Name _____

Math 240 FINAL EXAM (5/12/11)

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(1) Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 2 & -2 & -1 & 0 & -1 \\ -3 & 3 & 0 & -3 & -3 \end{bmatrix}$$

(a) Find a basis for the column space of A.

(b) Find a basis for the nullspace of A.

(c) Find a basis for the row-space of A.

(d) What is the rank of A? Explain.

(2) Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

(a) Find the eigenvalues of A.

(b) Find a corresponding eigenvector for each eigenvalue.

(c) Find a matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(3) Let $L : P_2 \to P_1$ be defined by $L\big(p(t)\big) = p'(t) + tp(0)$. Let $S = \{t^2, t, 1\}$ and $S' = \{t^2 + 1, t + 1, t - 1\}$ be ordered bases for P_2, and let $T = \{t, 1\}$ and $T' = \{t + 2, t + 1\}$ be ordered bases for P_1.

(a) Find the representation of L with respect to S and T.

(b) Find the representation of L with respect to S' and T'.

(c) Use your answers to a) and b) to find $L(2t^2 + 2)$ and check them with the formula for $L(2t^2 + 2)$.

(4) Let $\mathbf{w}_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$.

(a) Show that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is an orthonormal set with respect to the usual inner product on $\mathbb{R}^3$.

(b) Let $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find the projection of $\mathbf{v}$ onto W.

(5) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$.

(a) Find A^{-1}.

(b) Solve the system $A\,\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ s \end{bmatrix}$ for any real number s.

(6) Let A be a nonsingular $n \times n$ matrix and let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be linearly independent vectors in $\mathbb{R}^n$. Prove that the vectors $A\,\mathbf{v}_1, \ldots, A\,\mathbf{v}_k$ are also linearly independent.

(7) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

(a) Find $\det A$.

(b) Find $\det(10A)$.

(c) Is A invertible? If so, find $\det(A^{-1})$.

(8) Let A be an $n \times n$ matrix.

(a) Show that A is invertible if and only if 0 is not an eigenvalue of A.

(b) Suppose $\mathbf{v}_1$ is an eigenvector of A with associated eigenvalue λ_1, and $\mathbf{v}_2$ is an eigenvector with associated eigenvalue λ_2. If $\lambda_1 \neq \lambda_2$, show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

(9) Let $L : V \to W$ be a linear transformation.

(a) Prove that $L(\mathbf{0}_V) = \mathbf{0}_W$.

(b) Prove that the kernel of L is a subspace of V.

(c) Prove that the range of L is a subspace of W.