

Do all of 8 problems. Each problem is worth 25 points. Show all details of your work. Please, return this copy with your solutions.

1. There are 5 repair stations on the Hong Kong-Beijing railway from south to north. We name the stations from south to north exactly as station 1, station 2, station 3, station 4 and station 5, respectively. Repair parts are being delivered to the stations with annual frequencies $N_1 = 4$, $N_2 = 3$, $N_3 = 1$, $N_4 = 8$ and $N_5 = 4$ from a PDC on the railway. Assume that the the cost of delivery per unit distance is constant.

(i) Where should the PDC be located so that the cost of delivery be minimized?

(ii) Lately China has been experiencing a climate shift and the number of snow storms has been increasing gradually in the south. Suppose that N_2 begins to increase due to increasing amount of repair works. At what value of N_2 the location of the PDC has to start to shift southward?

2. In the annual boat race between Oxford and Cambridge the drag force F experienced by a boat is proportional to $S^{1/4}v$ where S is the surface area of the hull of the boat and v is the velocity of the boat. From physics we know that the power P required to overcome F is given by $P = Fv$. Further, suppose that P is proportional to n , the number of boatmen on board and n is proportional to the length l of the boat. Find the relationship between v and n assuming the boats are geometrically similar. (**Hint:** Use geometric similarity and let l be the characteristic dimension where $S \propto l^2$.)

3. A DeKalb farmer owns 45 acres of land, which she plans to plant with either wheat or corn. Each acre of wheat yields \$200 in profit and each acre of corn yields \$300 in profit. The labor and fertilizer requirements for each acre of wheat are 3 workers and 2 tons of fertilizer while for each acre of corn 2 workers and 4 tons of fertilizer are needed. Suppose that 100 workers and 130 tons of fertilizer are available all together, how many acres should she plant for each crop in order to maximize the total profit?

4. Fit the quadratic model $y = ax^2 + bx$ to a given data set $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$ using the least-squares criterion. Obtain equations for the parameters a and b in terms of the data points.

5. A windmill is rotated by air flow to produce power and to pump water. Assume its power P depends on the density of the air ρ , the viscosity of the air μ , the diameter of the windmill D , the wind speed v and the rotational speed ω **measured as numbers of revolution per unit time**. Using dimensional analysis show that

$$P = \mu D v^2 \phi \left(\frac{\rho D v}{\mu}, \frac{D \omega}{v} \right)$$

for some function ϕ of two variables. (**Hint: Power is measured as ML^2T^{-3} , viscosity as $ML^{-1}T^{-1}$ and density as ML^{-3} .**)

6. A patient takes a drug at equal time intervals $T = 6$ with dosage $Q = 3$. The amount of drug $C(t)$ remaining in the blood stream at time t obeys the differential equation $\frac{dC}{dt} = -\frac{C}{t+4}$. Find the residuals R_1 and R_2 of the drug at the end of the first and second periods.

7. A certain object falling from rest has velocity $v(t)$ at time t which obeys the equation:

$$\frac{dv}{dt} = 16 - v^2.$$

(i) Sketch the family of solution curves of the above autonomous equation for different initial conditions $v(0) = v_0$.

(ii) What is the limiting value of $v(t)$ as $t \rightarrow \infty$ when $v(0) = 0$?

8. The trajectories of a toy car moving in the $x - y$ plane vary according to the system

$$\begin{aligned} \frac{dx}{dt} &= 4y \\ \frac{dy}{dt} &= 2 - x. \end{aligned}$$

(i) Find all rest points (equilibria).

(ii) Sketch the trajectories of the system by setting up and solving the differential equation for the trajectories.

(iii) Indicate clearly the directions of motion along the trajectories.

(iv) Determine whether the rest points are stable, asymptotically stable or unstable.