Math 240
Final Exam
Fall 2004

1. Find the general solution of the system of equations

$$\begin{aligned}
&2x - 4y + z + 6w = 0 \\
&3x - 6y + 6z + 15w = -3 \\
&x - 2y - z + w = 1
\end{aligned}$$

2. Find the determinant $\begin{vmatrix} 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & -2 \\ 2 & 5 & -3 & 1 \\ 0 & 0 & 0 & 4 \end{vmatrix}$.

3. Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ is such that

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y - z \\ 2x + y + 4z \\ 3x - y + 3z \end{bmatrix}$$

(a) Find a basis for ker L.

(b) Find a basis for range L.

4. (a) Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ independent?

(b) Is the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & -1 \end{bmatrix}$ invertible? Explain.

5. Let P_2 denote the collection polynomials of degree ≤ 2 and the 0 polynomial. Define $L : P_2 \to \mathbb{R}^2$ by

$$L(p) = \begin{bmatrix} p(1) \\ p(2) \end{bmatrix}.$$

Let $S = \{1, 2+t, 1+t+t^2\}$, a basis of P_2 and $T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, a basis for $\mathbb{R}^2$.

Find the matrix of L for the bases S and T.

6. Let $\mathbb{R}^4$ have the usual (dot) inner product. Find an orthonormal basis for the subspace spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

7. Let $A = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) Find a matrix P so that $P^{-1}AP$ is diagonal.

8. (a) Let A and B be $n \times n$ matrices. Define "A is similar to B".

(b) Prove that if A is similar to B and B is similar to C, then A is similar to C.

(c) Suppose that A is similar to B and λ is a scalar. Show that $A - \lambda I$ is similar to $B - \lambda I$.

9. Suppose that v_1 and v_2 are linearly independent eigenvectors of a matrix A with corresponding eigenvalues λ_1 and λ_2. If $\lambda_1 \neq \lambda_2$, show $v_1 + v_2$ is <u>not</u> an eigenvector of A.

10. Show that if an $n \times n$ matrix A satisfies $A^T A = I$, then $\det(A) = \pm 1$.

11. Let V and W be vector spaces and $L : V \to W$ a linear map. Assume $S = \{v_1, ..., v_n\}$ is a basis of V and set $T = \{L(v_1), ..., L(v_n)\}$.

    (a) Show that if T spans W, then L is onto.

    (b) Show that if L is $1-1$, then T is independent.

12. Let W be a subspace of the inner product space V. Define

$$W^\perp = \{v \in V | (v, w) = 0 \text{ for all } w \text{ in } W\}.$$

(a) Show that $W^\perp$ is a subspace of V.

(b) Show that the only vector in both W and $W^\perp$ is 0.

13. Let V be an inner product space. Suppose that $\{v_1, v_2\}$ is an orthonormal set. Show $||c_1 v_1 + c_2 v_2||^2 = c_1^2 + c_2^2$