \in (-\infty, 0) \cup (0, +\infty).$$

Endpoint classification in $L^2(-\infty, 0) \cup L^2(0, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$0-$	WR
$0+$	WR
$+\infty$	LP

This is a WR form of Example 4; the singularity at zero has been regularized using quasi-derivatives. There is a close relationship between the examples 4 and

5; in particular they have the same eigenvalues - see [2]. For a general discussion of regularization using non-principal solutions see [26]. For numerical results see [5, Example 3].

(6) **The Sears-Titchmarsh equation**

$$-(xy'(x))' - xy(x) = \lambda x^{-1} y(x) \text{ for all } x \in (0, +\infty).$$

Endpoint classification in $L^2(0, \infty)$:

Endpoint	Classification
0	LP
$+\infty$	LCO

For the endpoint $+\infty$

$$u(x) = x^{-1/2} (\cos(x) + \sin(x)) \quad v(x) = x^{-1/2} (\cos(x) - \sin(x)) \text{ for all } x \in (0, +\infty).$$

This differential equation has one LP and one LCO endpoint. For details of boundary value problems on $[1, \infty)$ see [5, Example 4]. The equation was studied originally in [31, Chapter IV]; but see [30].

For problems on $[1, \infty)$ the spectrum is simple and discrete but unbounded both above and below.

Numerical results are given in [5, Example 4].

(7) **The BEZ equation**

$$-(xy'(x))' - x^{-1} y(x) = \lambda y(x) \text{ for all } x \in (-\infty, 0) \cup (0, +\infty).$$

Endpoint classification in $L^2(-\infty, 0) \cup$ $L^2(0, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$0-$	LCO
$0+$	LCO
$+\infty$	LP

For both endpoints $0-$ and $0+$:

$$u(x) = \cos(\ln(|x|)) \qquad v(x) = \sin(\ln(|x|)) \text{ for all } x \in (-\infty, 0) \cup (0, +\infty).$$

This example is similar to the differential equation of Example 6. On the interval $(0, 1]$ there is a singularity at 0 in LCO; the equation is R at 1.

For numerical results see [5, Example 5].

(8) **The Laplace tidal wave equation**

$$-(x^{-1} y'(x))' + \left(kx^{-2} + k^2 x^{-1}\right) y(x) = \lambda y(x) \text{ for all } x \in (0, +\infty)$$

where the parameter $k \in (-\infty, 0) \cup (0, +\infty)$

Endpoint classification in $L^2(0, \infty)$:

Endpoint	Classification
0	LCNO
$+\infty$	LP

For the endpoint 0:

$$u(x) = x^2 \qquad v(x) = x - k^{-1} \text{ for all } (0, +\infty).$$

This equation is a particular case of the more general equation with this name; for details and references see [16].

There are no representations for solutions of this differential equation in terms of the well-known special functions. Thus to determine boundary conditions at the LCNO endpoint 0 use has to be made of maximal domain functions; see the u, v functions given above. Numerical results for some boundary value problems and certain values of the parameter k, are given in [5, Example 8].

(9) **The Latzko equation**

$$-((1 - x^7)y'(x))' = \lambda x^7 y(x) \text{ for all } x \in (0, 1].$$

Endpoint classification in $L^2(0, 1]$:

Endpoint	Classification
0	WR
1	LCNO

For the endpoint 1:

$$u(x) = 1 \qquad v(x) = -\ln(1 - x) \text{ for all } (0, 1).$$

This differential equation has a long and celebrated history; see [14, Pages 43 to 45]. There is a LCNO singularity at the endpoint 1 which requires the use of maximal domain functions; see the u, v functions given above. The endpoint 0 is WR due to the fact that $w(0) = 0$.

This example is similar in some respects to the Legendre equation of Example 1 above.

For numerical results see [5, Example 7].

(10) **A weakly regular equation**

$$-(x^{1/2}y'(x))' = \lambda x^{-1/2} y(x) \text{ for all } x \in (0, +\infty).$$

Endpoint classification in $L^2(0, 1]$:

Endpoint	Classification
0	WR
$+\infty$	LP

This is a devised example to illustrate the computational difficulties of weakly regular problems.

The differential equation gives $p(0) = 0$ and $w(0) = \infty$ but nevertheless 0 is a regular endpoint in the Lebesgue integral sense; however 0 has to be classified as weakly regular in the computational sense.

The Liouville normal form of this equation is the Fourier equation, see Example 21 below; thus numerical results for this WR problem can be checked against numerical results from (i) a R problem, (ii) the roots of trigonometrical equations, and (iii) a LCNO problem (see below).

There are explicit solutions of this equation given by

$$\cos(2x^{1/2}\sqrt{\lambda}) \;\; ; \; \sin(2x^{1/2}\sqrt{\lambda})/\sqrt{\lambda}.$$

If 0 is treated as a LCNO endpoint then $u,\ v$ boundary condition functions are

$$u(x) = 2x^{1/2} \qquad v(x) = 1.$$

The regular Dirichlet condition $y(0) = 0$ is equivalent to the singular condition $[y,u](0) = 0$. Similarly the regular Neumann condition $(py')(0) = 0$ is equivalent to the singular condition $[y,\ v](0) = 0$.

The following indicated boundary value problems have the given explicit formulae for the eigenvalues:

$$y(0) = 0 \text{ or } [y,\ u](0) = 0, \text{ and } y(1) = 0 \text{ gives}$$
$$\lambda_n = ((n+1)\pi)^2/4 \ (n = 0, 1, ...)$$

$$(py')(0) = 0 \text{ or } [y,v](0) = 0, \text{ and } (py')(1) = 0 \text{ gives}$$
$$\lambda_n = \left((n+\tfrac{1}{2})\pi\right)^2/4 \ (n = 0, 1, ...).$$

(11) **The Plum equation**

$$-(y'(x))' + 100\cos^2(x)y(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty).$$

Endpoint classification in $L^2(-\infty, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

Plum [28] computed the first seven eigenvalues for periodic eigenvalues on the interval $[0,\pi]$,*i.e.*

$$y(0) = y(\pi) \qquad y'(0) = y'(\pi),$$

using a numerical homotopy method together with interval arithmetic, and obtained rigorous bounds for these seven computed eigenvalues. In double precision the SLEIGN2 computed eigenvalues are in good agreement with these guaranteed bounds.

(12) **The Mathieu equation**

$$-y''(x) + 2k\cos(2x)y(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty)$$

where that parameter $k \in (-\infty, 0) \cup (0, +\infty)$.

Endpoint classification in $L^2(-\infty, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

The classical Mathieu equation has a celebrated history and voluminous literature. There are no eigenvalues for this problem on $(-\infty, +\infty)$. There may be one negative eigenvalue of the problem on $[0, \infty)$ depending on the boundary condition at the endpoint 0. The continuous (essential) spectrum is the same for the whole line or half-line problems and consists of an infinite number of disjoint closed intervals. The endpoints of these - and thus the spectrum of the problem - can be characterized in terms of periodic and semi-periodic eigenvalues of Sturm-Liouville problems on the compact interval $[0, 2\pi]$; these can be computed with SLEIGN2.

The above remarks also apply to the general Sturm-Liouville equation with periodic coefficients of the same period; the so-called Hill's equation.

Of special interest is the starting point of the continuous spectrum - this is also the oscillation number of the equation. For the Mathieu equation ($p = 1, q = \cos(x), w = 1$) on both the whole line and the half line it is approximately -0.378; this result may be obtained by computing the first eigenvalue λ_0 of the periodic problem on the interval $[0, 2\pi]$.

(13) **The hydrogen atom equation**

It is convenient to take this equation in two forms:

$$-y''(x) + (kx^{-1} + hx^{-2})y(x) = \lambda y(x) \text{ for all } x \in (0, +\infty) \tag{1}$$

where the two independent parameters $h \in [-1/4, +\infty)$ and $k \in \mathbb{R}$, and

$$-y''(x) + (kx^{-1} + hx^{-2} + 1)y(x) = \lambda y(x) \text{ for all } x \in (0, +\infty) \tag{2}$$

where the two independent parameters $h \in (-\infty, -1/4)$ and $k \in \mathbb{R}$.

Note that form (2) is introduced as a device to aid the numerical computations in the difficult LCO case; it forces the boundary value problem to have a non-negative eigenvalue.

Endpoint classification, for both forms (1) and (2), in $L^2(0, +\infty)$:

Endpoint	Form	Parameters	Classification
0	1	$h = k = 0$	R
0	1	$h = 0, k \in \mathbb{R} \setminus \{0\}$	LCNO
0	1	$-1/4 \le h < 3/4, h \neq 0, k \in \mathbb{R}$	LCNO
0	1	$h \ge 3/4, k \in \mathbb{R}$	LP
0	2	$h < -1/4, k \in \mathbb{R}$	LCO
$+\infty$	1 and 2	$h, k \in \mathbb{R}$	LP

This is the two parameter version of the classical one-dimensional equation for quantum modelling of the hydrogen atom; see [18, Section 10].

For form (1) and all h, k there are no positive eigenvalues; form (2) is best considered in the single LCO case when some eigenvalues are positive; in form (1) there is a continuous spectrum on $[0, \infty)$; in form (2) there is a continuous spectrum on $[1, \infty)$.

If $k = 0$ the equation reduces to Bessel, see Example 2 above with $h = \nu^2 - 1/4$.

Results for form (1)

In all cases below ρ is defined by

$$\rho := (h + 1/4)^{1/2} \text{ for } h \ge -1/4.$$

(a) For $h \ge 3/4$ and $k \ge 0$ no boundary conditions are required; there is at most one negative eigenvalue and $\lambda = 0$ may be an eigenvalue; for $h \ge 3/4$ and $k < 0$

there are infinitely many negative eigenvalues given by

$$\lambda_n = \frac{-k^2}{(2n+2\rho+1)^2},\ \rho = (h+1/4)^{1/2} > 0,\ n = 0,1,2,3,\ldots$$

and $\lambda = 0$ is not an eigenvalue.

(b) For $h = 0$ and $k \in \mathbb{R} \setminus \{0\}$ a boundary condition is required at 0 for which

$$u(x) = x \qquad v(x) = 1 + k\,x\ln(x).$$

For some computed eigenvalues see [5] and [18, Section 10].

(c) For $-1/4 < h < 3/4$, *i.e.* $0 < \rho < 1$, and $h \neq 0$, *i.e.* $\rho \neq 1/2$, then a boundary condition is required at 0 for which, for all $x \in (0,+\infty)$,

$$u(x) = x^{\frac{1}{2}+\rho} \qquad v(x) = x^{\frac{1}{2}-\rho} + \frac{k}{1-2\rho}x^{\frac{3}{2}-\rho};$$

the following results hold for the non-Friedrichs boundary condition $[y,v](0) = 0$, *i.e.* $A1 = 0, A2 = 1$:

(i) $k > 0,\ 0 < \rho < 1/2$ there are no negative eigenvalues

(ii) $k > 0,\ 1/2 < \rho < 1$ there is exactly one negative eigenvalue given by

$$\lambda_0 = \frac{-k^2}{(2\rho-1)^2}$$

(iii) if $k < 0,\ 0 < \rho < 1/2$ there are infinitely many negative eigenvalues given by

$$\lambda_n = \frac{-k^2}{(2n-2\rho+1)^2},\ n = 0,1,2,3,\ldots$$

(iv) if $k < 0,\ 1/2 < \rho < 1$ there are infinitely many negative eigenvalues given by

$$\lambda_n = \frac{-k^2}{(2n-2\rho+3)^2}\ ,\ n = 0,1,2,3,\ldots$$

(v) for $k = 0$ and $(A1)(A2) < 0$ there is exactly one negative eigenvalue given by:

$$\lambda_0 = \frac{4\,(A1)\,\Gamma(1+\rho)}{(A2)\,\Gamma(1-\rho)^{1/\rho}}.$$

(d) For $h = -1/4,\ k \in R$, the LCNO classification at 0 prevails and a boundary condition is required for which, for all $x \in (0,+\infty)$,

$$u(x) = x^{1/2} + kx^{3/2} \qquad v(x) = 2x^{1/2} + \left(x^{1/2} + kx^{3/2}\right)\ln(x).$$

For $k = 0$ and $(A1)(A2) < 0$ there is exactly one negative eigenvalue given by:

$$\lambda_0 = -c\exp(2(A1)/A2), \quad c = 4\exp(4-2\gamma)$$

where γ is Euler's constant: $\gamma = 0.5772156649\ldots$.

Results for form (2)

(e) For $h < -1/4,\ k \in R$, the equation is LCO at 0 (recall that we added 1 to the coefficient $q(\cdot)$ for this case, thus moving the start of the continuous spectrum from 0 to 1) for which, defining

$$\sigma := (-h - 1/4)^{1/2},$$

then, for all $x \in (0, +\infty)$,

$$u(x) = x^{1/2}\left[(1 - (4h)^{-1}kx)\cos(\sigma \ln(x)) + k\sigma x \sin(\sigma \ln(x))/2\right]$$
$$v(x) = x^{1/2}\left[(1 - (4h)^{-1}kx)\sin(\sigma \ln(x)) + k\sigma x \cos(\sigma \ln(x))/2\right];$$

(i) when $k = 0$ this equation reduces to the Krall equation Example 20 below (but note that the notation is different).

(ii) When $k \neq 0$ explicit formulas for the eigenvalues are not available; however we report here on the qualitative properties of the spectrum for any boundary condition at 0:
(α) for all $k \in R$ there are infinitely many negative eigenvalues tending exponentially to $-\infty$
(β) for $k > 0$ there are only a finite number of eigenvalues in any bounded interval, in particular they do not accumulate at 1
(γ) for $k \leq 0$ the eigenvalues accumulate also at 1.
(δ) for $k = 0$ and $(A1)(A2) < 0$ there is exactly one negative eigenvalue given by:

$$\lambda_0 = \frac{4\,(A1)\,\Gamma(1+\rho)}{(A2)\,\Gamma(1-\rho)^{1/\rho}}.$$

Most of these results are due to Jörgens, see [18, Section 10]; a few new results were established by the authors.

(14) **The Marletta equation**

$$-y''(x) + \frac{3(x-31)}{4(x+1)(x+4)^2}y(x) = \lambda y(x) \text{ for all } x \in [0, +\infty).$$

Endpoint classification in $L^2(0, +\infty)$:

Endpoint	Classification
0	R
$+\infty$	LP

Since $q(x) \to 0$ as $x \to \infty$ the continuous spectrum consists of $[0, \infty)$ and every negative number is an eigenvalue for some boundary condition at 0.

For the boundary condition $A1 = 5, A2 = 8$ there is a negative eigenvalue λ_0 near -1.185. However the equation with $\lambda = 0$ has a solution

$$y(x) = \frac{1 - x^2}{(1 + x/4)^{5/2}} \text{ for all } x \in [0, \infty)$$

that satisfies this boundary condition which is NOT in $L^2(0, \infty)$ but is "nearly" in this space. This solution deceives SLEIGN and SLEIGN2 in single precision, and SLEDGE in double precision, into reporting $\lambda = 0$ as a second eigenvalue; in double precision SLEIGN and SLEIGN2 correctly report that λ_0 is the only eigenvalue, and SLEIGN2 reports the start of the continuous spectrum at 0.

Additional details of this example are to be found in the Marletta certification report on SLEIGN (not SLEIGN2) [23].

(15) **The harmonic oscillator equation**

$$-y''(x) + x^2 y(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty).$$

Endpoint classification in $L^2(-\infty, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

This is another classical equation; it is also the Liouville normal form of the differential equation for the Hermite orthogonal polynomials. On the whole real line the boundary value problem requires no boundary conditions at the endpoints of $\pm\infty$. Thus there is a unique self-adjoint extension with discrete spectrum given by :

$$\{\lambda_n = 2n + 1;\ n = 0, 1, 2, ...\}.$$

For a classical treatment see [31, Chapter IV, Section 2].

(16) **The Jacobi equation**

$$-\left((1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x)\right)' = \lambda(1-x)^{\alpha}(1+x)^{\beta}y(x) \text{ for all } x \in (-1, +1)$$

where the parameters $\alpha, \beta \in (-\infty, +\infty)$.

Endpoint classification in the weighted space $L^2((-1, +1); (1-x)^{\alpha}(1+x)^{\beta}))$:

Endpoint	Parameter	Classification
-1	$\beta \leq -1$	LP
-1	$-1 < \beta < 0$	WR
-1	$0 \leq \beta < 1$	LCNO
-1	$1 \leq \beta$	LP

Endpoint	Parameter	Classification
$+1$	$\alpha \leq -1$	LP
$+1$	$-1 < \alpha < 0$	WR
$+1$	$0 \leq \alpha < 1$	LCNO
$+1$	$1 \leq \alpha$	LP

For the endpoint -1 and for the WR and LCNO cases the boundary condition functions u, v are determined by

Parameter	u	v
$-1 < \beta < 0$	$(1+x)^{-\beta}$	1
$\beta = 0$	1	$\ln\left(\frac{1+x}{1-x}\right)$
$0 < \beta < 1$	1	$(1+x)^{-\beta}$

For the endpoint $+1$ and for the WR and LCNO cases the boundary condition functions u, v are determined by

Parameter	u	v
$-1 < \alpha < 0$	$(1-x)^{-\alpha}$	1
$\alpha = 0$	1	$\ln\left(\frac{1+x}{1-x}\right)$
$0 < \alpha < 1$	1	$(1-x)^{-\alpha}$

To obtain the classical Jacobi orthogonal polynomials it is necessary to take $-1 < \alpha, \beta$; then note the required boundary conditions:

Endpoint -1:

Parameter	Boundary condition
$-1 < \beta < 0$	$(py')(-1) = 0$ or $[y, v](-1) = 0$
$0 \leq \beta < 1$	$[y, u](-1) = 0$

Endpoint $+1$:

Parameter	Boundary condition
$-1 < \alpha < 0$	$(py')(+1) = 0$ or $[y, v](+1) = 0$
$0 \leq \alpha < 1$	$[y, u](+1) = 0$

For the classical Jacobi orthogonal polynomials the eigenvalues are given by:

$$\lambda_n = n(n + \alpha + \beta + 1) \text{ for } n = 0, 1, 2, \ldots$$

and this explicit formula can be used to give an independent check on the accuracy of the results from the SLEIGN2 code.

It is interesting to note that the required boundary condition for these Jacobi polynomials is the Friedrichs condition in the LCNO case but not in the WR case.

(17) **The rotation Morse oscillator equation**

$$-y''(x) + (2x^{-2} - 2000(2e(x) - e(x)^2))y(x) = \lambda y(x) \text{ for all } x \in (0, +\infty)$$

where

$$e(x) = \exp(-1.7(x - 1.3)) \text{ for all } x \in (0, +\infty).$$

Endpoint classification in the space $L^2(0, +\infty)$

Endpoint	Classification
0	LP
$+\infty$	LP

This classical problem has continuous spectrum on $[0, \infty)$ and exactly 26 negative eigenvalues. Enter NUMEIG1 = 0, NUMEIG2 = 28 and observe the 26 eigenvalues and the start of the continuous spectrum at 0.

(18) **The Dunsch equation**

$$-\left((1 - x^2)y'(x)\right)' + \left(\frac{2\alpha^2}{(1 + x)} + \frac{2\beta^2}{(1 - x)}\right) y(x) = \lambda y(x) \text{ for all } x \in (-1, +1)$$

where the independent parameters $\alpha, \beta \in [0, +\infty)$.

Boundary value problems for this differential equation are discussed in [11, Chapter VIII, Pages 1515-20].

Endpoint classification in the space $L^2(-1, +1)$ for -1:

Parameter	Classification
$0 \leq \alpha < 1/2$	LCNO
$1/2 \leq \alpha$	LP

Endpoint classification in the space $L^2(-1,+1)$ for $+1$:

Parameter	Classification
$0 \leq \beta < 1/2$	LCNO
$1/2 \leq \beta$	LP

For the LCNO cases the boundary condition functions u, v are given by

Endpoint	Parameter	u	v
-1	$\alpha = 0$	1.0	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$
-1	$0 < \alpha < 1/2$	$(1+x)^{\alpha}$	$(1+x)^{-\alpha}$
$+1$	$\beta = 0$	1.0	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$
$+1$	$0 < \beta < 1/2$	$(1-x)^{\beta}$	$(1-x)^{-\beta}$

Note that these u and v are not solutions of the differential equation but maximal domain functions. In [11, Page 1519] it is stated that the boundary value problem determined by the boundary conditions

$$[y,u](-1) = 0 = [y,u](1)$$

has eigenvalues given by the explicit formula

$$\lambda_n = (n+\alpha+\beta+1)(n+\alpha+\beta) \text{ for } n = 0,1,2,\ldots$$

(19) **The Donsch equation**

$$-\left((1-x^2)y'(x)\right)' + \left(\frac{-2\gamma^2}{(1+x)} + \frac{2\beta^2}{(1-x)}\right) y(x) = \lambda y(x) \text{ for all } x \in (-1,+1)$$

where the independent parameters $\gamma, \beta \in [0,+\infty)$.

Endpoint classification in the space $L^2(-1,+1)$ for -1:

Parameter	Classification
$\gamma = 0$	LCNO
$0 < \gamma$	LCO

Endpoint classification in the space $L^2(-1,+1)$ for $+1$:

Parameter	Classification
$0 \leq \beta < 1/2$	LCNO
$1/2 \leq \beta$	LP

For these LCNO/LCO cases the boundary condition functions u, v are given by

Endpoint	Parameter	u	v
-1	$\gamma = 0$	1	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$
-1	$0 < \gamma$	$\cos(\gamma\ln(1+x))$	$\sin(\gamma\ln(1+x))$
$+1$	$\beta = 0$	1	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$
$+1$	$0 < \beta < 1/2$	$(1-x)^{\beta}$	$(1-x)^{-\beta}$

This is a modification of Example 18 above which illustrates an LCNO/LCO mix obtained by replacing α with $i\gamma$; this changes the singularity at -1 from LCNO to LCO.

Again these u and v are not solutions of the differential equation but maximal domain functions.

(20) **The Krall equation**

$$-y''(x) + (1 - (k^2 + 1/4)x^{-2})y(x) = \lambda y(x) \text{ for all } x \in (0, +\infty)$$

where the parameter $k \in (0, +\infty)$.

Endpoint classification in the space $L^2(0, +\infty)$:

Endpoint	Classification
0	LCO
$+\infty$	LP

This example should be seen as a special case of the Bessel Example 2 above; solutions can be obtained in terms of the modified Bessel functions.

To help with the computations for this example the spectrum is translated by a term $+1$; this simple device is used for numerical convenience.

For problems with separated boundary conditions at endpoints 0 and ∞ there is a continuous spectrum on $[1, \infty)$ with a discrete (and simple) spectrum on $(-\infty, 1)$. This discrete spectrum has cluster points at both $-\infty$ and 1.

For the LCO endpoint at 0 the boundary condition functions are given by

$$u(x) = x^{1/2}\cos(k\ln(x)) \qquad v(x) = x^{1/2}\sin(k\ln(x)).$$

For the boundary value problem with boundary condition $[y, u](0) = 0$ the eigenvalues are given explicitly by:

(i) suppose $\Gamma(1 + i) = \alpha + i\beta$ and $\mu > 0$ satisfies $\tan\left(\ln(\frac{1}{2}\mu)\right) = -\alpha/\beta$

(ii) $\theta = \mathrm{Im}(\log(\Gamma(1 + i)))$

(iii) $\ln(\frac{1}{2}\mu) = \frac{1}{2}\pi + \theta + s\pi$ for $s = 0, \pm 1, \pm 2, ...$

(iv) $\mu_s^2 = \left(2\exp(\theta + \frac{1}{2}\pi)\right)^2 \exp(2s\pi)$ $s = 0, \pm 1, \pm 2, ...$

then the eigenvalues are $\lambda_n = -\mu^2_{-(n+1)} + 1$ $(n = 0, \pm 1, \pm 2, ...)$.

SLEIGN2 can compute only six of these eigenvalues in a normal UNIX server, even in double precision, λ_{-3} to λ_2; other eigenvalues are, numerically, too close to 1 or too close to $-\infty$. Here we list these SLEIGN2 computed eigenvalues in double precision in a normal UNIX server and compare them with the same eigenvalues computed from the transcendental equation; for the problem on $(0, \infty)$ with $k = 1$ and $A1 = 1.0,\ A2 = 0.0$.

NUMEIG	eig from SLEIGN2	eig from trans. equ.	iflag
-3	$-276,562.5$	$-14,519,130$	4
-2	$-27,114.48$	$-27,114.67$	2
-1	-49.62697	-49.63318	2
0	0.9054452	0.9054454	1
1	0.9998234	0.9998234	1
2	0.9999997	0.9999997	3

.

(21) **The Fourier equation**

$$-y''(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty)$$

Endpoint classification in $L^2(-\infty, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

This is a simple constant coefficient equation whose eigenvalues, for any self-adjoint boundary condition, can be characterized in terms of a transcendental equation involving only trigonometric functions.

(22) **The Laguerre equation**

$$-(x^{\alpha+1}\exp(-x)y'(x))' = \lambda x^{\alpha}\exp(-x)y(x) \text{ for all } x \in (0, +\infty)$$

where the parameter $\alpha \in (-\infty, +\infty)$.

Endpoint classification in the weighted space $L^2((0, +\infty); x^{\alpha}\exp(-x))$:

Endpoint	Parameter	Classification
0	$\alpha \leq -1$	LP
0	$-1 < \alpha < 0$	WR
0	$0 \leq \alpha < 1$	LCNO
0	$1 \leq \alpha$	LP
$+\infty$	$\alpha \in (-\infty, +\infty)$	LP

For these WR/LCNO cases the boundary condition functions u, v are given by:

Endpoint	Parameter	u	v
0	$-1 < \alpha < 0$	$x^{-\alpha}$	1
0	$\alpha = 0$	1	$\ln(x)$
0	$0 < \alpha < 1$	1	$x^{-\alpha}$

This is the classical form of the differential equation which for parameter $\alpha > -1$ produces the classical Laguerre polynomials as eigenfunctions; for the boundary condition $[y, 1](0) = 0$ at 0, when required, the eigenvalues are then (remarkably!) independent of α and given by $\lambda_n = n$ $(n = 0, 1, 2, ...)$; see [1, Chapter 22, Section 22.6].

SLEIGN2 does not compute eigenvalues well with this differential equation on $(0, \infty)$, with the code in a UNIX server; this appears to be due to numerical problems resulting from the exponentially small coefficients; however, see Example 23 below.

(23) **The Laguerre/Liouville equation**

$$-y''(x) + \left(\frac{\alpha^2 - 1/4}{x^2} - \frac{\alpha + 1}{2} + \frac{x^2}{16}\right) y(x) = \lambda y(x) \text{ for all } x \in (0, +\infty)$$

where the parameter $\alpha \in (-\infty, +\infty)$.

Endpoint classification in the space $L^2(0,+\infty)$:

Endpoint	Parameter	Classification
0	$\alpha \leq -1$	LP
0	$-1 < \alpha < 1$, but $\alpha^2 \neq 1/4$	LCNO
0	$\alpha^2 = 1/4$	R
0	$1 \leq \alpha$	LP
$+\infty$	$\alpha \in (-\infty,+\infty)$	LP

For these WR/LCNO cases the boundary condition functions u, v are given by:

Endpoint	Parameter	u	v
0	$-1 < \alpha < 0$ but $\alpha \neq -1/2$	$x^{\frac{1}{2}-\alpha}$	$x^{\frac{1}{2}+\alpha}$
0	$\alpha = -1/2$	x	1
0	$\alpha = 0$	$x^{1/2}$	$x^{1/2}\ln(x)$
0	$0 < \alpha < 1$ but $\alpha \neq 1/2$	$x^{\frac{1}{2}+\alpha}$	$x^{\frac{1}{2}-\alpha}$
0	$\alpha = 1/2$	x	1

This is the Liouville normal form of the Laguerre equation; the two forms are unitarily equivalent so that the spectrum and the eigenfunctions of equivalent boundary value problems are identical. This Liouville form is more suitable for eigenvalue computations in contrast to the previous example.

The Laguerre polynomials are produced as eigenfunctions only when $\alpha > -1$. For $\alpha \geq 1$ the LP condition holds at 0. For $0 \leq \alpha < 1$ the appropriate boundary condition is the Friedrichs condition: $[y,u](0) = 0$; for $-1 < \alpha < 0$ use the non-Friedrichs condition: $[y,\ v](0) = 0$. In all these cases $\lambda_n = n$ for $n = 0, 1, 2, ...$

(24) **The Jacobi/Liouville equation**

$$-y''(x) + q(x)y(x) = \lambda y(x) \text{ for all } x \in (-\pi/2,+\pi/2)$$

where the coefficient q is given by, for all $x \in (-\pi/2,+\pi/2)$,

$$q(x) = \frac{\beta^2 - 1/4}{4\tan^2((x+\pi)/2)} + \frac{\alpha^2 - 1/4}{4\tan^2((x-\pi)/2)} - \frac{4\alpha\beta + 4\beta + 4\alpha + 3}{8}.$$

Here the parameters $\alpha, \beta \in (-\infty+,\infty)$.

Endpoint classification in the space $L^2(-\pi/2,+\pi/2)$:

Endpoint	Parameter	Classification
$-\pi/2$	$\beta \leq -1$	LP
$-\pi/2$	$-1 < \beta < 1$ but $\beta^2 \neq 1/4$	LCNO
$-\pi/2$	$\beta^2 = 1/4$	R
$-\pi/2$	$1 \leq \beta$	LP

Endpoint	Parameter	Classification
$+\pi/2$	$\alpha \leq -1$	LP
$+\pi/2$	$-1 < \alpha < 1$ but $\alpha^2 \neq 1/4$	LCNO
$+\pi/2$	$\alpha^2 = 1/4$	R
$+\pi/2$	$1 \leq \alpha$	LP

For the endpoint $-\pi/2$ and for LCNO cases the boundary condition functions u, v are determined by, here $b(x) = 2\tan^{-1}(1) + x$ for all $x \in (-\pi/2, +\pi/2)$,

Parameter	u	v
$-1 < \beta < 0$	$b(x)^{\frac{1}{2}-\beta}$	$b(x)^{\frac{1}{2}+\beta}$
$\beta = 0$	$\sqrt{b(x)}$	$\sqrt{b(x)}\ln(b(x))$
$0 < \beta < 1$	$b(x)^{\frac{1}{2}+\beta}$	$b(x)^{\frac{1}{2}-\beta}$

For the endpoint $+\pi/2$ and for LCNO cases the boundary condition functions u, v are determined by, here $a(x) = 2\tan^{-1}(1) - x$ for all $x \in (-\pi/2, +\pi/2)$,

Parameter	u	v
$-1 < \alpha < 0$	$a(x)^{\frac{1}{2}-\alpha}$	$a(x)^{\frac{1}{2}+\alpha}$
$\alpha = 0$	$\sqrt{a(x)}$	$\sqrt{a(x)}\ln(a(x))$
$0 < \alpha < 1$	$a(x)^{\frac{1}{2}+\alpha}$	$a(x)^{\frac{1}{2}-\alpha}$

This is the Liouville normal form of the Jacobi equation of Example 16.

The classical Jacobi orthogonal polynomials are produced only when both $\alpha, \beta > -1$. For $\alpha, \beta > +1$ the LP condition holds and no boundary condition is required to give the polynomials. If $-1 < \alpha, \beta < 1$ then the LCNO condition holds and boundary conditions are required to produce the Jacobi polynomials; these conditions are as follows:

Endpoint $-\pi/2$

Parameter	Boundary condition
$-1 < \beta < 0$	$[y, v](-\pi/2) = 0$
$0 \leq \beta < 1$	$[y, u](-\pi/2) = 0$

Endpoint $+\pi/2$

Parameter	Boundary condition
$-1 < \alpha < 0$	$[y, v](+\pi/2) = 0$
$0 \leq \alpha < 1$	$[y, u](+\pi/2) = 0$

Recall from Example 16 for the classical orthogonal Jacobi polynomials the eigenvalues are given explicitly by:

$$\lambda_n = n(n + \alpha + \beta + 1) \text{ for } n = 0, 1, 2, \ldots$$

(25) **The Meissner equation**

$$-y''(x) = \lambda w(x)y(x) \text{ for all } x \in (-\infty, +\infty)$$

where the weight coefficient w is defined by

$$\begin{aligned} w(x) &= 1 \text{ for all } x \in (-\infty, 0] \\ &= 9 \text{ for all } x \in (0, +\infty). \end{aligned}$$

Endpoint classification in the space $L^2(-\infty, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

This equation arose in a model of a one dimensional crystal. For this constant coefficient equation with a weight function which has a jump discontinuity the eigenvalues can be characterized as roots of a transcendental equation involving only trigonometrical and inverse trigonometrical functions. There are infinitely many simple eigenvalues and infinitely many double ones for the periodic case; they are given by:

Periodic boundary conditions on $(-1/2, +1/2)$**,** *i.e.*

$$y(-1/2) = y(+1/2) \qquad y'(-1/2) = y'(+1/2).$$

We have $\lambda_0 = 0$ and for $n = 0, 1, 2, \ldots$

$$\lambda_{4n+1} = (2m\pi + \alpha)^2; \;\; \lambda_{4n+2} = (2(n+1)\pi - \alpha))^2;$$

$$\lambda_{4n+3} \;\;=\;\; \lambda_{4n+4} = (2(n+1)\pi))^2.$$

where $\alpha \;=\; \cos^{-1}(-7/8)$

Semi-periodic boundary conditions on $(\;-1/2, +1/2)$**,** *i.e.*

$$y(-1/2) = -y(+1/2) \qquad y'(-1/2) = -y'(+1/2).$$

With $\beta = \cos^{-1}((1+\sqrt(33))/16)$ and $\gamma = \cos^{-1}((1-\sqrt(33))/16)$ these are all simple and given by, for $n = 0, 1, 2, \ldots$

$$\lambda_{4n} \;=\; (2n\pi \;+\; \beta)^2; \; \lambda_{4n+1} \;=\; (2n\pi \;+\; \gamma)^2;$$

$$\lambda_{4n+2} \;=\; (2(n+1)\pi \;-\; \gamma)^2; \; \lambda_{4n+3} \;=\; (2(n+1)\pi \;-\; \beta)^2.$$

See [12] and [17].

(26) **The Lohner equation**

$$-y''(x) - 1000xy(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty)$$

Endpoint classification in the space $L^2(-\infty, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

In [22] Lohner computed the Dirichlet eigenvalues of index (in SLEIGN2 notation) 0, 9, 49 and 99 using interval arithmetic and obtained rigorous bounds. In double precision SLEIGN2 computed eigenvalues are in good agreement with these guaranteed bounds.

(27) **The Jörgens equation**

$$-y''(x) + (\exp(2x)/4 - k\exp(x))y(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty)$$

where the parameter $k \in (-\infty, +\infty)$.

Endpoint classification in the space $L^2(-\infty, +\infty)$, for all $k \in (-\infty, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

This is a remarkable example from Jörgens and SLEIGN2 obtains excellent results. Details of this problem are given in [18, Part II, Section 10]. For all $k \in (-\infty, +\infty)$ the boundary value problem on the interval $(-\infty, +\infty)$ has a continuous spectrum on $[0, +\infty)$; for $k \leq 1/2$ there are no eigenvalues; for $h = 0, 1, 2, 3, \ldots$ and then k chosen by $h < k - 1/2 \leq h + 1$, there are exactly $h + 1$ eigenvalues and these are all below the continuous spectrum; these eigenvalues are given explicitly by

$$\lambda_n = -(k - 1/2 - n)^2, \quad n = 0, 1, 2, 3, \ldots, h.$$

(28) **The Behnke-Goerisch equation**

$$-y''(x) + k\cos^2(x)y(x) = \lambda y(x) \text{ for all } x \in (-\infty, +\infty)$$

where the parameter $k \in (-\infty, +\infty)$,

Endpoint classification in the space $L^2(-\infty, +\infty)$, for all $k \in (-\infty, +\infty)$:

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

This is a form of the Mathieu equation. In [9] these authors computed a number of Neumann eigenvalues of this problem using interval arithmetic with rigorous bounds. In double precision SLEIGN2 computed eigenvalues are in good agreement with these guaranteed bounds.

(29) **The Whittaker equation**

$$-y''(x) + \left(\frac{1}{4} + \frac{k^2 - 1}{x^2}\right) y(x) = \lambda \frac{1}{x} y(x) \text{ for all } x \in (0, +\infty)$$

where the parameter $k \in [1, +\infty)$.

Endpoint classification in the space $L^2(0, +\infty)$, for all $k \in [1, +\infty)$:

Endpoint	Classification
0	LP
$+\infty$	LP

This equation is studied in [18, Part II, Section 10]. There it is shown that the LP case holds at $+\infty$ and also at 0 for $k \geq 1$. The spectrum is discrete and is given explicitly by:

$$\lambda_n = n + (k + 1)/2, \quad n = 0, 1, 2, 3, \ldots.$$

(30) **The Littlewood-McLeod equation**

$$-y''(x) + x\sin(x)y(x) = \lambda y(x) \text{ for all } x \in [0, +\infty).$$

Endpoint classification in the space $L^2(0,+\infty)$:

Endpoint	Classification
0	R
$+\infty$	LP

The spectral analysis of this differential equation is considered in [21] and [24]; the equation is R at 0 and LP at $\pm\infty$. All self-adjoint operators in $L^2[0,\infty)$ have a simple, discrete spectrum $\{\lambda_n : n = 0, \pm 1, \pm 2, \ldots\}$ that is unbounded both above and below, *i.e.*

$$\lim_{n\to-\infty} \lambda_n = -\infty \qquad \lim_{n\to+\infty} \lambda_n = +\infty.$$

Every eigenfunction has infinitely many zeros in $(0,\infty)$.

SLEIGN2, and other codes, fail to compute the eigenvalues for this type of LP oscillatory problem. However there is qualitative information to be obtained by considering regular problems on $[0,X]$ with, say, Dirichlet boundary conditions $y(0) = Y(X) = 0$.

(31) **The Morse equation**

$-y''(x) + (9\exp(-2x) - 18\exp(-x))y(x) = \lambda y(x)$ for all $x \in (-\infty,+\infty)$

Endpoint classification in the space $L^2(-\infty,+\infty)$

Endpoint	Classification
$-\infty$	LP
$+\infty$	LP

This differential equation is studied in [3, Example 6]; the spectrum has exactly three negative, simple eigenvalues, and a continuous spectrum on $[0,\infty)$; the eigenvalues are given explicitly by

$$\lambda_n = -(n-2.5)^2 \text{ for } n = 0,1,2.$$

(32) **The Heun equation**

$$-(py')' + qy = \lambda wy \text{ on } (0,1)$$

where the coefficients p, q, w are given explicitly by, for all $x \in (0,1)$,

$$\begin{aligned} p(x) &= x^c(1-x)^d(x+s)^e \\ q(x) &= abx^c(1-x)^{d-1}(x+s)^{e-1} \\ w(x) &= x^{c-1}(1-x)^{d-1}(x+s)^{e-1}. \end{aligned}$$

The parameters a, b, c, d, e and s are all real numbers and satisfy the following two conditions

$$(i)\ s > 0 \text{ and } c \geq 1, d \geq 1, a \geq b$$

and

$$(ii)\ a + b + 1 - c - d - e = 0.$$

¿From these conditions it follows that

$$a \geq 1, b \geq 1, e \geq 1 \text{ and } a + b - d \geq 1.$$

The differential equation above is a special case of the general Heun equation

$$\frac{d^2w(z)}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right)\frac{dw(z)}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}w(z) = 0$$

with the general parameters $\alpha, \beta, \gamma, \delta, \varepsilon$ replaced by the real numbers a, b, c, d, e, a replaced by $-s$, and q replaced by the spectral parameter λ. For general information concerning the Heun equation see the compendium [29]; for the special form of the Heun equation considered here, and for the connection with confluence of singularities and applications, see the recent paper [20].

We note that the coefficients of the Sturm-Liouville differential equation above satisfy the conditions

(*i*) $q, w \in C[0,1]$ and $w(x) > 0$ for all $x \in (0,1)$

(*ii*) $p^{-1} \in L^1_{\text{loc}}(0,1), p(x) > 0$ for all $x \in (0,1)$

(*iii*) $p^{-1} \notin L^1(0,1/2]$ and $p^{-1} \notin L^1[1/2,1)$.

Thus both endpoints 0 and 1 are singular for the differential equation. Analysis shows that the endpoint classification for this equation is

Endpoint	Parameter	Classification
0	$c \in [1,2)$	LCNO
0	$c \in [2,+\infty)$	LP
1	$d \in [1,2)$	LCNO
1	$d \in [2,+\infty)$	LP

For the endpoint 0 and for LCNO cases the boundary condition functions u, v are determined by:

Parameter	u	v
$c = 1$	1	$\ln(x)$
$1 < c < 2$	1	x^{1-c}

For the endpoint 1 and for LCNO cases the boundary condition functions u, v are determined by:

Parameter	u	v
$d = 1$	1	$\ln(1-x)$
$1 < d < 2$	1	$(1-x)^{1-d}$

Further it may be shown that the spectrum of any self-adjoint problem on $(0,1)$, with the parameters a, b, c, d, e and s satisfying the above conditions, and considered in the space $L^2((0,1);w)$ with either separated or coupled boundary conditions, is bounded below and discrete. For the analytic properties, and proofs of the spectral properties of this Heun differential equation, see the paper [4].

In xamples.f, under Example 32, the user can enter a choice for the parameters a, b, c, d, e and s, subject to the conditions above being satisfied, and obtain numerical results for the eigenvalues.

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