MATH 434
FINAL EXAM

Professor Biswa Datta | Name ____________________

FALL05 | Z-Number ____________________

1. (a) **(5 pts.)** Define 2-norm of a matrix. Prove that if x is an n-vector and Q is an $n \times n$ orthogonal matrix, then $||Qx||_2 = ||x||_2$.

   (b) prove the following properties:

      i. **(5 pts.)** $||QA||_F = ||A||_F$, where Q is an orthogonal matrix.

      ii. **(5 pts.)** For any eigenvalue λ of a matrix $|\lambda| \leq ||A||$

2. (a) **(5 pts.)** Define the growth factor. Compute the growth factor of

$$A \begin{pmatrix} 10^{-10} & 1 \\ 1 & 2 \end{pmatrix},$$

using Gaussian elimination without pivoting with 4-digit arithmetic.

(b) **(5 pts.)** State a mathematical result on the stability of Gaussian elimination involving the growth factor. Using this result, show that Gaussian elimination without pivoting for the matrix A in 2(a) is unstable.

(c) **(5 pts.)** Perform Gaussian elimination with partial pivoting on the matrix A in 2(a) and prove that the process is numerically stable for this matrix by computing the growth factor.

(d) **(5 pts.)** Given

$$A = \begin{pmatrix} a_1 & b_1 & 0 \\ b_2 & a_2 & b_3 \\ 0 & b_4 & a_3 \end{pmatrix}$$

Write down a set of inequalities that will guarantee that Gaussian elimination without pivoting will be stable.

3. (a) **(5 pts.)** Define a Householder matrix. Prove that a Householder matrix is orthogonal.

   (b) **(5 pts.)** Given an n-vector x, Show how to implicitly compute the matrix-vector product Hx, where H is an $n \times n$ Householder matrix.

   (c) **(5 pts.)** Prove that the matrix product HA, where A is an $n \times n$ arbitrary matrix, can be compute in $O(n^2)$ flops.

(d) **(5 pts.)** Given $x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, find a Householder matrix H such that the **third** component of Hx is zero.

4. (a) **(5 pts.)** Define the pseudoinverse of a rectangular matrix A. Compute the pseudoinverse of $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$.

   (b) **(5 pts.)** Define the condition number of an $m \times n$ rectangular matrix ($m > n$) using the pseudoinverse. Compute the condition number of the matrix A in 4(a).

(c) **(5 pts.)** State an expression for the least-squares solution to a full-rank overdetermined system $Ax = b$ using the pseudoinverse of A. Using this result, compute the least-squares solution to $Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, with A as given 4(a).

(d) **(5 pts.)** prove that if A is a full-rank $m \times n$ matrix ($m \geq n$), then $A^T A$ is symmetric positive definite.

(e) **(5 pts.)** Based on the result (d) and using normal equations, prove that the least-squares solution to full-rank overdetermined system $Ax = b$ is unique.

5. (a) **(5 pts.)** State the power method for computing the dominant eigenvalue of a matrix A.

(b) **(5 pts.)** Consider the following sequence of matrices $\{A_k\}$ defined by:

$$
\begin{aligned}
A_0 &\equiv A \\
A_0 &= Q_0 R_0 \\
A_1 &= R_0 Q_0 = Q R_1 \\
A_2 &= R_1 Q_1 = Q_2 R_2
\end{aligned}
$$

and so on.

Prove that A_1 has the same eigenvalue as A.

(c) **(5 pts.)** Given the factorization of a matrix in the form:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the singular values of A.

(5 pts.) Find the least-squares solution to $Ax = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ using the above decomposition.

6. **BONUS PROBLEM**

1. (a) **(5 pts.)** State the result on the necessary and sufficient condition for convergence of the iteration:

$$x^{(k+1)} = Bx^{(k)} + d.$$

(No Proof Needed).

(b) **(5 pts.)** Assuming the result in (a) Prove that the Jacobi iterative method convergens for any arbitrary initial choice of the solution if the matrix A is a strictly row diagonally dominant matrix.