**Math 680 Fall 2017**

A Theorem of Hardy

Our goal here is to prove the following theorem due to Hardy.

**Theorem** (Hardy): There are infinitely many zeros $\rho$ of $\zeta(s)$ with $\Re(s) = 1/2$.

The proof will require some preparatory steps.

**Theorem** (Mellin): For all $z \in \mathbb{C}$ with $\Re(z) > 0$ and all $\sigma_0 > 0$,

$$\frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s) z^{-s}\, ds = e^{-z}.$$

Proof: For any positive rational number of the form $k = n + 1/2$, $n \in \mathbb{Z}$, consider the contour integral

$$\frac{1}{2\pi i}\oint_{R_k} \Gamma(s) z^{-s}\, ds$$

where $R_k$ is the rectangle with vertices $-k \pm ik$ and $\sigma_0 \pm ik$. Note that the only poles of the integrand come from poles of the Gamma function, and we know these are located along the real axis at 0 and the negative integers. In particular, due to our choice of $k$, our contour integral avoids these poles. By exercise 26, the integrals along the top, bottom, and left sides tend to zero as $k \to \infty$. Given this, by Cauchy's Theorem and exercise 27 we see that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \Gamma(s) z^{-s}\, ds &= \sum_{n \le 0} \operatorname{res}_{s=n}\big(\Gamma(s) z^{-s}\big) \\
&= \sum_{n \le 0} z^{-n} \frac{(-1)^n}{(-n)!} \\
&= \sum_{m \ge 0} z^{m} \frac{(-1)^m}{m!} \\
&= e^{-z}.
\end{aligned}$$

**Lemma** 1: If $\Re(z) > 0$ and $\sigma_0 > 1$, then

$$\frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \zeta(s)\Gamma(s/2)(\pi z)^{-s/2}\, ds = 2\sum_{n \ge 1} e^{-\pi n^2 z}.$$

Proof: Set $w = \pi n^2 z$ (and use the obvious change of variables) in Mellin's Theorem above to get

$$\begin{aligned}
2e^{-\pi n^2 z} &= 2e^{-w} \\
&= \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \Gamma(s/2) w^{-s/2}\, ds \\
&= \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \Gamma(s/2)(\pi z)^{-s/2} \frac{1}{n^s}\, ds,
\end{aligned}$$

whence

$$2\sum_{n\geq 1} e^{-\pi n^2 z} = \sum_{n\geq 1} \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0+i\infty} \Gamma(s/2)(\pi z)^{-s/2}\frac{1}{n^s}\,ds.$$

By Stirling's formula $|\Gamma\big((\sigma_0+it)/2\big)| \gg\ll (|t|/2)^{(\sigma_0-1)/2}e^{-\pi|t|/4}$ for $|t|\geq 1$, say. Writing $z = re^{i\theta}$ we get $|\Gamma\big((\sigma_0+it)/2\big)(\pi z)^{-(\sigma_0+it)/2}| \ll |t|^{(\sigma_0-1)/2}e^{-\pi|t|/4}e^{t\theta/2}$ for $|t|\geq 1$, where the implicit constant now depends on $\sigma_0$ and $|z|$ but is independent of $t$. Since $\Re(z) > 0$ we have $-\pi/2 < \theta < \pi/2$. We thus see that

$$\int_{-\infty}^{\infty} |\Gamma\big((\sigma_0+it)/2\big)||(\pi z)^{-(\sigma_0+it)/2}|\,dt$$

is convergent. Now since $\sigma_0 > 1$ we may interchange the summation and the integration above to get the lemma.

**Lemma** 2: For $T \geq 2$

$$\int_1^T \zeta(1/2+it)\,dt = T + O(T^{1/2}),$$

where the implicit constant is absolute.

Proof: Let $R$ be the rectangle with vertices $1/2+i,\ 2+i,\ 1/2+iT$, and $2+iT$. Since $\zeta(s)$ is analytic in the complex plane except for the pole at $s=1$, it is analytic within this rectangle, so that

(1)
$$\oint_R \zeta(s)\,ds = 0.$$

Obviously the integral along the bottom of the rectangle

$$\int_{1/2}^{2} \zeta(\sigma+i)\,d\sigma$$

is a constant. For the top, using exercise 28 we have

$$\begin{aligned}\left|\int_{1/2}^{2}\zeta(\sigma+iT)\,d\sigma\right| &\leq \int_{1/2}^{2}|\zeta(\sigma+iT)|\,d\sigma\\ &\ll \int_{1/2}^{2}(1+T^{1-\sigma})\log T\,d\sigma\\ &\ll \log T + \int_{1/2}^{2} T^{1-\sigma}\log T\,d\sigma\\ &= \log T - T^{1-\sigma}\Big|_{\sigma=1/2}^{2}\\ &\ll T^{1/2}.\end{aligned}$$

For the integral along the right side of $R$ we have

$$\begin{aligned}
\int_1^T \zeta(2+it)\,dt &= \int_1^T \sum_{n\geq 1} \frac{1}{n^{2+it}}\,dt \\
&= \sum_{n\geq 1} \frac{1}{n^2} \int_1^T n^{-it}\,dt \\
&= \int_1^T 1\,dt + \sum_{n\geq 2} \frac{1}{n^2} \int_1^T n^{-it}\,dt \\
&= (T-1) + \sum_{n\geq 2} \frac{n^{-i} - n^{-iT}}{in^2 \log n} \\
&= T + O(T^{1/2})
\end{aligned}$$

The lemma follows from these estimates and (1).

Proof of Theorem: Fix a $T$ for the moment (a "large" $T$) and consider the contour integral

$$\frac{1}{2\pi i} \oint_C \zeta(s)\Gamma(s/2)(\pi z)^{-s/2}\,ds,$$

where $C$ is the rectangle with vertices $1/2 \pm iT$ and $2 \pm iT$. As in Lemma 1, we will assume $z$ is satisfies $\Re(z) > 0$. In particular, $z \neq 0$ so that the $(\pi z)^{-s/2}$ portion of the integrand has no poles. Certainly the Gamma factor has no poles within the contour, so that we just have the one (simple) pole at $s = 1$. Since the residue of $\zeta(s)$ at $s = 1$ is 1, we get

$$\frac{1}{2\pi i} \oint_C \zeta(s)\Gamma(s/2)(\pi z)^{-s/2}\,ds = \Gamma(1/2)(\pi z)^{-1/2} = z^{-1/2}.$$

Estimating with Stirling's formula exactly as in the proof of Lemma 1 and using exercise 28 to estimate the zeta function, we see that the integrals along the top and bottom portion tend to 0 as $T \to \infty$. Thus

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(s)\Gamma(s/2)(\pi z)^{-s/2}\,ds - \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \zeta(s)\Gamma(s/2)(\pi z)^{-s/2}\,ds = z^{-1/2}.$$

Via this and Lemma 1,

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta(1/2+it)\Gamma(1/4+it/2)\pi^{-1/4-it/2} z^{-it/2} &= \frac{z^{1/4}}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \zeta(s)\Gamma(s/2)(\pi z)^{-s/2}\,ds \\
&= \frac{z^{1/4}}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(s)\Gamma(s/2)(\pi z)^{-s/2}\,ds - z^{-1/4} \qquad (2) \\
&= z^{1/4} 2 \sum_{n=1}^{\infty} e^{-\pi n^2 z} - z^{-1/4}.
\end{aligned}$$

For notational convenience set $\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(s/2)\pi^{-s/2}$; this is handy since the functional equation now reads

$$\xi(s) = \xi(1-s). \qquad (3)$$

Note that for $s = \sigma > 0$

$$\xi(s) = \xi(\sigma) = \frac{1}{2}\sigma(\sigma-1)\zeta(\sigma)\Gamma(\sigma/2)\pi^{-\sigma/2} = \frac{1}{2}\sigma(\sigma-1)(1-2^{1-\sigma})^{-1}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^\sigma}\Gamma(\sigma/2)\pi^{-\sigma/2} \in \mathbb{R}.$$

Since this holds for all positive $\sigma$, the functional equation (3) imples that $\xi(\sigma) \in \mathbb{R}$ for all $\sigma \in \mathbb{R}$. Hence by the Schwartz reflection principle $\xi(\overline{s}) = \overline{\xi(s)}$, so that in particular

$$\xi(1/2+it) = \overline{\xi(1/2-it)} = \overline{\xi\big(1-(1/2-it)\big)} = \overline{\xi(1/2+it)}.$$

In other words, $\xi(s) \in \mathbb{R}$ whenever $\Re(s) = 1/2$. But for $s = 1/2 + it$ we also have

$$s(s-1) = (1/2+it)(it-1/2) = -(t^2+1/4) \in \mathbb{R}.$$

We therefore conclude that the $\zeta(1/2+it)\Gamma(1/4+it/2)\pi^{-1/4-it/4}$ portion of the integrand on the left side of (2) is real.

For real $t$ write

$$A(t) = \frac{\zeta(1/2+it)\Gamma(1/4+it/2)\pi^{-1/4-it/2}}{|\Gamma(1/4+it/2)\pi^{-1/4-it/2}|}, \qquad B(t) = \frac{|\Gamma(1/4+it/2)\pi^{-1/4-it/2}|}{z^{it/2}}.$$

We saw above that the numerator in $A(t)$ is real, so that $A(t) \in \mathbb{R}$. We set $z = e^{i\theta}$ for $\theta \in \mathbb{R}$ to be determined. Then $z^{it/2} = e^{-\theta t/2} \in \mathbb{R}$, so that $B(t) \in \mathbb{R}$ as well. We now have

(4) $$\int_{-\infty}^{\infty} \zeta(1/2+it)\Gamma(1/4+it/2)\pi^{-1/4-it/2}z^{-it/2}\,dt = \int_{-\infty}^{\infty} A(t)B(t)\,dt,$$

where $A(t), B(t) \in \mathbb{R}$ with $B(t) > 0$ assuming that $z$ is of the form $z = e^{i\theta}$. Further, we clearly see that $A(t) = 0$ in the integrand precisely when $\zeta(1/2+it) = 0$.

We now take $\theta = \pi/2 - \delta$ for a "small" $\delta$ (note that this implies $\Re(z) > 0$, as required). Then $z^{-it/2} = e^{(\pi/2-\delta)t/2}$. By Stirling's formula we have $|\Gamma(1/4+it/2)| \gg\ll |t|^{-1/4}e^{-\pi|t|/4}$ for "large" $|t|$. In particular, we use this for $1/2\delta \le t \le 1/\delta$ to see that $B(t) \gg \delta^{1/4}$ for $1/2\delta \le t \le 1/\delta$. Recalling that $B(t) > 0$, we have by the above (with sufficiently small $\delta$)

(5) $$\begin{aligned}\int_{-\infty}^{\infty} |A(t)|B(t)\,dt &\ge \int_{1/2\delta}^{1/\delta} |A(t)|B(t)\,dt \\ &\gg \delta^{1/4}\int_{1/2\delta}^{1/\delta} |\zeta(1/2+it)|\,dt \\ &\gg \delta^{-3/4}.\end{aligned}$$

For the last inequality here we are using Lemma 2 with $T = 1/\delta$ and $T = 1/2\delta$, with $1/\delta$ "large" (i.e., $\delta$

"small"). On the other hand, $|z| \gg\ll 1$ since $\delta$ is "small", so that by (2) and (4) (setting $v = u\sqrt{\pi\sin(\delta)}$)

$$\begin{aligned} \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} A(t)B(t)\,dt\right| &= \left|-z^{-1/4} + z^{1/4}2\sum_{n=1}^{\infty} e^{-\pi n^2 z}\right| \\ &\ll 1 + \sum_{n\geq 1} |e^{-\pi n^2 z}| \\ &= 1 + \sum_{n\geq 1} e^{-\pi n^2 \cos(\pi/2-\delta)} \\ &= 1 + \sum_{n\geq 1} e^{-\pi n^2 \sin(\delta)} \\ &\leq 1 + \int_0^{\infty} e^{-\pi u^2 \sin(\delta)}\,du \\ &= 1 + \frac{1}{\sqrt{\pi\sin(\delta)}}\int_0^{\infty} e^{-v^2}\,dv \\ &\ll \delta^{-1/2}. \end{aligned} \tag{6}$$

Finally, we suppose by contradiction that there are only finitely many zeros $\rho$ of the zeta function with $\Re(\rho) = 1/2$. In particular, for $C$ sufficently large $A(t)$ does not change sign for $|t| \geq C$. Thus

$$\int_{|t|\geq C} |A(t)|B(t)\,dt = \left|\int_{|t|\geq C} A(t)B(t)\,dt\right|.$$

Since $C$ is fixed at this point, we get

$$\left|\int_{-\infty}^{\infty} A(t)B(t)\,dt\right| = \int_{-\infty}^{\infty} |A(t)|B(t)\,dt + O(1),$$

where the implicit constant is independent of $\delta$ (it will depend on $C$). Since this equation contradicts our inequalities (5) and (6) for sufficiently small $\delta$, we conclude that there must be infinitely many zeros $\rho$ with $\Re(\rho) = 1/2$.