1. (30 pts.) Prove the following norm properties:

i. \( \|A^T A\|_2^2 = \|A\|_2^2 \).

ii. \( \|A^{-1}\|_2 = \frac{1}{\sigma_{\text{min}}} \).

iii. \( \|I\|_F = \sqrt{n} \), where \( I \) is an \( n \times n \) identity matrix.

iv. \( \|Qx\|_2 = \|x\|_2 \), where \( Q \) is an \( n \times n \) orthogonal matrix and \( x \) is an \( n \)-vector.

v. \( \|OA\|_F = \|A\|_F \), where \( O \) is an orthogonal matrix.
2. (a) (5 pts.) Define the backward stability of an algorithm.

(b) (5 pts.) Prove that the computation $fI(x(y + z))$ is backward stable.

(c) (5 pts.) State a mathematical result to show that Householder’s method for QR factorization of a matrix is backward stable.
(d) **(10 pts.)** Define an ill-conditioned problem. Prove that $\text{Cond}_2(A) = \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}}$. 

(e) **(5 pts.)** Using the result of (d), construct a $3 \times 3$ example of an ill-conditioned matrix.

(f) **(10 pts.)** Let $U = (u_{ii})$ be an nonsingular upper triangular matrix. Then show that

$$\text{Cond}_2(U) \geq \frac{\max(u_{ii})}{\min(u_{ii})}$$
3. (a) **(60 pts.)** State a numerically effective algorithm for computing a Householder matrix $H$ such that $Hx$ is a multiple of $e_1$, where $x$ is an $n$-vector.

(b) Apply your algorithm to the vector $x = (0, 1, 2)^T$.

(c) Using the result of 3(b), find the QR factorization of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}.$$
(d) Find the least-squares solution to $Ax = b$, where $A$ is same as in 3(c), and $b = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$.

(e) Prove that the matrix product $HA$, where $A$ is an $n \times n$ arbitrary matrix and $H$ is an $n \times n$ Householder matrix, can be computed in $O(n^2)$ flops.

(f) Let $A$ be $m \times n$ and have full-rank. Then prove that the normal equation

$$A^T Ax = A^T b$$

has a unique solution.
4. (a) (10 pts) Prove that two similar matrices have the same eigenvalues.

(b) (5 pts.) State the real Schur triangularization theorem.

(c) (10 pts.) Consider the following basic QR Iteration Algorithm:

\[ A_0 = A \]

For \( k = 1, 2, \ldots \) do

\[ A_{k-1} = Q_k R_k \]

\[ A_k = R_k Q_k. \]

End

Show that \( A_k \) has the same eigenvalues as \( A \).
(d) (10 pts.) Apply Geršgorin theorem to find regions of the complete plane containing the eigenvalues of

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
3 & 4 & 9 \\
1 & 1 & 1
\end{pmatrix}.
\]

5. (a) (35 pts.) Given

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 3 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

Compute the followings without explicitly computing the matrix \(A\):

i. (5 pts.) \(\|A\|_2 = \)

ii. (5 pts.) \(\text{Cond}_2(A) = \)

iii. (5 pts.) \(\|A\|_F = \)

iv. (5 pts.) Singular values of \(A = \)

v. (5 pts.) Singular vectors of \(A = \)

vi. (10 pts.) Find the least-squares solution of \(Ax = b\) using SVD of \(A\) where \(A\) is as given in (a), and \(b = (1, 1, 1)^T\).