MATH 434
FINAL EXAM

Professor Biswa Datta — Name ______________________

FALL06 — Z-Number ______________________

1. **(30 pts.)** Prove the following norm properties:

   i. $||A^TA||_2^2 = ||A||_2^2$.

   ii. $||A^{-1}||_2 = \dfrac{1}{\sigma_{min}}$

   iii. $||I||_F = \sqrt{n}$, where I is an $n \times n$ identity matrix.

   iv. $||Qx||_2 = ||x||_2$, where Q is an $n \times n$ orthogonal matrix and x is an $n-vector$.

   v. $||OA||_F = ||A||_F$, where O is an orthogonal matrix.

2. (a) **(5 pts.)** Define the backward stability of an algorithm.

(b) **(5 pts.)** Prove that the computation $fl(x(y+z))$ is backward stable.

(c) **(5 pts.)** State a mathematical result to show that Householder's method for QR factorization of a matrix is backward stable.

(d) **(10 pts.)** Define an ill-conditioned problem. Prove that $\mathrm{Cond}_2(A) = \dfrac{\sigma \max}{\sigma_{\min}}$.

(e) **(5 pts.)** Using the result of (d), construct a 3×3 example of an ill-conditioned matrix.

(f) **(10 pts.)** Let $U = (u_{ii})$ be an nonsingular upper triangular matrix. Then show that

$$\mathrm{Cond}_2(U) \geq \frac{\max(u_{ii})}{\min(u_{ii})}$$

3. (a) **(60 pts.)** State a numerically effective algorithm for computing a Householder matrix H such that Hx is a multiple of e_1, where x is an n-vector.

(b) Apply your algorithm to the vector $x = (0, 1, 2)^T$.

(c) Using the result of 3(b), find the QR factorization of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

(d) Find the least-squares solution to $Ax = b$, where A is same as in 3(c), and $b = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$.

(e) Prove that the matrix product HA, where A is an $n \times n$ arbitrary matrix and H is an $n \times n$ Householder matrix, can be computed in $O(n^2)$ flops.

(f) Let A be $m \times n$ and have full-rank. Then prove that the normal equation

$$A^T Ax = A^T b$$

has a unique solution.

4. (a) **(10 pts)** Prove that two similar matrices have the same eigenvalues.

(b) **(5 pts.)** State the real schur triangularization theorem.

(c) **(10 pts.)** Consider the following basic QR Iteration Algorithm:

$$A_0 = A$$
For $k = 1, 2, \cdots$ do
$$A_{k-1} = Q_k R_k$$
$$A_k = R_k Q_k.$$
End

Show that A_k has the same eigenvalues as A.

(d) **(10 pts.)** Apply Geršgorin theorem to find regions of the complete plane containing the eigenvalues of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 9 \\ 1 & 1 & 1 \end{pmatrix}.$$

5. (a) **(35 pts.)** Given

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Compute the followings without explicitly computing the matrix A:

i. **(5 pts.)** $||A||_2 =$

ii. **(5 pts.)** $\text{Cond}_2(A) =$

iii. **(5 pts.)** $||A||_F =$

iv. **(5 pts.)** Singular values of $A =$

v. **(5 pts.)** Singular vectors of $A =$

vi. **(10 pts.)** Find the least-squares solution of $Ax = b$ using SVD of A where A is as given in (a), and $b = (1, 1, 1)^T$.