Fall 2003 | Final Exam | Name ____________
Professor K. Datta | MATH 434 |

1.(a) [30 pts.] Develop an algorithm to compute Hx, where H is a $n \times n$ Householder matrix and x is a $n \times 1$ vector without explicitly computing the product.

(b) Given $x = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$, find a Householder matrix H such that Hx is multiple of $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

(c) Using the results in (b), find the QR factorization of the matrix $A = \begin{pmatrix} 0 & 1 \\ 2 & 2 \\ 0 & 1 \end{pmatrix}$.

2.(a) [30 pts.] Let A be $m \times n, m \geq n$. Then using the QR factorization of A, prove that $A^T A$ is positive definite if and only if A has rank n.

(b) Let $A = \begin{pmatrix} 0 & 1 \\ 2 & 2 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$. Find the least square solution to $Ax = b$ using QR factorization. (Use 1(c)).

(c) Does the least square solution to $Ax = b$ always exist? If the solution exists, is it unique? Explain.

3.(a) [25 pts.] Show that similar matrices have the same set of eigenvalues.

(b) State the Real Schur Triangularization Theorem.

(c) The basic QR iteration algorithm applied to a $n \times n$ arbitrary matrix A, generates the following sequences of matrices $\{A_k\}$:

$$
\begin{aligned}
&A = A_0 \\
&\text{for } k = 1, 2, \ldots \text{ do} \\
&A_{k-1} = Q_k R_k \\
&A_k = R_k Q_K
\end{aligned}
$$

Show that A_k is orthogonally similar to A_{k-1}.

4.(a) [30 pts.] Using the Geršgorin disc theorem, prove that a diagonally dominant matrix is non-singular.

(b) Apply two iterations of the Power method to compute the largest (in magnitude) eigenvalue of $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, starting with $x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(c) Apply Geršgorin disc Theorem to find regions of Complex plane containing the eigenvalues of the matrix in 2(b). Find the smallest eigenvalue of matrix A (in 2(b)) using this result.

5.(a) [30 pts.] Define Condition number of a matrix.

(b) Let 0 be an orthogonal matrix. Then show that Cond(0) with respect to 2-norm is 1.

(c) Let $A = \begin{pmatrix} .54 & .55 \\ .55 & .54 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} -49.5413 & 50.4587 \\ 50.4587 & -49.5413 \end{pmatrix}$.

(i) Find condition number of A with respect to the infinity norm.

(ii) Let A be the same as above. Let $A(x + \delta x) = b + \delta b$ where $\delta b = \begin{pmatrix} .01 \\ 0 \end{pmatrix}$. Find the upper bound of the relative error $\frac{||\delta x||}{||x||}$ (without computing the solution).

6. [25 pts.] Let T be a $n \times n$ symmetric positive definite tridiagonal matrix.

(a) Write an algorithm to compute the Cholesky decomposition of T.

(b) Using the algorithm in 6(a), solve $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$.

7.(a) [30 pts.] Given $A = \begin{pmatrix} .0001 & 2 \\ 1 & 1 \end{pmatrix}$. Find LU factorization of A using Gaussian Elimination (use 4 decimal digits arithmetic).

(b) Using the results of 7(a), compute the solution to the system

$$Ax = b, \text{ where } b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

(c) Compute the growth factor.