

## Partial Derivatives for Math 229

Our purpose here is to explain how one computes “partial derivatives.” We will not attempt to explain how they arise or why one would use them; that is left to other courses (e.g., multivariate calculus or electromagnetism).

Unlike the usual situation in first semester calculus, we start with a function of two or more variables. Two examples of such functions are  $f(x, y) = \sin(xy)$  and  $g(x, y, z) = x^2 + xy^2z - 4xyz^{-2}$ . For each variable there is a corresponding partial derivative. For example, for a function  $f(x, y)$  of the two variables  $x$  and  $y$  there are two partial derivatives  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ . Technically speaking these partial derivatives are defined by

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

and

$$\frac{\partial f}{\partial y} = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

When it comes to computing such things, you need only remember the following: *To compute the partial derivative with respect to a variable, you use the same derivative rules and formulas as usual while treating the other variables as constants.* The following two examples should help show how one computes partial derivatives.

**Example 1:** Set  $f(x, y) = \sin(xy)$ . To compute the partial derivative  $\frac{\partial f}{\partial x}$  we will use the chain rule, constant multiple rule, the formula for the derivative of the sine function and the formula for the derivative of a power. Setting  $u = xy$  for the chain rule and treating  $y$  as a constant, we have

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial \sin(xy)}{\partial x} \\ &= \frac{\partial \sin u}{\partial x} && \text{replacing } xy \text{ with } u \\ &= \frac{d \sin u}{du} \cdot \frac{\partial xy}{\partial x} && \text{using the chain rule} \\ &= \cos u \cdot y \frac{\partial x}{\partial x} && \text{using a formula and the constant multiple rule} \\ &= \cos(xy) \cdot y && \text{replacing } u \text{ with } xy \text{ and using another formula.} \end{aligned}$$

Compare this with how you would compute the derivative of  $\sin(\pi x)$ . (Hint: you would use *the exact same steps!*)

**Example 2:** Set  $g(x, y, z) = x^2 + xy^2z - 4xyz^{-2}$ . To compute the partial derivative  $\frac{\partial g}{\partial y}$  we will use the sum/difference rule, constant multiple rule, the formula for the derivative of a constant,

and the formula for the derivative of a power. Treating both  $x$  and  $z$  as constants (so  $x^2$  and  $z^{-2}$  are constants, too), we have

$$\begin{aligned}
 \frac{\partial g}{\partial y} &= \frac{\partial(x^2 + xy^2z - 4xyz^{-2})}{\partial y} \\
 &= \frac{\partial x^2}{\partial y} + \frac{\partial(xy^2z)}{\partial y} - \frac{\partial(4xyz^{-2})}{\partial y} && \text{using the sum/difference rule} \\
 &= 0 + xz \frac{\partial y^2}{\partial y} - 4xz^{-2} \frac{\partial y}{\partial y} && \text{by formula and the constant multiple rule} \\
 &= 0 + xz \cdot 2y - 4xz^{-2} \cdot 1 && \text{by the power formula.}
 \end{aligned}$$

Compare this with how you would compute the derivative of  $e^2 + ey^2\pi - 4ey\pi^{-2}$ .

Just as with “usual” derivatives in first semester calculus, we have higher order partial derivatives as well. A big difference is that with partial derivatives there are several “second derivatives” instead of just one. For instance, if we start with a function  $f(x, y)$  of two variables, we have not one, but *four* ways to partially differentiate twice:

$$\begin{aligned}
 \frac{\partial^2 f}{\partial x^2} & \text{ partially differentiate with respect to } x \text{ twice,} \\
 \frac{\partial^2 f}{\partial y^2} & \text{ partially differentiate with respect to } y \text{ twice,} \\
 \frac{\partial^2 f}{\partial x \partial y} & \text{ partially differentiate with respect to } y \text{ then } x, \\
 \frac{\partial^2 f}{\partial y \partial x} & \text{ partially differentiate with respect to } x \text{ then } y.
 \end{aligned}$$

In a more expanded notation,

$$\begin{aligned}
 \frac{\partial^2 f}{\partial x^2} &= \frac{\partial(\partial f / \partial x)}{\partial x} \\
 \frac{\partial^2 f}{\partial y^2} &= \frac{\partial(\partial f / \partial y)}{\partial y} \\
 \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial(\partial f / \partial y)}{\partial x} \\
 \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial(\partial f / \partial x)}{\partial y}.
 \end{aligned}$$

**Example 3:** Set  $f(x, y) = xye^x - 4x^2 \cos(x + y)$ . We will compute the four second order

partial derivatives. First, (using  $u$  for  $x + y$  in the chain rule) we have

$$\begin{aligned}
 \frac{\partial f}{\partial x} &= \frac{\partial(xy e^x - 4x^2 \cos(x+y))}{\partial x} \\
 &= \frac{\partial(xy e^x)}{\partial x} - \frac{\partial(4x^2 \cos(x+y))}{\partial x} \\
 &= y \frac{\partial(x e^x)}{\partial x} - 4 \frac{\partial(x^2 \cos(x+y))}{\partial x} \\
 &= y \left( x \frac{\partial e^x}{\partial x} + e^x \frac{\partial x}{\partial x} \right) - 4 \left( x^2 \frac{\partial \cos(x+y)}{\partial x} + \cos(x+y) \frac{\partial x^2}{\partial x} \right) \\
 &= y(xe^x + e^x) - 4 \left( x^2 \frac{\partial \cos u}{\partial x} + \cos(x+y) 2x \right) \\
 &= y(xe^x + e^x) - 4 \left( x^2 \frac{d \cos u}{du} \cdot \frac{\partial(x+y)}{\partial x} + \cos(x+y) 2x \right) \\
 &= y(xe^x + e^x) - 4 \left( x^2 (-\sin u) \left( \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x} \right) + \cos(x+y) 2x \right) \\
 &= y(xe^x + e^x) - 4(-x^2 \sin(x+y)(1+0) + \cos(x+y) 2x) \\
 &= y(xe^x + e^x) + 4x^2 \sin(x+y) - 8x \cos(x+y).
 \end{aligned}$$

(Remember we treat  $y$  as a constant here.) Next

$$\begin{aligned}
 \frac{\partial f}{\partial y} &= \frac{\partial(xy e^x - 4x^2 \cos(x+y))}{\partial y} \\
 &= \frac{\partial(xy e^x)}{\partial y} - \frac{\partial(4x^2 \cos(x+y))}{\partial y} \\
 &= x e^x \frac{\partial y}{\partial y} - 4x^2 \frac{\partial \cos(x+y)}{\partial y} \\
 &= x e^x - 4x^2 \frac{\partial \cos u}{\partial y} \\
 &= x e^x - 4x^2 \frac{d \cos u}{du} \frac{\partial(x+y)}{\partial y} \\
 &= x e^x - 4x^2 (-\sin u) \left( \frac{\partial x}{\partial y} + \frac{\partial y}{\partial y} \right) \\
 &= x e^x + 4x^2 \sin(x+y)(0+1) \\
 &= x e^x + 4x^2 \sin(x+y).
 \end{aligned}$$

(Remember we treat  $x$  as a constant here.)

Before we compute the second order partial derivatives, we note from above that

$$\frac{\partial(xe^x)}{\partial x} = xe^x + e^x, \quad \frac{\partial \cos(x+y)}{\partial x} = -\sin(x+y), \quad \frac{\partial \cos(x+y)}{\partial y} = -\sin(x+y),$$

and in an entirely similar fashion

$$\frac{\partial \sin(x+y)}{\partial x} = \cos(x+y) \quad \text{and} \quad \frac{\partial \sin(x+y)}{\partial y} = \cos(x+y).$$

We have

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial(\partial f/\partial x)}{\partial x} \\
&= \frac{\partial(yxe^x + e^x) + 4x^2 \sin(x+y) - 8x \cos(x+y)}{\partial x} \\
&= \frac{\partial(yxe^x + e^x)}{\partial x} + \frac{\partial(4x^2 \sin(x+y))}{\partial x} - \frac{\partial(8x \cos(x+y))}{\partial x} \\
&= y \frac{\partial(xe^x + e^x)}{\partial x} + 4 \frac{\partial(x^2 \sin(x+y))}{\partial x} - 8 \frac{\partial(x \cos(x+y))}{\partial x} \\
&= y \left( \frac{\partial(xe^x)}{\partial x} + \frac{\partial e^x}{\partial x} \right) + 4 \left( x^2 \frac{\partial \sin(x+y)}{\partial x} + \sin(x+y) \frac{\partial x^2}{\partial x} \right) - 8 \left( x \frac{\partial \cos(x+y)}{\partial x} + \cos(x+y) \frac{\partial x}{\partial x} \right) \\
&= y(xe^x + e^x + e^x) + 4(x^2 \cos(x+y) + \sin(x+y)2x) - 8(-x \sin(x+y) + \cos(x+y)) \\
&= y(xe^x + 2e^x) + 4x^2 \cos(x+y) + 16x \sin(x+y) - 8 \cos(x+y),
\end{aligned}$$

using our formulas above. The next three are a little bit simpler:

$$\begin{aligned}
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial(\partial f/\partial y)}{\partial y} \\
&= \frac{\partial(xe^x + 4x^2 \sin(x+y))}{\partial y} \\
&= \frac{\partial(xe^x)}{\partial y} + \frac{\partial(4x^2 \sin(x+y))}{\partial y} \\
&= 0 + 4x^2 \frac{\partial \sin(x+y)}{\partial y} \\
&= 4x^2 \cos(x+y),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial(\partial f/\partial y)}{\partial x} \\
&= \frac{\partial(xe^x + 4x^2 \sin(x+y))}{\partial x} \\
&= \frac{\partial(xe^x)}{\partial x} + \frac{\partial(4x^2 \sin(x+y))}{\partial x} \\
&= xe^x + e^x + 4 \frac{\partial(x^2 \sin(x+y))}{\partial x} \\
&= xe^x + e^x + 4 \left( x^2 \frac{\partial \sin(x+y)}{\partial x} + \sin(x+y) \frac{\partial x^2}{\partial x} \right) \\
&= xe^x + e^x + 4(x^2 \cos(x+y) + \sin(x+y)2x) \\
&= xe^x + e^x + 4x^2 \cos(x+y) + 8x \sin(x+y),
\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial(\partial f / \partial x)}{\partial y} \\ &= \frac{\partial(y(xe^x + e^x) + 4x^2 \sin(x + y) - 8x \cos(x + y))}{\partial y} \\ &= \frac{\partial(y(xe^x + e^x))}{\partial y} + \frac{\partial(4x^2 \sin(x + y))}{\partial y} - \frac{\partial(8x \cos(x + y))}{\partial y} \\ &= (xe^x + e^x) \frac{\partial y}{\partial y} + 4x^2 \frac{\partial \sin(x + y)}{\partial y} - 8x \frac{\partial \cos(x + y)}{\partial y} \\ &= xe^x + e^x + 4x^2 \cos(x + y) + 8x \sin(x + y).\end{aligned}$$

Notice how the two “mixed partials”  $\frac{\partial^2 f}{\partial x \partial y}$  and  $\frac{\partial^2 f}{\partial y \partial x}$  are equal. According to Clairaut’s Theorem, this will always be the case when these two mixed partial derivatives are continuous. Any functions you come across in this class will be this way, so you can just assume that  $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ .

### Exercises

Find the first partial derivatives of the given function.

1.  $f(x, y) = y^5 - 3xy$
2.  $f(x, y) = x^4 y^{-2} + 9x^2 y$
3.  $g(x, y) = \tan(xy^2)$
4.  $h(x, y) = \frac{x^2 + 3y}{(x + y)^2}$
5.  $h(x, y, z) = x \ln(y + z^2) + e^{xyz}$

Find all of the second partial derivatives of the given function.

6.  $g(x, y) = \frac{xy}{x - y}$
7.  $h(x, y, z) = xyz + xy^2 z^3$

Verify the conclusion of Clairaut’s Theorem, i.e., show that the two mixed partial derivatives are equal, for the following functions.

8.  $f(x, y) = e^{xy} \sin y$
9.  $g(x, y) = \tan^{-1}(yx^2 + y^2)$