Our goal here is to prove the following theorem due to Hardy.

**Theorem (Hardy):** There are infinitely many zeros $\rho$ of $\zeta(s)$ with $\Re(s) = 1/2$.

The proof will require some preparatory steps.

**Theorem (Mellin):** For all $z \in \mathbb{C}$ with $\Re(z) > 0$ and all $\sigma_0 > 0$,

$$
\frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Gamma(s) z^{-s} \, ds = e^{-z}.
$$

**Proof:** For any positive rational number of the form $k = n + 1/2$, $n \in \mathbb{Z}$, consider the contour integral

$$
\frac{1}{2\pi i} \oint_{R_k} \Gamma(s) z^{-s} \, ds
$$

where $R_k$ is the rectangle with vertices $-k \pm ik$ and $\sigma_0 \pm ik$. Note that the only poles of the integrand come from poles of the Gamma function, and we know these are located along the real axis at 0 and the negative integers. In particular, due to our choice of $k$, our contour integral avoids these poles. By exercise 26, the integrals along the top, bottom, and left sides tend to zero as $k \to \infty$. Given this, by Cauchy’s Theorem and exercise 27 we see that

$$
\frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Gamma(s) z^{-s} \, ds = \sum_{n \leq 0} \text{res}_{s=n} (\Gamma(s) z^{-s})
$$

$$
= \sum_{n \leq 0} z^{-n} \frac{(-1)^n}{(-n)!}
$$

$$
= \sum_{m \geq 0} z^m \frac{(-1)^m}{m!}
$$

$$
= e^{-z}.
$$

**Lemma 1:** If $\Re(z) > 0$ and $\sigma_0 > 1$, then

$$
\frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \zeta(s) \Gamma(s/2)(\pi z)^{-s/2} \, ds = 2 \sum_{n \geq 1} e^{-\pi n^2 z}.
$$

**Proof:** Set $w = \pi n^2 z$ (and use the obvious change of variables) in Mellin’s Theorem above to get

$$
2e^{-\pi n^2 z} = 2e^{-w}
$$

$$
= \frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Gamma(s/2) w^{-s/2} \, ds
$$

$$
= \frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Gamma(s/2)(\pi z)^{-s/2} \frac{1}{n^s} \, ds,
$$
whence
\[ 2 \sum_{n \geq 1} e^{-\pi n^2 z} = \sum_{n \geq 1} \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \Gamma(s/2)(\pi z)^{-s/2} \frac{1}{n^s} \, ds. \]

By Stirling’s formula \(|\Gamma((\sigma_0 + it)/2)| \gg (|t|/2)^{(\sigma_0-1)/2}e^{-\pi|t|^2/4} \) for \(|t| \geq 1\), say. Writing \(z = re^{i\theta}\) we get \(|\Gamma((\sigma_0 + it)/2)(\pi z)^{-(\sigma_0+it)/2}| \ll |t|(|\sigma_0-1|/2)e^{-\pi|t|^2/4}e^{i\theta/2} \) for \(|t| \geq 1\), where the implicit constant now depends on \(\sigma_0\) and \(|z|\) but is independent of \(t\). Since \(\Re(z) > 0\) we have \(-\pi/2 < \theta < \pi/2\). We thus see that
\[ \int_{-\infty}^{\infty} |\Gamma((\sigma_0 + it)/2)||(\pi z)^{-(\sigma_0+it)/2}| \, dt \]
is convergent. Now since \(\sigma_0 > 1\) we may interchange the summation and the integration above to get the lemma.

**Lemma 2:** For \(T \geq 2\)
\[ \int_{1/2}^{T} \zeta(1/2 + it) \, dt = T + O(T^{1/2}), \]
where the implicit constant is absolute.

**Proof:** Let \(R\) be the rectangle with vertices \(1/2 + i, 2 + i, 1/2 + iT,\) and \(2 + iT\). Since \(\zeta(s)\) is analytic in the complex plane except for the pole at \(s = 1\), it is analytic within this rectangle, so that
\[ (1) \quad \oint_R \zeta(s) \, ds = 0. \]

Obviously the integral along the bottom of the rectangle
\[ \int_{1/2}^{2} \zeta(\sigma + i) \, d\sigma \]
is a constant. For the top, using exercise 28 we have
\[
\left| \int_{1/2}^{2} \zeta(\sigma + iT) \, d\sigma \right| \leq \int_{1/2}^{2} |\zeta(\sigma + iT)| \, d\sigma
\ll \int_{1/2}^{2} (1 + T^{1-\sigma}) \log T \, d\sigma
\ll \log T + \int_{1/2}^{2} T^{1-\sigma} \log T \, d\sigma
= \log T - T^{1-\sigma} \big|_{\sigma=1/2}^{\sigma=2}
\ll T^{1/2}.
\]
For the integral along the right side of $R$ we have
\[
\int_1^T \zeta(2 + it) \, dt = \int_1^T \sum_{n \geq 1} \frac{1}{n^{2+it}} \, dt \\
= \sum_{n \geq 1} \frac{1}{n^2} \int_1^T \frac{t}{n^2} \, dt \\
= \int_1^T 1 \, dt + \sum_{n \geq 2} \frac{1}{n^2} \int_1^T \frac{t}{n^2} \, dt \\
= (T - 1) + \sum_{n \geq 2} \frac{n^{-i} - n^{-iT}}{in^2 \log n} \\
= T + O(T^{1/2})
\]

The lemma follows from these estimates and (1).

Proof of Theorem: Fix a $T$ for the moment (a “large” $T$) and consider the contour integral
\[
\frac{1}{2\pi i} \oint_C \zeta(s) \Gamma(s/2)(\pi z)^{-s/2} \, ds,
\]
where $C$ is the rectangle with vertices $1/2 \pm iT$ and $2 \pm iT$. As in Lemma 1, we will assume $z$ is satisfies $\Re(z) > 0$. In particular, $z \neq 0$ so that the $(\pi z)^{-s/2}$ portion of the integrand has no poles. Certainly the Gamma factor has no poles within the contour, so that we just have the one (simple) pole at $s = 1$. Since the residue of $\zeta(s)$ at $s = 1$ is 1, we get
\[
\frac{1}{2\pi i} \oint_C \zeta(s) \Gamma(s/2)(\pi z)^{-s/2} \, ds = \Gamma(1/2)(\pi z)^{-1/2} = z^{-1/2}.
\]

Estimating with Stirling’s formula exactly as in the proof of Lemma 1 and using exercise 28 to estimate the zeta function, we see that the integrals along the top and bottom portion tend to 0 as $T \to \infty$. Thus
\[
\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(s) \Gamma(s/2)(\pi z)^{-s/2} \, ds - \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \zeta(s) \Gamma(s/2)(\pi z)^{-s/2} \, ds = z^{-1/2}.
\]

Via this and Lemma 1,
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta(1/2 + it) \Gamma(1/4 + it/2) \pi^{-1/4-it/2} z^{-it/2} \, ds = \frac{z^{1/4}}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \zeta(s) \Gamma(s/2)(\pi z)^{-s/2} \, ds \\
= \frac{z^{1/4}}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(s) \Gamma(s/2)(\pi z)^{-s/2} \, ds - z^{-1/4} \\
= z^{1/4} 2 \sum_{n=1}^{\infty} e^{-\pi n^2 z} - z^{-1/4}.
\]

For notational convenience set $\zeta(s) = \frac{1}{2} s(s-1) \zeta(s) \Gamma(s/2) \pi^{-s/2}$; this is handy since the functional equation now reads
\[
\zeta(s) = \xi(1-s).
\]
Note that for \( s = \sigma > 0 \)
\[
\xi(s) = \xi(\sigma) = \frac{1}{2} \sigma(\sigma - 1) \zeta(\sigma) \Gamma(\sigma/2) \pi^{-\sigma/2} = \frac{1}{2} \sigma(\sigma - 1)(1 - 2^{1-\sigma})^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^\sigma} \Gamma(\sigma/2) \pi^{-\sigma/2} \in \mathbb{R}.
\]
Since this holds for all positive \( \sigma \), the functional equation (3) implies that \( \xi(\sigma) \in \mathbb{R} \) for all \( \sigma \in \mathbb{R} \). Hence by the Schwartz reflection principle \( \xi(\overline{\sigma}) = \overline{\xi(\sigma)} \), so that in particular
\[
\xi(1/2 + it) = \overline{\xi(1/2 - it)} = \overline{\xi(1 - (1/2 - it))} = \overline{\xi(1/2 + it)}.
\]
In other words, \( \xi(s) \in \mathbb{R} \) whenever \( \Re(s) = 1/2 \). But for \( s = 1/2 + it \) we also have
\[
s(s - 1) = (1/2 + it)(it - 1/2) = -(t^2 + 1/4) \in \mathbb{R}.
\]
We therefore conclude that the \( \zeta(1/2 + it)\Gamma(1/4 + it/2)\pi^{-1/4-it/4} \) portion of the integrand on the left side of (2) is real.

For real \( t \) write
\[
A(t) = \frac{\zeta(1/2 + it)\Gamma(1/4 + it/2)\pi^{-1/4-it/2}}{|\Gamma(1/4 + it/2)\pi^{-1/4-it/2}|}, \quad B(t) = \frac{|\Gamma(1/4 + it/2)\pi^{-1/4-it/2}|}{\pi^{it/2}}.
\]
We saw above that the numerator in \( A(t) \) is real, so that \( A(t) \in \mathbb{R} \). We set \( z = e^{i\theta} \) for \( \theta \in \mathbb{R} \) to be determined. Then \( z^{it/2} = e^{-\theta t/2} \in \mathbb{R} \), so that \( B(t) \in \mathbb{R} \) as well. We now have
\[
(1) \quad \int_{-\infty}^{\infty} \zeta(1/2 + it)\Gamma(1/4 + it/2)\pi^{-1/4-it/2} z^{-it/2} dt = \int_{-\infty}^{\infty} A(t)B(t) dt,
\]
where \( A(t), B(t) \in \mathbb{R} \) with \( B(t) > 0 \) assuming that \( z \) is of the form \( z = e^{i\theta} \). Further, we clearly see that \( A(t) = 0 \) in the integrand precisely when \( \zeta(1/2 + it) = 0 \).

We now take \( \theta = \pi/2 - \delta \) for a “small” \( \delta \) (note that this implies \( \Re(z) > 0 \), as required). Then \( z^{-it/2} = e^{(\pi/2-\delta)t/2} \). By Stirling’s formula we have \( |\Gamma(1/4 + it/2)| \ll |t|^{-1/4} e^{-\pi|t|/4} \) for “large” \( |t| \). In particular, we use this for \( 1/2\delta \leq t \leq 1/\delta \) to see that \( B(t) \gg \delta^{1/4} \) for \( 1/2\delta \leq t \leq 1/\delta \). Recalling that \( B(t) > 0 \), we have by the above (with sufficiently small \( \delta \))
\[
\int_{-\infty}^{\infty} |A(t)||B(t)| dt \geq \int_{1/2\delta}^{1/\delta} |A(t)||B(t)| dt \gg \delta^{1/4} \int_{1/2\delta}^{1/\delta} |\zeta(1/2 + it)| dt \gg \delta^{-3/4}.
\]
For the last inequality here we are using Lemma 2 with \( T = 1/\delta \) and \( T = 1/2\delta \), with 1/\( \delta \) “large” (i.e., \( \delta \)
“small”). On the other hand, $|z| \gg 1$ since $\delta$ is “small”, so that by (2) and (4) (setting $v = u\sqrt{\pi \sin(\delta))}$

$$
\left| \frac{1}{2\pi} \int_{-\infty}^{\infty} A(t)B(t) \, dt \right| = \left| -z^{-1/4} + z^{1/4} 2 \sum_{n=1}^{\infty} e^{-\pi n^2 z} \right|
\ll 1 + \sum_{n \geq 1} |e^{-\pi n^2 z}|
= 1 + \sum_{n \geq 1} e^{-\pi n^2 \cos(\pi/2 - \delta)}
= 1 + \sum_{n \geq 1} e^{-\pi n^2 \sin(\delta)}
\leq 1 + \int_{0}^{\infty} e^{-\pi u^2 \sin(\delta)} \, du
= 1 + \frac{1}{\sqrt{\pi \sin(\delta)}} \int_{0}^{\infty} e^{-u^2} \, dv
\ll \delta^{-1/2}.
$$

Finally, we suppose by contradiction that there are only finitely many zeros $\rho$ of the zeta function with $\Re(\rho) = 1/2$. In particular, for $C$ sufficiently large $A(t)$ does not change sign for $|t| \geq C$. Thus

$$
\int_{|t| \geq C} |A(t)|B(t) \, dt = \left| \int_{|t| \geq C} A(t)B(t) \, dt \right|.
$$

Since $C$ is fixed at this point, we get

$$
\left| \int_{-\infty}^{\infty} A(t)B(t) \, dt \right| = \int_{-\infty}^{\infty} |A(t)|B(t) \, dt + O(1),
$$

where the implicit constant is independent of $\delta$ (it will depend on $C$). Since this equation contradicts our inequalities (5) and (6) for sufficiently small $\delta$, we conclude that there must be infinitely many zeros $\rho$ with $\Re(\rho) = 1/2$. 

5