

Improving the Trans-Siberian Railway Model

In the sample project we considered in class, we determined that the cost of supplying city x_j with material for a year depended on the location x of the Parts Distribution Center (PDC), the number N_j of visits which city x_j required, and the cost-per-mile k :

$$C_j(x) = kN_j|x - x_j|.$$

We used this observation to determine the total cost of a decision to place the PDC at location x , and we deduced (perhaps surprisingly) that the optimal location of the PDC was always at one of the cities and the choice of the best city is independent of the values of the numbers x_j .

But in that model we assumed there *was* such a concept as “cost-per-mile”, i.e. we assumed the per-mile shipping costs were the same everywhere on the railway. In this project we will investigate what happens when we make the model more realistic by assuming that k varies across the track. For example, an improved model would take into account the fact that the unit cost (per mile) is likely to be higher where the track climbs a mountain than where the track is flat.

We will continue to use the other assumptions of the sample model: we still assume the track has no loops nor branches, there are a finite number of cities x_1, \dots, x_n to be served, and the cities must be supplied with separate supply runs. We will assume for the first few questions that the cost per mile maintains a constant value k_i between the i -th and $(i+1)$ -th cities, but even this assumption will be weakened later.

It is always helpful to consider a specific example first. (This is just an example; do not assume particular values in your answers to any other questions besides the first.)

1. Suppose there are four cities at locations $x_1 = 2$, $x_2 = 5$, $x_3 = 9$, and $x_4 = 11$, requiring respectively $N_i = 6, 2, 3, 7$ visits, respectively. Suppose that the cost-per-mile to ship goods is \$4 between x_1 and x_2 , but \$8 between x_2 and x_3 , and \$10 between x_3 and x_4 . What is the cost of keeping the PDC at $x = 6$? What about at $x = 10$?

Now repeat the ideas you used in Problem 1 in a general setting. We want a formula for the cost of servicing city x_j for a year, but that cost depends on the location x of the PDC. Let i be the number of cities to the left (west) of the PDC, so that $x_i < x \leq x_{i+1}$. Let's write $C_{ij}(x)$ for the cost of servicing city x_j instead of $C_j(x)$, to emphasize that our calculations depend on the fact that x lies in the i -th interval. Let k_i be the unit cost to ship goods between x_i and x_{i+1} .

2. Show that when $j \leq i$ we have

$$C_{ij}(x) = N_j \{k_j(x_{j+1} - x_j) + \cdots + k_{i-1}(x_i - x_{i-1}) + k_i(x - x_i)\},$$

while if $j \geq i + 1$ then we have

$$C_{ij}(x) = N_j \{k_i(x_{i+1} - x) + k_{i+1}(x_{i+2} - x_{i+1}) + \cdots + k_{j-1}(x_j - x_{j-1})\}.$$

3. Using the results in problem 2, find expressions for numbers A_i and S_i so that whenever the PDC is located between cities i and $i + 1$ the total cost is $C(x) = S_i x + A_i$. Find similar expressions for the cost function which are valid for $x < x_1$ and $x > x_n$, respectively. Then explain why the cost function is piecewise linear on the railway. What does the graph of $C(x)$ look like? (Note: one may show that $C(x)$ is continuous. You do not have to prove this, but you may use this fact if you wish.)

4. Derive an efficient algorithm for finding the optimal location for the PDC similar to the one in the sample project given by $T_i, i = 1, \dots, n - 1$. Remember that when all the unit costs k_i were assumed equal, we found that there could never be a better location for the PDC than at one of the cities x_j . Is that still true in the more general problem we are now considering? How important is the variation in cost to the problem of determining the optimal location?

Finally we consider a case in which the cost-per-mile k varies continuously with position. That means we are assuming there is a continuous function k with the property that at each point s on the train line, the unit cost at s is $k(s)$. In particular, the cost of a short trip near s will be approximately $k(s)$ times the length of the trip. More precisely, from the calculus we know that the cost of a single journey between two points α and β on this train line will be $|\int_{\alpha}^{\beta} k(s) ds|$.

5. Use the above result to express $C_j(x)$ as an integral for $j = 1, \dots, n$. (You need to consider the two cases when $x_j \leq x$ and $x_j > x$.) Add those together to write the whole cost function $C(x)$ as a sum of integrals. Use a fundamental result from calculus to determine a formula for $C'(x)$ which is valid for $x < x_1, x \in (x_1, x_2), \dots, x \in (x_{n-1}, x_n)$, and $x > x_n$. Can you determine the sign of this derivative?

Can there be a case in which it is cheaper to put the PDC somewhere other than one of the original cities? Derive an efficient algorithm for finding the optimal location for the PDC.