Math 680 Fall 2017

A Particular Infinite Product

Let $\{p_i\}$ be the sequence of primes in ascending order: $p_1 = 2,\ p_2 = 3$, etc... Note that $p_{n+1} \geq 2 + p_n$ for all $n > 1$. In particular, $p_n > n + 1$ for all $n > 1$.

For a complex number s we consider the infinite product

$$\begin{aligned} P(s) :&= \prod_{n=1}^{\infty} (1 - p_n^{-s})^{-1} \\ &= \prod_{n=1}^{\infty} \left(\frac{1 - p_n^{-s} + p_n^{-s}}{1 - p_n^{-s}} \right) \\ &= \prod_{n=1}^{\infty} \left(1 + \frac{p_n^{-s}}{1 - p_n^{-s}} \right) \\ &= \prod_{n=1}^{\infty} \left(1 + \frac{1}{p_n^s - 1} \right). \end{aligned}$$

Setting $u_n(s) = (p_n^s - 1)^{-1}$, we have

$$|u_n(s)| = \frac{1}{|p_n^s - 1|} \leq \frac{1}{|p_n^s| - 1}.$$

Write $s = \sigma + it$ as before. Then

$$|p_n^s| = p_n^\sigma > (n+1)^\sigma > n^\sigma + 1$$

for all $n > 1$. In particular, $|u_n(s)| < n^{-\sigma}$ for all $n > 1$. Thus, by a previous result the infinite product $P(s)$ converges absolutely whenever $\Re(s) > 1$.

Denote the n^{th} partial product by

$$P_n(s) := (1 - p_1^{-s})^{-1} \cdots (1 - p_n^{-s})^{-1}.$$

Recalling that $\sum_{m \geq 0} x^m = (1 - x)^{-1}$ whenever $|x| < 1$, we see that

$$P_n(s) = \left(\sum_{m=0}^{\infty} p_1^{-ms} \right) \cdots \left(\sum_{m=0}^{\infty} p_n^{-ms} \right)$$

whenever $\Re(s) > 1$. Note that each factor here is absolutely convergent whenever $\Re(s) > 1$, in which case we may rearrange the product any way we deem useful. Set J_n to be the subset of positive integers m such that all prime factors of m are in the set $\{p_1, \ldots, p_n\}$. We then have

$$P_n(s) = \sum_{m \in J_n} m^{-s}.$$

We clearly have (with $\sigma = \Re(s)$)

$$
\begin{aligned}
|\zeta(s) - P_n(s)| &= \left| \sum_{\substack{m\geq 1 \\ m\notin J_n}} m^{-s} \right| \\
&\leq \sum_{\substack{m\geq 1 \\ m\notin J_n}} m^{-\sigma} \\
&< \sum_{m\geq p_{n+1}} m^{-\sigma} \\
&\leq \sum_{m\geq n+2} m^{-\sigma},
\end{aligned}
$$

so that

$$\lim_{n\to\infty} |\zeta(s) - P_n(s)| = 0$$

for all s with $\sigma > 1$.

Theorem (Euler Product): For all $s = \sigma + it$ with $\sigma > 1$ we have

$$\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.$$

This product is absolutely convergent for such s. Further, for all $\sigma_0 > 1$, the product is uniformly and absolutely convergent on the set of $s = \sigma + it$ with $\sigma \geq \sigma_0$, $\zeta(s) \neq 0$ on this set, and

$$\frac{-\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Proof: By our results for infinite products, all that remains is to compute the term-wise derivative of the logarithm. We have

$$
\begin{aligned}
\log\left(\prod_{p \text{ prime}} (1-p^{-s})^{-1} \right) &= -\sum_{p \text{ prime}} \log(1-p^{-s}) \\
&= \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{p^{-ms}}{m}.
\end{aligned}
$$

Taking derivatives term-wise yields

$$
\begin{aligned}
\frac{\zeta'(s)}{\zeta(s)} &= \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{dp^{-ms}/m}{ds} \\
&= -\sum_{p \text{ prime}} \sum_{m=1}^{\infty} p^{-ms} \log p \\
&= -\sum_{\substack{n=p^m \\ p \text{ prime} \\ m\geq 1}} \frac{\log(p)}{n^s} \\
&= -\sum_{n\geq 1} \frac{\Lambda(n)}{n^s}.
\end{aligned}
$$

We've seen previously that $\zeta(s)$ is analytic on the right half-plane $\Re(s) > 0$ with the sole exception of a simple pole at $s = 1$ where the residue is 1. This implies that $\zeta(s) - (s-1)^{-1}$ is analytic on the right half-plane, which in turn shows that $\zeta'(s) + (s-1)^{-2}$ is analytic there. These two facts combine to show that the quotient $-\zeta'(s)/\zeta(s) \to \infty$ as $s \to 1$. Now via the Euler product we have

$$\sum_{n \geq 1} \frac{\Lambda(n)}{n^\sigma} \to \infty$$

as $\sigma \to 1^+$. With this in hand, it is rather easy to see that

$$\sum_{p \text{ prime}} \frac{\log p}{p^\sigma} \to \infty$$

as $\sigma \to 1^+$, trivially implying that there are infinitely many primes. This argument may seem ridiculously convoluted, but it captures some core ideas. In particular, we will use a similar argument to prove Dirichlet's theorem on primes in an arithmetic progression: for any $m > 1$ and a relatively prime to m,

$$\lim_{\sigma \to 1^+} \sum_{\substack{p \text{ prime} \\ p \equiv a \mod m}} \frac{\log p}{p^\sigma} = \infty,$$

so that there are infinitely many primes p in the arithmetic progression a modulo m.