Math 680 Fall 2017

Chebyshev's Estimates

Here we will prove Chebyshev's estimates for the prime counting function $\pi(x)$. These estimates are superseded by the Prime Number Theorem, of course, but are interesting from both a historical perspective and in the methods involved.

Before we get to the actual estimates themselves, it's useful to recall two relevant definitions from advanced calculus.

Definition: Suppose $\{x_n\}$ is a real-valued sequence. The *limit superior* or *limsup* of the sequence is L and we write $\limsup_{n\to\infty} x_n = L$ if for every $\epsilon > 0$ there is an N such that $x_n < L + \epsilon$ for all $n > N$ and $x_n > L - \epsilon$ for infinitely many $n > N$. In other words, L is an accumulation point for the sequence, and is actually the supremum of all accumulation points of the sequence.

There is an analogous notion for *limit inferior*. Moreover, both can be extended to include $L = \pm\infty$ in the obvious manner.

Lemma: Both the limit inferior and limit superior of a sequence exist and we have $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$. Morever $\lim_{n\to\infty} x_n$ exists if and only if the liminf and limsup are equal, in which case the limit is equal to the liminf.

Recall that the von Mangoldt Lambda function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^e \text{ for some prime } p \text{ and } e \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We will investigate the following two functions that are closely related to the prime counting function $\pi(x)$:

$$\Theta(x) := \sum_{\substack{p\leq x \\ p \text{ prime}}} \log p, \qquad \Psi(x) := \sum_{1\leq n\leq x} \Lambda(n) = \sum_{\substack{p^m\leq x \\ p \text{ prime} \\ m\geq 1}} \log p.$$

Proposition: We have

$$\liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} = \liminf_{x\to\infty} \frac{\Psi(x)}{x} = \liminf_{x\to\infty} \frac{\Theta(x)}{x}$$

and

$$\limsup_{x\to\infty} \frac{\pi(x)}{x/\log x} = \limsup_{x\to\infty} \frac{\Psi(x)}{x} = \limsup_{x\to\infty} \frac{\Theta(x)}{x}.$$

Proof: We first make some obvious observations. Clearly $\Theta(x) \leq \Psi(x)$, so that

(1) $$\limsup_{x\to\infty} \frac{\Psi(x)}{x} \geq \limsup_{x\to\infty} \frac{\Theta(x)}{x}.$$

Also, if p is a prime and $p^m \le x < p^{m+1}$, then $\log p$ occurs in the sum for $\Psi(x)$ exactly m times. Thus

$$\begin{aligned}
\Psi(x) &= \sum_{\substack{p^m \le x \\ p \text{ prime} \\ m \ge 1}} \log p \\
&= \sum_{\substack{p \le x \\ p \text{ prime}}} \left[\frac{\log x}{\log p}\right] \log p \\
&\le \sum_{\substack{p \le x \\ p \text{ prime}}} \log x \\
&= \pi(x) \log x.
\end{aligned}$$

(Here $[\cdot]$ denotes the greatest integer function, as usual.) This immediately implies that

$$\limsup_{x\to\infty} \frac{\Psi(x)}{x} \le \limsup_{x\to\infty} \frac{\pi(x)}{x/\log x}. \tag{2}$$

Now fix an $\alpha \in (0,1)$. Assuming that $x > 1$ we have

$$\Theta(x) = \sum_{\substack{p \le x \\ p \text{ prime}}} \log p \ge \sum_{\substack{x^\alpha < p \le x \\ p \text{ prime}}} \log p.$$

Notice that all primes p occurring in the second sum here necessarily satisfy $\log p > \alpha \log x$. Hence

$$\begin{aligned}
\Theta(x) &> \alpha \log x \sum_{\substack{x^\alpha < p \le x \\ p \text{ prime}}} 1 \\
&= \alpha \log x \big(\pi(x) - \pi(x^\alpha)\big) \\
&> \alpha \log x \big(\pi(x) - x^\alpha\big),
\end{aligned}$$

so that

$$\frac{\Theta(x)}{x} > \frac{\alpha \pi(x)}{x/\log x} - \frac{\alpha \log x}{x^{1-\alpha}}. \tag{3}$$

But for all $\alpha \in (0,1)$ we have

$$\lim_{x\to\infty} \frac{\alpha \log x}{x^{1-\alpha}} = 0.$$

This observation together with (3) yields

$$\limsup_{x\to\infty} \frac{\Theta(x)}{x} \ge \limsup_{x\to\infty} \frac{\alpha \pi(x)}{x/\log x}.$$

Since this is true for all $\alpha \in (0,1)$, we get

$$\limsup_{x\to\infty} \frac{\Theta(x)}{x} \ge \limsup_{x\to\infty} \frac{\pi(x)}{x/\log x}. \tag{4}$$

We now turn to the liminfs. Similar to (1) and (2), we have

(1')
$$\liminf_{x\to\infty} \frac{\Psi(x)}{x} \geq \liminf_{x\to\infty} \frac{\Theta(x)}{x}$$

and

(2')
$$\liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} \geq \liminf_{x\to\infty} \frac{\Psi(x)}{x}.$$

Using (3) once more, we get

(4')
$$\liminf_{x\to\infty} \frac{\Theta(x)}{x} \geq \liminf_{x\to\infty} \frac{\pi(x)}{x/\log x}.$$

All together, (1), (2), (4), (1'), (2') and (4') combine to prove the proposition.

Theorem(Chebyshev): We have

$$4\log 2 \geq \limsup_{x\to\infty} \frac{\pi(x)}{x/\log x} \geq \liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} \geq \log 2.$$

In particular, there exist positive real numbers m and M such that

$$M > \frac{\pi(x)}{x/\log x} > m$$

for all x sufficiently large.

Proof: We will consider the binomial coefficient

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \frac{(2n)(2n-1)\cdots(n+1)}{n!}$$

for positive integers n. We note that

$$2^{2n} = (1+1)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} > \binom{2n}{n}.$$

Moreover, $\binom{2n}{n}$ is the greatest summand here, so that

$$2^{2n} < (2n+1)\binom{2n}{n}.$$

Therefore

(5)
$$2^{2n} > \binom{2n}{n} > \frac{2^{2n}}{2n+1}.$$

Now suppose p is a prime satisfying $n < p \leq 2n$. Such a prime is clearly not a factor of $n!$, but is a factor of $(2n)!$. Thus every such prime is a divisor of $\binom{2n}{n}$. In particular, we have

$$\binom{2n}{n} \geq \prod_{\substack{n<p\leq 2n \\ p \text{ prime}}} p$$

and

$$\log\left(\binom{2n}{n}\right) \geq \sum_{\substack{n<p\leq 2n \\ p \text{ prime}}} \log p = \Theta(2n) - \Theta(n).$$

(Here, as usual, empty products are interpreted to be 1 and empty sums are interpreted to be zero.) Combining this with (5) gives

$$\Theta(2n) - \Theta(n) < 2n\log 2. \tag{6}$$

Next for all positive integers m we see via (6) that

$$\begin{aligned}
\Theta(2^m) = \Theta(2^m) - \Theta(1) &= \sum_{i=0}^{m-1} \Theta(2\cdot 2^i) - \Theta(2^i) \\
&< \sum_{i=0}^{m-1} 2\cdot 2^i \log 2 \\
&= 2\log 2 \sum_{i=0}^{m-1} 2^i \\
&= 2\log 2(2^m - 1) \\
&< 2^{m+1}\log 2.
\end{aligned}$$

We note that $\Theta(x)$ is a non-decreasing function. Thus for any $x > 1$ we may write $2^{m-1} < x \leq 2^m$ for some positive integer m and get

$$\Theta(x) \leq \Theta(2^m) < 2^{m+1}\log 2 \leq 4x\log 2.$$

Now by the Proposition

$$\limsup_{x\to\infty} \frac{\pi(x)}{x/\log x} = \limsup_{x\to\infty} \frac{\Theta(x)}{x} \leq 4\log 2.$$

We next consider the sum

$$S(x) := \sum_{1\leq n\leq x} \log n - 2\sum_{1\leq n\leq x/2} \log n.$$

Recalling that $\log n = \sum_{d|n} \Lambda(d)$, we have

$$S(x) = \sum_{1\leq n\leq x} \sum_{d|n} \Lambda(d) - 2\sum_{1\leq n\leq x/2} \sum_{d|n} \Lambda(d).$$

Writing $x = qd + r$ with $0 \leq r < d$, we see that $\{d,\ 2d, \dots, qd\}$ is the set of n satisfying $1 \leq n \leq x$ and $d|n$. This implies that

$$\begin{aligned}
S(x) &= \sum_{1\leq d\leq x} \Lambda(d)\left[\frac{x}{d}\right] - 2\sum_{1\leq d\leq x/2} \Lambda(d)\left[\frac{x}{2d}\right] \\
&= \sum_{1\leq d\leq x/2} \Lambda(d)\left(\left[\frac{x}{d}\right] - 2\left[\frac{x}{2d}\right]\right) + \sum_{(x/2)<d\leq x} \Lambda(d)\left[\frac{x}{d}\right] \\
&\leq \sum_{1\leq d\leq x/2} \Lambda(d) + \sum_{(x/2)<d\leq x} \Lambda(d) \\
&= \Psi(x).
\end{aligned}$$

Hence

$$\frac{\Psi(x)}{x} \geq \frac{S(x)}{x} = \frac{1}{x} \sum_{1 \leq n \leq x} \log n - \frac{2}{x} \sum_{1 \leq n \leq x/2} \log n. \tag{7}$$

Now since $\log t$ is an increasing function,

$$\int_1^{x+1} \log t \, dt \geq \sum_{1 \leq n \leq x} \log n$$

and

$$\int_1^{x} \log t \, dt \leq \sum_{1 \leq n \leq x} \log n.$$

In particular,

$$\begin{aligned} \frac{S(x)}{x} &\geq \frac{1}{x} \int_1^{x} \log t \, dt - \frac{2}{x} \int_1^{(x/2)+1} \log t \, dt \\ &= \frac{1}{x}\big(x \log x - x + 1\big) - \frac{2}{x}\left(\frac{x+2}{2} \log\big((x+2)/2\big) - \frac{x+2}{2} + 1\right) \\ &= \log x + \frac{1}{x} - \frac{x+2}{x} \log(x+2) + \frac{x+2}{x} \log 2 \\ &> \log\big(x/(x+2)\big) - \frac{2}{x} \log(x+2) + \log 2. \end{aligned}$$

Combining this with (7) and the Proposition gives

$$\begin{aligned} \liminf_{x \to \infty} \frac{\pi(x)}{x/\log x} &= \liminf_{x \to \infty} \frac{\Psi(x)}{x} \\ &\geq \liminf_{x \to \infty} \frac{S(x)}{x} \\ &> \lim_{x \to \infty} \log\big(x/(x+2)\big) - \frac{2}{x} \log(x+2) + \log 2 \\ &= \log 2. \end{aligned}$$