Northern Illinois University, Math 680

September 25, 2020
More on Dirichlet Series

Corollary 1

If $D(s)$ is a Dirichlet series, then there is a $\sigma_c$ (possibly $\pm \infty$) such that $D(s)$ converges for all $s = \sigma + it$ with $\sigma > \sigma_c$ and for no $s$ with $\sigma < \sigma_c$.

Further, if $\sigma_0 > \sigma_c$, then there is a neighborhood of $s_0 = \sigma_0 + it_0$ in which $D(s)$ converges uniformly.

Corollary 2

If $D(s)$ is a Dirichlet series with $\sigma_c < \infty$, then $D(s)$ is analytic for all $s = \sigma + it$ with $\sigma > \sigma_c$, and $D'(s) = -\infty \sum_{n=1}^{\infty} a_n \log n n s$ uniformly in the half-plane given by $\sigma > \sigma_c$. 
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Corollary 2

If $D(s)$ is a Dirichlet series with $\sigma_c < \infty$, then $D(s)$ is analytic for all $s = \sigma + it$ with $\sigma > \sigma_c$, and

$$D'(s) = - \sum_{n=1}^{\infty} \frac{a_n \log n}{n^s}$$

uniformly in the half-plane given by $\sigma > \sigma_c$. 
Theorem 3

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$$D(s) = s \int_1^\infty A(x)x^{-(s+1)} \, dx$$

for all $s = \sigma + it$ with $\sigma > 0$. 
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$$\limsup_{x \to \infty} \frac{\log |A(x)|}{\log x} = \sigma_c$$
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for all $s = \sigma + it$ with $\sigma > \sigma_c$. 
Proof:

In place of the equation

\[ N_0 (R(N - 1) - R(N)) = aN \]

we used in the proof of the Theorem from Wednesday, we instead use

\[ A(N) - A(N - 1) = aN. \]

Arguing almost exactly as in the proof of Wednesday's Theorem, we get

\[ \sum_{n=1}^{N} a_n = A(N) \]

\[ -s + \int_{N}^{1} A(x)dx \]

This is analogous to the equation

\[ \sum_{n=M+1}^{N} a_n = R(M) \]

\[ M_0 - s - R(N) \]

\[ \int_{N}^{M} R(u)du \]

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\[ \sum_{n=1}^{N} \frac{a_n}{n^s} = A(N)N^{-s} + s \int_1^N A(x)x^{-(s+1)} \, dx. \]
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This is analogous to the equation

\[ \sum_{n=M+1}^{N} \frac{a_n}{n^s} = R(M)M^{s_0-s} - R(N)N^{s_0-s} + (s_0 - s) \int_{M}^{N} R(u)u^{s_0-s-1} \, du \]

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whenever $$\Re(s) = \sigma > \theta.$$
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By our equation

$$\sum_{n=M+1}^{N} \frac{a_n}{n^s} = R(M)M^{s_0-s} - R(N)N^{s_0-s} + (s_0 - s) \int_{M}^{N} R(u)u^{s_0-s-1} \, du$$

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A(N) = -R(N)N^{\sigma_0} + \sigma_0 \int_{0}^{N} R(u)u^{\sigma_0-1} \, du,
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Definition 1
For two real-valued functions $f(x)$ and $g(x)$ defined and positive on some ray $(a, \infty)$, we write $f(x) \gg g(x)$ and $g(x) \ll f(x)$ if for some $C > 0$ we have $f(x) \geq Cg(x)$ for all $x \in (a, \infty)$. 
By the hypothesis that $\sigma_0 > \sigma_c$, $R(u)$ is bounded as a function of $u$. This shows that $|A(N)| \ll N^{\sigma_0}$ where the implicit constant doesn’t depend on $N$. Whence

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Definition 2

Given a Dirichlet series, the quantity $\sigma_c$ above is called the *abscissa of convergence*. 

The *abscissa of absolute convergence*, $\sigma_a$, is defined to be

$$\sigma_a = \inf \{ \sigma : D(s) \text{ converges absolutely for all } s \in \mathbb{C} \text{ of the form } s = \sigma + it \}.$$ 

Though a Dirichlet series may converge at a given value of $s$, that doesn't imply it converges absolutely. For example, consider the Dirichlet series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^s},$$

which converges (as we've seen) for all $s$ with $\Re(s) > 0$, but only converges absolutely when $\Re(s) > 1$. 


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**Lemma 4**

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Proof: Let \( \epsilon > 0 \), so that \( \sum a_n n^{-\sigma_c - \epsilon} \) converges. This implies that \( |a_n| \ll n^{\sigma_c + \epsilon} \), since the summands must go to zero. Thus \( \sum |a_n| n^{-\sigma_c - 1 - 2\epsilon} \) is convergent, implying that \( \sigma_a \leq \sigma_c + 1 + 2\epsilon \). Since \( \epsilon \) was arbitrary, we see that \( \sigma_a \leq \sigma_c + 1 \).
Theorem 5

Suppose $D(s)$ is a Dirichlet series with $\sigma_c < \infty$.
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$$|D(s)| \ll (1 + |t|)^{1 - \delta + \epsilon}$$

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Proof:

\[ s_0 = \sigma_c + \epsilon \]

in the equation

\[ N \sum_{n=1}^{M+1} a_n s_n = R(M) s_0 - s_0 - R(N) N s_0 + (s_0 - s) \int N M R(u) u \sigma_c + \epsilon - s - 1 \, du. \]

Assuming that \( s = \sigma + \epsilon \) with \( \sigma \geq \sigma_c + \delta > \sigma_c + \epsilon \), we may let \( N \rightarrow \infty \) and get

\[ D(s) - M \sum_{n=1}^{M+1} a_n n s_n = \infty \sum_{n=M+1}^{\infty} a_n n s_n = R(M) M \sigma_c + \epsilon - s - (\sigma_c + \epsilon - s - 1) \int M R(u) u \sigma_c + \epsilon - s - 1 \, du. \]
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$$\left| \sum_{n=1}^{M} \frac{a_n}{n^s} \right| \leq \sum_{n=1}^{M} \frac{|a_n|}{|n^s|}$$
Now $D(\sigma_c + \epsilon)$ is convergent, so we know that $|a_n| \ll n^{\sigma_c + \epsilon}$ and $|R(u)| \ll 1$. Hence

$$\left| \sum_{n=1}^{M} \frac{a_n}{n^s} \right| \leq \sum_{n=1}^{M} \frac{|a_n|}{|n^s|} = \sum_{n=1}^{M} \frac{|a_n|}{n^\sigma}$$
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\left| \sum_{n=1}^{M} \frac{a_n}{n^s} \right| \leq \sum_{n=1}^{M} \frac{|a_n|}{|n^s|} = \sum_{n=1}^{M} \frac{|a_n|}{n^\sigma} \ll \sum_{n=1}^{M} n^{\sigma_c + \epsilon - \sigma}
\]
Now \( D(\sigma_c + \epsilon) \) is convergent, so we know that \( |a_n| \ll n^{\sigma_c + \epsilon} \) and \( |R(u)| \ll 1 \). Hence

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Now $D(\sigma_c + \epsilon)$ is convergent, so we know that $|a_n| \ll n^{\sigma_c + \epsilon}$ and $|R(u)| \ll 1$. Hence

$$\left| \sum_{n=1}^{M} \frac{a_n}{n^s} \right| \leq \sum_{n=1}^{M} \frac{|a_n|}{|n^s|} \leq \sum_{n=1}^{M} \frac{|a_n|}{n^\sigma} \ll \sum_{n=1}^{M} n^{\sigma_c + \epsilon - \sigma} \ll M^{1 + \sigma_c + \epsilon - \sigma} \ll M^{1 + \epsilon - \delta},$$
Now $D(\sigma_c + \epsilon)$ is convergent, so we know that $|a_n| \ll n^{\sigma_c + \epsilon}$ and $|R(u)| \ll 1$. Hence

$$\left| \sum_{n=1}^{M} \frac{a_n}{n^s} \right| \leq \sum_{n=1}^{M} \frac{|a_n|}{|n^s|}$$

$$= \sum_{n=1}^{M} \frac{|a_n|}{n^\sigma}$$

$$\ll \sum_{n=1}^{M} n^{\sigma_c + \epsilon - \sigma}$$

$$\ll M^{1+\sigma_c + \epsilon - \sigma}$$

$$\ll M^{1+\epsilon - \delta}.$$

$$|R(M)M^{\sigma_c + \epsilon - s}| \ll M^{\sigma_c + \epsilon - \sigma} \leq M^{\epsilon - \delta},$$
and
and

\[\left| (\sigma_c + \epsilon - s) \int_{\infty}^{\infty} R(u) u^{\sigma_c + \epsilon - s - 1} \, du \right| \leq |(\sigma_c + \epsilon - s)| \int_{\infty}^{\infty} |R(u)| u^{\sigma_c + \epsilon - \sigma - 1} \, du\]
and

\[
\left| (\sigma_c + \epsilon - s) \int_M^\infty R(u) u^{\sigma_c + \epsilon - s - 1} \, du \right|
\]

\[
\leq |(\sigma_c + \epsilon - s)| \int_M^\infty |R(u)| u^{\sigma_c + \epsilon - \sigma - 1} \, du
\]

\[
\ll (\sigma + |t| - \sigma_c - \epsilon) \int_M^\infty u^{\sigma_c + \epsilon - \sigma - 1} \, du
\]
and

\[
\left| (\sigma_c + \epsilon - s) \int_M^\infty R(u) u^{\sigma_c + \epsilon - s - 1} \, du \right|
\leq \left| (\sigma_c + \epsilon - s) \right| \int_M^\infty |R(u)| u^{\sigma_c + \epsilon - \sigma - 1} \, du
\ll (\sigma + |t| - \sigma_c - \epsilon) \int_M^\infty u^{\sigma_c + \epsilon - \sigma - 1} \, du
= \left( 1 + \frac{|t|}{\sigma - \sigma_c - \epsilon} \right) M^{\sigma_c + \epsilon - \sigma}
\]
\[
\left| (\sigma_c + \epsilon - s) \int_M^\infty R(u) u^{\sigma_c + \epsilon - s - 1} \, du \right|
\]
\[
\leq \left| (\sigma_c + \epsilon - s) \right| \int_M^\infty |R(u)| u^{\sigma_c + \epsilon - \sigma - 1} \, du
\]
\[
\ll (\sigma + |t| - \sigma_c - \epsilon) \int_M^\infty u^{\sigma_c + \epsilon - \sigma - 1} \, du
\]
\[
= \left(1 + \frac{|t|}{\sigma - \sigma_c - \epsilon} \right) M^{\sigma_c + \epsilon - \sigma}
\]
\[
\ll (1 + |t|) M^{\epsilon - \delta},
\]
and

\[ \left| (\sigma_c + \epsilon - s) \int_{M}^{\infty} R(u) u^{\sigma_c+\epsilon-s-1} \, du \right| \]

\[ \leq \left| (\sigma_c + \epsilon - s) \right| \int_{M}^{\infty} |R(u)| u^{\sigma_c+\epsilon-\sigma-1} \, du \]

\[ \ll (\sigma + |t| - \sigma_c - \epsilon) \int_{M}^{\infty} u^{\sigma_c+\epsilon-\sigma-1} \, du \]

\[ = \left( 1 + \frac{|t|}{\sigma - \sigma_c - \epsilon} \right) M^{\sigma_c+\epsilon-\sigma} \]

\[ \ll (1 + |t|) M^{\epsilon-\delta}, \]

where the implicit constant depends on \( \epsilon \) and \( \delta \).
The above together with the equation
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\[ D(s) - \sum_{n=1}^{M} \frac{a_n}{n^s} = \sum_{n=M+1}^{\infty} \frac{a_n}{n^s} = R(M)M^{\sigma_c+\epsilon-s} + (\sigma_c + \epsilon - s) \int_{M}^{\infty} R(u)u^{\sigma_c+\epsilon-s-1} \, du \]

gives
The above together with the equation

\[
D(s) - \sum_{n=1}^{M} \frac{a_n}{n^s} = \sum_{n=M+1}^{\infty} \frac{a_n}{n^s} = R(M)M^{\sigma_c + \epsilon - s} + (\sigma_c + \epsilon - s) \int_{M}^{\infty} R(u)u^{\sigma_c + \epsilon - s - 1} \, du
\]

gives

\[
|D(s)| \ll \left| \sum_{n=1}^{M} \frac{a_n}{n^s} \right| + |R(M)M^{\sigma_c + \epsilon - s}| + \left| (\sigma_c + \epsilon - s) \int_{M}^{\infty} R(u)u^{\sigma_c + \epsilon - s - 1} \, du \right|
\]
The above together with the equation

\[ D(s) - \sum_{n=1}^{M} \frac{a_n}{n^s} = \sum_{n=M+1}^{\infty} \frac{a_n}{n^s} \]

\[ = R(M) M^{\sigma_c + \epsilon - s} + (\sigma_c + \epsilon - s) \int_{M}^{\infty} R(u) u^{\sigma_c + \epsilon - s - 1} \, du \]

gives

\[ |D(s)| \ll \left| \sum_{n=1}^{M} \frac{a_n}{n^s} \right| + |R(M) M^{\sigma_c + \epsilon - s}| \]

\[ + \left| (\sigma_c + \epsilon - s) \int_{M}^{\infty} R(u) u^{\sigma_c + \epsilon - s - 1} \, du \right| \]

\[ \ll M^{1+\epsilon-\delta} + (1 + |t|) M^{\epsilon-\delta}. \]
The above together with the equation

\[ D(s) - \sum_{n=1}^{M} \frac{a_n}{n^s} = \sum_{n=M+1}^{\infty} \frac{a_n}{n^s} \]

\[ = R(M)M^{\sigma_c+\epsilon-s} + (\sigma_c + \epsilon - s) \int_{M}^{\infty} R(u)u^{\sigma_c+\epsilon-s-1} \, du \]

gives

\[ |D(s)| \ll \left| \sum_{n=1}^{M} \frac{a_n}{n^s} \right| + |R(M)M^{\sigma_c+\epsilon-s}| \]

\[ + \left| (\sigma_c + \epsilon - s) \int_{M}^{\infty} R(u)u^{\sigma_c+\epsilon-s-1} \, du \right| \]

\[ \ll M^{1+\epsilon-\delta} + (1 + |t|)M^{\epsilon-\delta}. \]

Setting \( M = [1 + |t|] \) completes the proof.