Fall 2004 | Final Exam | Name ____________________
Professor K. Datta | MATH 434 |

1.(a) (30 pts) Define the condition number of a matrix.

(b) Let $A = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}, c \neq 1.$

Calculate $\text{Cond}_\infty(A)$. When does $A$ become ill-conditioned?

(c) Show that $\text{Cond}_2(A) \geq 1$.

(d) Suppose that in the problem $Ax = b$, the vector $b$ is subject to perturbation but not the matrix $A$. Let $A$ be a nonsingular matrix, then prove the following inequality:

$$\frac{||\delta x||}{||x||} \leq \text{Cond }(A) \cdot \frac{||\delta b||}{||b||}$$

where $A(x + \delta x) = b + \delta b$.

2.(a) (30 pts.) Let $A$ be a symmetric Tridiagonal matrix. State an algorithim to find $LU$ factorizations of $A$ using Ganssian elemination without pivoting.

(b) When does the above algorithim fail? State conditions under which $LU$ factorization is unique.

(c) Using the above algorithim, solve the following system

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix}$$

(d) Express the inverse of $A$ in a factored form.

3.(a) Let $A$ be $m$x$n$, $m \geq n$ and rank $(A) = n$. Let $Q^T A = R = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}$ be the $QR$ factorization of $A$ where Rank $(R_1) = n$.

Show that $x$ is the least squares solution of the system $Ax = b$ if $x$ is chosen such that $R_1 x = c$ where $Q^T b = \begin{pmatrix} c \\ d \end{pmatrix}$.

(b) Does the least squares solution to an overdetermined problem always exist? If the solution exists, is it unique? Give the reasons for your answers.

(c) Let $A = \begin{pmatrix} 0 & 2 \\ 2 & 3 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Find the least square solution to $Ax = b$ using the method in (a).

(d) Find the residential $||Ax - b||_2$.

4.(a) (30 pts.) State the Real Schur Triagularization Theorem.

(b) Using (a) show that all the eigenvalues of a real symmetric matrix are real.

(c) Show that similar matrices have the same set of eigenvalues.

(d) State the basic $QR$ iteration algorithim for eigenvalue computation of a matrix $A$. Show that every member of this iteration has the same eigenvalues as of $A$.

5.(a) (20 pts.) Using the Gersgorin disc theorem, prove that a diagonally dominant matrix in nonsingular.

(b) Apply 2 iterations of power method to compute the largest (in magnitude) eigenvalue of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 0 & 0 & 1 \end{pmatrix}, \text{ starting with } x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

6.(a) (30 pts.) Define a Householder matrix.

(b) Show that Householder matrix is symmetric and orthogonal.

(c) Given $x = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$, find a householder matrix $H$ such that $Hx = \begin{pmatrix} 0 \\ * \\ 0 \\ 0 \end{pmatrix}$