1. Prove that for any $C \in \mathbb{R}$, the equation

$$x^3 + 3x + C = 0$$

has at most one solution in the interval $[-1, 1]$.

2. Use L'Hôpital's rule to evaluate

$$\lim_{x \to 0} \frac{x^2 \sin x}{\sin x - x \cos x}$$
3. Use the definition of the derivative to find the derivative of \( f(x) = 2\sqrt{x} + 1 \) at \( x_0 \in (0, \infty) \). Does \( f'(0) \) exist? Explain.

4. Prove, using \( \varepsilon \) and \( \delta \) that \( f(x) = x^2 \) is continuous at any \( x \in [10, 21) \). Is it uniformly continuous on this interval? State any theorems that you use.
5. Prove that the set \( S = \left\{ \left( \frac{x}{2}, n^2 \right) \mid x \in \mathbb{Q}, \ n \in J \right\} \) is countable.

6. Let \( S \) be a non-empty subset of \( \mathbb{R} \) which is bounded below. Let \( x = \inf \{ S \} \). If \( x \not\in S \), prove that \( x \) is an accumulation point of \( S \).
7. The set \((0, 1]\) is not compact. Find an infinite open cover with no finite subcover.

8. Let \(f : \mathbb{R} \to \mathbb{R}\) be differentiable. If \(f'(x) > 0\) for all \(x \in \mathbb{R}\), prove that \(F(x) = f(x) + 8\) is increasing. What, if any, extra conditions would need to be imposed for \(G(x) = (f(x))^2\) to be increasing?
9. If $c > 1$ prove that \( \left\{ \sqrt[n]{c} \right\} \) is convergent. What is it convergent to?

10. Prove directly that \( f(x) = x \) is Riemann integrable on \([0, 1]\).
11. If $a_n, b_n > 0$ for all $n \in J$, $\{a_n^2\}$ is convergent and $\{b_n^2\}$ is Cauchy, prove that $\{a_n b_n\}$ is convergent. State any theorems that you use.

12. Must every bounded sequence in $\mathbb{R}$ be convergent? Must every bounded sequence contain a convergent subsequence? Is every convergent sequence bounded? Give examples.
13. Show that

$$2^x = 6x$$ for some $$x \in (0, 1)$$

14. Let $$f : [a, b] \to \mathbb{R}$$ be continuous on $$[a, b]$$, differentiable on $$(a, b)$$ with $$f(a) = f(b) = 0$$. Prove that for each $$k \in \mathbb{R}$$ there is a $$c \in (a, b)$$ with $$f'(c) = kf(c)$$. Hence, or otherwise, show that it is always possible to solve the equation $$\cos x = k \sin x$$ for $$x \in (0, \pi)$$ for all $$k \in \mathbb{R}$$
15. $f : [1, 2] \rightarrow [1, 2]$ is continuous with $f([1, 2]) = [1, 2]$. Prove that there exists $c \in (1, 2)$ with $f(c) = c$. 
