

## Modeling of a single species population

In this project we will study a fish population in a lake. By setting up a differential equation, we will investigate the dynamics of this population and show that the population will eventually approach the so called carrying capacity of the environment if the initial population is larger than a “threshold” and become extinct if it is smaller than the threshold. For the purpose of protection of this population, we will set up a scheme for fishing.

1. Denote by  $P(t)$  the fish population at time  $t$ . Assume the birth rate  $\beta(P)$  and death rate  $\delta(P)$  of  $P(t)$  per individual per year are given by,

$$\beta(P) = a \text{ and } \delta(P) = bP + \frac{c}{P},$$

respectively, where  $a, b, c$  are positive constants such that  $a^2 - 4bc > 0$ . Apply the principle

$$\frac{dP}{dt} = [\beta(P) - \delta(P)]P$$

to show that  $P(t)$  satisfies the differential equation

$$\frac{dP}{dt} = k(M - P)(P - m), \tag{1}$$

where  $k = b$ ,  $m = \frac{a - \sqrt{a^2 - 4bc}}{2b}$ , and  $M = \frac{a + \sqrt{a^2 - 4bc}}{2b}$ . We observe that the logistic equation is the special form of (1) when  $m = 0$ . Let the initial population at  $t_0$  be given by

$$P(t_0) = P_0 \tag{2}$$

2. Explain why  $P(t) \equiv m$  and  $P(t) \equiv M$  are solutions of (1) and why they are the only constant solutions of (1). Without solving the equation, discuss the behavior of the other solutions  $P(t)$  using the signs of  $\frac{dP}{dt}$  and  $\frac{d^2P}{dt^2}$  for the cases when  $P(t) < m$ ,  $m < P(t) < M$ , and  $P(t) > M$ , respectively.
3. It is known that the solution curves  $P = P(t)$  do not intersect one another in the  $t - P$  plane. Use this fact to solve the initial value problem (1), (2) for the case when  $P_0 \neq m, M$ , to obtain

$$\frac{P - m}{M - P} = \frac{P_0 - m}{M - P_0} e^{(M-m)k(t-t_0)}. \tag{3}$$

Change (3) further to derive

$$P = \frac{M(P_0 - m) + m(M - P_0)e^{-(M-m)k(t-t_0)}}{(P_0 - m) + (M - P_0)e^{-(M-m)k(t-t_0)}}. \tag{4}$$

4. (i) For the case where  $P_0 < m$ , explain why Part 2 implies that the population  $P(t)$  will become extinct at a finite time  $t^*$ . Find the value of  $t^*$ .  
 (ii) For the case where  $m < P_0 < M$  and  $P_0 > M$ , explain why Part 2 implies that  $\lim_{t \rightarrow \infty} P(t)$  exists. Use (4) to show that  $\lim_{t \rightarrow \infty} P(t) = M$ .

(iii) Show that the solutions  $P(t) \equiv m$  and  $P(t) \equiv M$  are included in the form (4). Discuss the significance of the numbers  $m$  and  $M$  in determining the behavior of  $P(t)$  and explain why we call  $m$  the threshold and  $M$  the carrying capacity of the environment.

5. Based on the information obtained above, sketch the solution curves for different values of  $P_0$ .
6. In order to protect this population from being extinct by any unexpected reasons, we want to require that  $P(t) \geq \frac{M+m}{2}$  for all  $t \geq t_0$ . The number  $\frac{M+m}{2}$  is called the safety level of this population. Derive the formula for  $P(t)$  for the case when the initial condition  $P(t_0) = \frac{M+m}{2}$ .
7. The following are collected data on the birth and death rates  $\beta(P)$  and  $\delta(P)$  of this population:

$P$	8200	11300	12500	14400	16100	18900
$\beta(P)$	0.49	0.50	0.52	0.48	0.49	0.51
$P\delta(p)$	3345	4554	5125	6147	7184	9144

Use Excel to find  $\beta(P)$  and  $P\delta(P)$ , and then find  $\delta(P)$ . Determine the values of  $m$  and  $M$  using the relations in Part 1.

Suppose an epidemic disease attacked the fish in the lake, and the survived population is 4000. By comparing this value with  $m$  and  $M$ , show what will happen without taking any precautions.

To prevent this species from being extinct in the lake, the government decides to stock the same type of fish into the lake. How much is needed to raise the population to the safety level?

One year after the stocking is done, fishing is allowed, but only once a year. Compute the maximum amount for fishing each year in order to keep the population above the safety level.