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There are infinitely many zeros $\rho$ of $\zeta(s)$ with $\Re(s) = 1/2$.

The proof will require some preparatory steps, the first of which is itself a named theorem.

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For all $z \in \mathbb{C}$ with $\Re(z) > 0$ and all $\sigma_0 > 0$,

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Hardy’s Theorem Part I: Mellin’s Theorem

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Our proof of Mellin’s Theorem requires a few technical estimates.

Lemma 3
Suppose $\Re(z)$ and $\sigma_0$ are both positive. Then
$$\lim_{k \to \infty} \int_{\sigma_0 - k} \left| \Gamma(\sigma \pm ik) z - (\sigma \pm ik) \right| d\sigma = 0.$$ 

Proof:
Via Stirling's formula $|\Gamma(s)| \ll |s| s - 1/2 e^{-s}$. Temporarily set $\delta = \pi/2 + |\text{Arg}(z)|^2$. Note that $\delta < \pi/2$ since $\Re(z) > 0$. A bit of computation shows that for $k$ sufficiently large in terms of $\sigma_0$ and $|\text{Arg}(z)|$ we get
$$|\Gamma(\sigma \pm ik)| \leq k \sigma \exp(-k \delta).$$
Also, $|z - (\sigma \pm ik)| = |z| - \sigma \exp(\mp k \text{Arg}(z))$. 

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Via these, we see that

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\int_0^\infty \left| \Gamma(\sigma \pm ik) z^{-k} \right| d\sigma < z, \sigma_0 k \sigma_0 + 1 \exp \left( -k \left( \frac{\pi}{2} - \delta \right) \right),
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Since \( \delta < \frac{\pi}{2} \), the lemma follows.

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Suppose \( \Re(z) > 0 \) and for \( n \in \mathbb{Z} \) set \( k = n + \frac{1}{2} \).

Then

\[
\lim_{n \to \infty} \int_{k-\infty}^{k} \left| \Gamma(-k + it) z^{k - it} \right| dt = 0.
\]

Proof:

We use the identity

\[
\Gamma(s) = \frac{\pi}{\sin(\pi s) \Gamma(1-s)},
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which holds for all \( s \not\in \mathbb{Z} \) by Exercise #16.
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Lemma 5

For all integers $n \leq 0$ the residue of $\Gamma(s)$ at $s = n$ is $(-1)^n/(-n)!$. 

Proof: Via Exercise #16 once more 

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\sin(\pi s) = \pi \Gamma(1-s) 
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Taking the limit as $s \to n$ and using Exercise #14 ($\Gamma(1-n) = (-n)!$ in this case) gives the result.
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Given all the above, by Cauchy’s Theorem we see that
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